# Production of electron-positron pairs by high-energy photons in oriented crystals

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The theory of pair production by a photon in single crystals is developed for arbitrary energies and entry angles. For small entry angles the theory describes the pair production in the field of the axes or planes of the crystal. In this case corrections to the constant-field approximation are found. For relatively large entry angles the general expressions go over to the standard theory of coherent pair production. A modification of the theory of coherent pair production applicable to a broader range of entry angles is obtained.

## I. INTRODUCTION

When a high-energy photon interacts with a crystal the mechanism by which electron-positron pairs are produced is considerably modified in comparison with an amorphous medium. Even in the 1950s it was known that when the photon momentum $\mathbf{k}$ lies near the direction of axes or planes there is interference of the contributions to pair production at different centers. For certain values of the photon angle of incidence and energy the probability of this mechanism (coherent pair production; see Refs. 1 and 2 and references cited therein) differs substantially from the probability of independent pair production at separate centers, which occurs in an amorphous medium (the Bethe-Heitler mechanism).

Also well known is the mechanism for production of a pair of particles by a high-energy photon in a constant external electromagnetic field (see, e.g., Section 11 of Ref. 3). Very recently it has been established that for photon energies which are readily available at the present time (tens of GeV) this mechanism can appear in crystals when the photon angle of incidence $\vartheta_0$ (the angle between $\mathbf{k}$ and the axis or plane) is small. The constant-field approximation is applicable if $\vartheta_0 \ll V_0/m$, where $V_0$ is the characteristic scale of the potential, for in this case the field in the axis (or plane) can be considered constant over the pair formation length.[4] The rest of the problem then reduces to selecting an adequate potential and performing appropriate averages.[4,5] This mechanism has been discussed previously also in Refs. 6 and 7. In Ref. 6, where the planar case was considered (for which the effect appears at a higher energy), an asymptotic expression was found for the pair production probability at low frequencies [in essence, Eq. (11.32) from Ref. 3] and an expression was obtained for the spectrum of one of the resulting particles of a polarized $e^+e^-$ pair in a constant field [we note that this result follows from Eq. (10.100) of Ref. 3 with use of substitution rules]. The subsequent analysis reduced to estimates which contain a number of inaccuracies. In Ref. 7 the problem for the axial case was solved numerically, without use of the results for a constant field, for the *⟨110⟩ axis of diamond*. The probabilities obtained by us in Ref. 4 (for the same conditions) was found to be several times lower than those calculated in Ref. 7. In Ref. 8 the constant-field results were used, but a discrepancy with Refs. 4 and 5 remains; after correcting errors in the calculations the probabilities were found to be in reasonable agreement with Refs. 4 and 5 (J. C. Kimball, private communication).

The behavior of the pair production probability in a constant field is determined by the parameter[1)]

$$\varkappa = \frac{e}{m^3}\left(|F_{\mu\nu}k^\nu|^2\right)^{1/2} \xrightarrow[H=0,\,\mathbf{E}\perp\mathbf{k}]{} \frac{\omega}{m}\frac{E}{E_0}, \tag{1.1}$$

where $K^\nu(\omega, \mathbf{k})$ is the photon 4-momentum, $m$ is the electron mass, $E$ is the local value of the electric field of the axis or plane, and $E_0 = m^2/e = m^2c^3/e\hbar = 1.32\cdot 10^{16}$ V/cm is the critical field. For $\varkappa \ll 1$ the probability of pair production in a constant field is exponentially small, $\propto \exp(-8/3\varkappa)$, and in single crystals only coherent pair production is a specific effect. For $\varkappa \sim 1$ the pair production probability in a constant field $W_e^{CF}$ rises rapidly and becomes equal to the probability of the Bethe-Heitler mechanism $W_{BH}$ at a photon energy $\omega = \omega_t$. For example, at room temperature for the ⟨111⟩ axis in tungsten $\omega_t \approx 22$ GeV and $\omega_t \approx 90$ GeV in diamond[4]; see Table I.

Many features of this process—pair production by a photon in a crystal—are similar to the production of pairs in the field of a plane electromagnetic wave. This is due to the fact that for photons (ultrarelativistic particles) moving near axes or planes, the crystal field can be reduced to a flux of incident equivalent photons. Since the description of pair production in the field of the wave is considerably simpler than in a single crystal, the analogy mentioned turns out to be extremely useful. We shall represent the crystal potential in the form

$$U(\mathbf{r}) = \sum_{\mathbf{q}} G(\mathbf{q})\, e^{-i\mathbf{q}\mathbf{r}}; \tag{1.2}$$

the explicit form of $G(\mathbf{q})$ will depend on the type of lattice and so far we do not need it. In the rest system of the crystal there is only an electric field $\mathbf{E}$. In the system of reference moving with a velocity $\mathbf{v} = \mathbf{n}v$ along the direction of entry of the photon $\mathbf{n}$, a magnetic field $\mathbf{H} = \gamma_v \mathbf{E}\times\mathbf{v}$ arises, where $\gamma_v = (1 - v^2)^{-1/2} \gg 1$. The resulting field in this system, as is well known, can be represented in the form of a flux of equivalent photons with frequency $\gamma_v |q_\parallel| v$, where $q_\parallel = \mathbf{qn}$, which with relativistic accuracy has the form

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TABLE I. Parameters of the potential and certain quantities characterizing the production of pairs by a photon in the field of crystal axes.

| Crystal | $u_1$ | $V_0$ | $\eta$ | $a_s$, Å | $x_0$ | $\omega_t$, GeV | $r^{max}$ | $\frac{\varkappa_m}{\varkappa_s}$ |
|---|---|---|---|---|---|---|---|---|
| C($d$) | 0.040 | 29 | 0.025 | 0.326 | 5.5 | 90 | 156 | 6.03 |
| Si($d$) | 0.075 | 54 | 0.150 | 0,299 | 15.1 | 150 | 66 | 2.02 |
| Fe | 0.068 | 180 | 0,145 | 0.276 | 19.8 | 50 | 33 | 2.07 |
| Ge($d$) | 0.085 | 91 | 0,130 | 0.300 | 16.3 | 60 | 25 | 2.24 |
| Ge($d$) (100) | 0.054 | 114,5 | 0.063 | 0.302 | 19,8 | 50 | 27 | 3.54 |
| W | 0.050 | 417 | 0.115 | 0.215 | 39.7 | 22 | 10.5 | 2,43 |
| W (77) | 0.030 | 348 | 0,027 | 0.228 | 35,3 | 13 | 10.8 | 5.78 |

*Note.* Data are given for the ⟨110⟩ axis in cooled Ge(100) and for the ⟨111⟩ axis in the remaining cases. The numbers in parentheses after the designation of the crystal are the values of temperature; where no numbers are given, $T = 293$ K. The quantities $V_0$, $a_s$, $\eta$, and $x_0$ are the parameters of the potential (3.8); $u_1$ is the amplitude of thermal vibrations; $w_t$ is the photon energy at which the probability of the process in the field of axes is comparable with the amorphous value; $r^{max} = W^{CF}_{e\,max}/W_{BH}$ is an estimate on the basis of Eq. (3.15) of the maximum value of the effect. The symbol (d) denotes a structure of the diamond type fcc(d).

$$\mathbf{J}_\mathbf{q} = -\mathbf{n}\frac{\gamma_v}{4\pi e^2}\frac{|G(\mathbf{q})|^2}{q_\parallel}\mathbf{q}_\perp^2, \tag{1.3}$$

where $\mathbf{q}_\perp = \mathbf{q} - \mathbf{n}(\mathbf{qn})$. In the interaction region, the transverse dimension of which is of the order $\lambda_c = 1/m = \hbar/mc$, and the longitudinal dimension is the formation length, which in the c.m.s. system of the incident and equivalent photons is of the order $2\pi/|q_\parallel|\gamma_v$, there are $N_\mathbf{q} \approx |\mathbf{J}_\mathbf{q}|2\pi\lambda_c^2/|q_\parallel|\gamma_v$ photons. The effective strength of the interaction is characterized by the parameter

$$\alpha N_{ph} = \alpha\sum_\mathbf{q} N_\mathbf{q} \approx \sum_\mathbf{q}\frac{|G(\mathbf{q})|^2\mathbf{q}_\perp^2}{m^2 q_\parallel^2}. \tag{1.4}$$

This parameter is purely classical (it does not contain $\hbar$) and always arises in problems with an external electromagnetic field. The parameter $\xi^2$ in the theory of processes with an intense plane wave has the same meaning (see for example Section 101 of Ref. 9 and also Ref. 10); the parameter $\rho$ in the problem of radiation in quasiperiodic motion also has the same meaning (see Ref. 11). For $\alpha N_{ph} \ll 1$ the external field can be taken into account in perturbation theory, and for $\alpha N_{ph} \gg 1$ we have the constant-field limit. For estimates we can assume that $|G(\mathbf{q})| \sim V_0$, $q_\parallel \sim q_\perp\vartheta_0$, in which case

$$\alpha N_{ph} \sim (V_0/m\vartheta_0)^2. \tag{1.5}$$

Therefore for $\vartheta_0 \ll V_0/m$ the constant-field approximation is applicable, and at $V_0 \gg V_0/m$ perturbation theory is valid, the first approximation of which is the theory of coherent pair production. These criteria were obtained previously by the present authors[4,5] from other considerations.

The probability of coherent pair production is greatest at photon entry angles

$$\vartheta_0 \sim \vartheta_m = m^2\Delta/\pi\omega, \tag{1.6}$$

where $\Delta$ is the typical distance between axes or planes. As the photon energy increases the position of the maximum shifts toward smaller angles, and when an angle $\vartheta_m \sim V_0/m$ is reached the theory of coherent pair production becomes inapplicable in the region of its maximum (and to the left of it). In this sense this theory is completely inadequate in the high-energy region.

The present work is devoted to construction of a unified description of the process of pair production by a photon in a crystal which is applicable for any photon entry angles and at any energies (Section II). For $\vartheta_0 \ll V_0/m$ and $\vartheta_0 \gg V_0/m$ the probabilities obtained go over to well-known limiting cases. In the case $\vartheta_0 \ll V_0/m$ a correction $\propto (m\vartheta_0/V_0)^2$ is also calculated. For $\varkappa \gg 1$ the probability for production of a pair of particles in a constant field $W_e^{CF}$ can exceed $W_{BH}$ by 10–100 times[5]; see also Table I and Section III.

In Section IV for the region $\vartheta_0 \gtrsim V_0/m$ we have obtained from the general formulas an expression for the probability which consists of a modification of the theory of coherent pair production which has a broader region of applicability in $\vartheta_0$ than the standard theory. Calculation of the probability on the basis of the general formula is a rather complicated computational problem. On the other hand, inclusion of corrections $\sim (m\vartheta_0/V_0)^2$ to the constant-field approximation and modification of the theory of coherent production permits one to describe the probability of the process by means of simple expressions for all entry angles except $\vartheta_0 \sim V_0/m$; in this region we used interpolation.

## II. QUASICLASSICAL DESCRIPTION OF PAIR PRODUCTION BY A PHOTON IN SINGLE CRYSTALS

A general expression for the probability of production of a pair of particles in an external field in the quasiclassical approximation was obtained in Section 11 of Ref. 3. The essential features of the problem considered appear only when we evaluate the magnitude of the discarded terms. As was shown, they are $\lesssim 1/ma \ll 1$, where $a$ is the size of the region of action of the potential, provided that

$$\rho_c \equiv V_0\omega/m^2 \gg 1. \tag{2.1}$$

Actually the production of electron-positron pairs in the field in an axis or plane becomes observable only if this condition is satisfied (see Ref. 4). Then the pair production probability (during the entire interaction time) is [compare with Eq. (2) in Ref. 5]

$$w_e = \frac{\alpha m^2}{(2\pi)^2\omega}\int\frac{d^3r_0}{V}\int\frac{d^3p_0}{\varepsilon\varepsilon'}$$

$$\times \int_{t_2} dt_1 \int dt_2 \left\{ 1 - \frac{\varepsilon^2+\varepsilon'^2}{4\varepsilon\varepsilon'} \gamma^2 [\mathbf{v}(t_1) - \mathbf{v}(t_2)]^2 \right\} e^{iA},$$

$$A = \frac{\varepsilon\omega}{2\varepsilon'} \int_{t_1} dt \left\{ \frac{1}{\gamma^2} + [\mathbf{n} - \mathbf{v}(t)]^2 \right\}. \tag{2.2}$$

Here $V$ is the volume of the crystal and $\mathbf{p}_0$ is the momentum of a particle of the resulting pair at the production point $\mathbf{r}_0$. Similar expressions for the radiation can be found in Section 2 of Ref. 11.

We shall represent the velocity as $\mathbf{v}(t) = \mathbf{v}_0 + \Delta\mathbf{v}(t, \mathbf{v}_0)$, and then the integrand in the phase $A$ in Eq. (2.2) can be written in the form

$$1/\gamma^2 + \mathbf{u}^2 + 2\mathbf{u}\Delta\mathbf{v} + (\Delta\mathbf{v})^2, \tag{2.3}$$

where

$$\mathbf{u} \equiv \mathbf{n} - \mathbf{v}_0 = \mathbf{n}(1 - \mathbf{n}\mathbf{v}_0) - \mathbf{v}_{0\perp} \approx -\mathbf{v}_{0\perp},$$

so that we can assume that the vector $\mathbf{u}$ lies in the plane perpendicular to $\mathbf{n}$. If the condition (2.1) is satisfied, the main contribution to the probability (2.2) will be from the portions of the trajectory in which the direction of the vector $\mathbf{v}_0$ is close to that of $\mathbf{n}$:

$$|\mathbf{u}| \sim 1/\gamma, \quad |\Delta\mathbf{v}(t, \mathbf{v}_0)| \sim 1/\gamma.$$

However, in that case we can make the substitution

$$\Delta\mathbf{v}(t, \mathbf{v}_0) \to \Delta\mathbf{v}(t, \mathbf{n}) \equiv \Delta\mathbf{v}(t).$$

Accordingly in the coefficient of the exponential expression in (2.2) we shall make the substitution

$$\mathbf{v}(t_1) - \mathbf{v}(t_2) \to \Delta\mathbf{v}(t_1) - \Delta\mathbf{v}(t_2).$$

With relativistic accuracy we can rewrite the phase space $d^3p_0 \approx \varepsilon^2 d\varepsilon d^2u$. After these transformations the integration over $\mathbf{u}$ can be carried out. As a result we obtain

$$w_e = \frac{i\alpha m^2}{2\pi\omega^2} \int \frac{d^3r_0}{V} \int^{\omega} d\varepsilon \int \frac{dt\, d\tau}{\tau + i0} \left\{ 1 - \frac{\varepsilon^2+\varepsilon'^2}{4\varepsilon\varepsilon'} \gamma^2 \left[ \Delta\mathbf{v}\left(t - \frac{\tau}{2}\right) - \Delta\mathbf{v}\left(t + \frac{\tau}{2}\right) \right]^2 \right\} e^{iA_1},$$

where

$$A_1 = \frac{\varepsilon\omega\tau}{2\varepsilon'} \left\{ \frac{1}{\gamma^2} + \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} dx [\Delta\mathbf{v}(t+x)]^2 - \left[ \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} dx\, \Delta\mathbf{v}(t+x) \right]^2 \right\} \tag{2.4}$$

and we have gone over to the variables $t$, $\tau$: $t_1 = t - \tau/2$, $t_2 = t + \tau/2$; the direction in which the point $\tau = 0$ is enclosed (the sign of the imaginary increment) follows from the condition of convergence of the integral over $u$.

All of the interesting effects occur at small photon-entry angles $\vartheta_0 \ll 1$. We shall consider the case in which $\vartheta_0 \gg \vartheta_c$, where $\vartheta_c = (2U_0/\omega)^{1/2}$ is the critical angle for a given axis or plane and $U_0$ is the depth of the corresponding potential well. Since $\rho_c \gg 1$, we have $\vartheta_c \gg 1/\gamma$. It was shown above that the particles of the pair which results are emitted into a narrow cone with an opening angle $\sim 1/\gamma$ about the direction of $n$. Then, if $\vartheta_0 \gg \vartheta_c$, the particles of the pair will move high above the barrier (from the point of view of the mode of their motion in the channels formed by the axes or planes) and to find $\Delta v(t)$ for them one can use the straight-line-trajectory approximation. In the crystal potential (1.2), using the equation of motion and the straight-line-trajectory approximation, we find

$$\Delta\mathbf{v}(t) = -\frac{1}{\varepsilon} \sum_{\mathbf{q}} G(\mathbf{q}) e^{-i\mathbf{q}\mathbf{r}} (e^{-iq_\parallel t} - 1) \frac{\mathbf{q}_\perp}{q_\parallel}. \tag{2.5}$$

We note that, since in (2.4) $\Delta\mathbf{v}$ enters quadratically, the result in the straight-line-trajectory approximation no longer depends on the sign of the potential, and the difference between electrons and positrons in the final state drops out. Substituting (2.5) into (2.4) and calculating the integrals which are obtained, we find the pair production probability:

$$w_e = \frac{i\alpha m^2}{2\pi\omega^2} \int \frac{d^3r_0}{V} \int^{\omega} d\varepsilon \int dt \int \frac{d\tau}{\tau + i0} \left\{ 1 + \frac{\varepsilon^2+\varepsilon'^2}{\varepsilon\varepsilon'} \times \left( \sum_{\mathbf{q}} G(\mathbf{q}) \frac{\mathbf{q}_\perp}{m q_\parallel} \exp[-i(\mathbf{q}\mathbf{r}_0 + q_\parallel t)] \sin\frac{q_\parallel \tau}{2} \right)^2 \right\} e^{iA_2}, \tag{2.6}$$

$$A_2 = \frac{m^2\omega\tau}{2\varepsilon\varepsilon'} \left\{ 1 + \sum_{\mathbf{q},\mathbf{q}'} \frac{G(\mathbf{q})G(\mathbf{q}')}{m^2} \frac{\mathbf{q}_\perp\mathbf{q}_\perp'}{q_\parallel q_\parallel'} \exp(-i[(\mathbf{q}+\mathbf{q}')\mathbf{r}_0 + (q_\parallel + q_\parallel')t]) \times \left[ 2\frac{\sin[(q_\parallel + q_\parallel')\tau/2]}{(q_\parallel + q_\parallel')\tau} - 4\frac{\sin(q_\parallel\tau/2)}{q_\parallel\tau} \frac{\sin(q_\parallel'\tau/2)}{q_\parallel'\tau} \right] \right\}. \tag{2.7}$$

In this expression the time $t$ enters only in the form of the combination $r_{0\parallel} + t$, and after the substitution $r_{0\parallel} + t \to r_{0\parallel}$ the dependence on time remains only in the limits of integration over $r_{0\parallel}$. This dependence can be neglected if the length of the crystal (in the longitudinal direction) is considerably greater than the formation length in the production of an electron-positron pair $l_f$. At an energy $\omega \approx 100$ GeV in germanium $l_f \sim 10^{-5}$ cm and increases slowly with increase of $\omega$ (asymptotically $\propto \omega^{1/3}$), so that this condition is essentially always satisfied. In this case we can omit the integration over $t$, and the region of integration over $\tau$, since the integrals converge rapidly, can be extended to infinity. As a result we obtain for the probability of production of an electron-positron pair per unit time

$$\frac{dw_e}{dt} \equiv W_e = \frac{i\alpha m^2}{2\pi\omega^2} \int \frac{d^3r_0}{V} \int^{\omega} d\varepsilon \int_{-\infty}^{\infty} \frac{d\tau}{\tau + i0} \left\{ 1 + \frac{\varepsilon^2+\varepsilon'^2}{\varepsilon\varepsilon'} \times \left[ \sum_{\mathbf{q}} \frac{G(\mathbf{q})\mathbf{q}_\perp}{m q_\parallel} e^{-i\mathbf{q}\mathbf{r}_0} \sin q_\parallel \tau \right]^2 \right\} e^{iA_3},$$

$$A_3 = \frac{m^2\omega\tau}{\varepsilon\varepsilon'} \left\{ 1 + \sum_{\mathbf{q},\mathbf{q}'} \frac{1}{m^2} G(\mathbf{q}) G(\mathbf{q}') \frac{\mathbf{q}_\perp\mathbf{q}_\perp'}{q_\parallel q_\parallel'} \exp[-i(\mathbf{q}+\mathbf{q}')\mathbf{r}_0] \times \left[ \frac{\sin(q_\parallel + q_\parallel')\tau}{(q_\parallel + q_\parallel')\tau} - \frac{\sin q_\parallel\tau}{q_\parallel\tau} \frac{\sin q_\parallel'\tau}{q_\parallel'\tau} \right] \right\}. \tag{2.8}$$

Although the expression for the probability $W_e$ was obtained for an entry angle $\vartheta_0 \gg \vartheta_c$, below we shall show that actually the formula (2.8) is valid right down to $\vartheta_0 = 0$ and is a general expression for the probability of pair production by a high-energy photon entering the crystal at a small angle to a crystallographic axis or plane. In the large-angle region $W_e$ (2.8) goes over to the well-known probability of coherent pair production, and at small angles it goes over to the probability of pair production in a constant field. These transitions will be traced in detail below. In the intermediate region it is necessary to use (2.8) directly.

### III. PROBABILITY OF THE PROCESS FOR $\vartheta_0 \ll V_0/m$. CORRECTIONS TO THE CONSTANT-FIELD APPROXIMATION

Let us turn to analysis of the general expression (2.8). For estimates we shall assume

$$G(\mathbf{q}) \sim V_0, \quad |\mathbf{q}_\perp| \sim 1/a, \quad q_\parallel \sim \vartheta_0/a,$$

where $a$ is the characteristic linear scale of action of the potential. Then the order of magnitude of the double sum in $A_3$ in (2.8) is $(V_0/m\vartheta_0)^2$. We shall discuss first the case of relatively small angles when $\vartheta_0 \ll V_0/m$ (as before we shall assume that $\vartheta_0 \gg \vartheta_c$, which is possible as a result of the condition $\rho_c \gg 1$). We shall choose the $z$ axis along the crystallographic axis with respect to which[2)] the angle $\vartheta_0$ is measured. Then for vectors $\mathbf{q}$ lying in the $xy$ plane we have $q_\parallel \tau \sim \vartheta_0 m/V_0 \ll 1$, and for any other $\mathbf{q}$ we will have $q_\parallel \tau \sim m/V_0 \gg 1$, while in the double sum in $A_3$ it is necessary to retain only terms with $q_z = q_z' = 0$; the contribution of the remaining terms with $q_z \neq 0$ is down by a power in the parameter $V_0/m$. Expanding the functions $q_\parallel \tau$ in powers of the argument, we obtain

$$A_3 = \frac{m^2\omega\tau}{\varepsilon\varepsilon'}\Big\{1 - \frac{\tau^2}{3}\sum_{\mathbf{q}_t,\mathbf{q}_t'} \frac{1}{m^2} G(\mathbf{q}_t)G(\mathbf{q}_t')\mathbf{q}_t\mathbf{q}_t' \times\exp[-i(\mathbf{q}_t+\mathbf{q}_t')\boldsymbol{\rho}_0] \times(1-{}^1/_{10}\tau^2[(\mathbf{q}_t\mathbf{n})^2+(\mathbf{q}_t'\mathbf{n})^2+{}^2/_3(\mathbf{q}_t\mathbf{n})(\mathbf{q}_t'\mathbf{n})])\Big\}, \tag{3.1}$$

where the subscript $t$ denotes the component of a vector lying in the $xy$ plane and $\boldsymbol{\rho}_0 = \mathbf{r}_{0t}$. We can rewrite Eq. (3.1) in terms of the potential along the axes, which depends only on the transverse ordinate [compare with Eq. (1.2)]

$$U(\boldsymbol{\rho}_0) = \sum_{\mathbf{q}_t} G(\mathbf{q}_t)\exp(-i\mathbf{q}_t\boldsymbol{\rho}_0), \tag{3.2}$$

namely:

$$A_3 = \frac{m^2\omega\tau}{\varepsilon\varepsilon'}\Big\{1 + \frac{\tau^2}{3}\mathbf{b}^2 + \frac{\tau^4}{15}\Big[\mathbf{b}(\mathbf{n}\nabla)^2\mathbf{b} + \frac{1}{3}((\mathbf{n}\nabla)\mathbf{b})^2\Big]\Big\}, \tag{3.3}$$

where $\mathbf{b} = \nabla U(\rho_0)/m$, $\nabla = \partial/\partial\boldsymbol{\rho}_0$. After a similar procedure the expression multiplying the exponential in the integrand of (2.8) takes the form

$$\{\ldots\} = 1 - \frac{\varepsilon^2+\varepsilon'^2}{\varepsilon\varepsilon'}\tau^2\Big[\mathbf{b}^2 + \frac{\tau^2}{3}\mathbf{b}(\mathbf{n}\nabla)^2\mathbf{b}\Big]. \tag{3.4}$$

Note that $\mathbf{b}$ is expressed in terms of the particle acceleration $\mathbf{b} = -\gamma\mathbf{v}(\rho_0)$, so that the entire expression is dependent on the local properties of the motion, which is characteristic for pair production by a photon in a constant field. The terms in (3.3) which contain $\mathbf{n}$ have a relative magnitude $\sim(m\vartheta_0/V_0)^2$, and we can expand these terms in $\exp iA_3$. After substituting (3.3) and (3.4) into (2.8) and performing this expansion, we evaluate the integrals over $\tau$ by means of well-known relations:

$$\int_{-\infty}^{\infty} d\tau \exp\Big[-i\lambda\Big(\tau+\frac{\tau^3}{3}\Big)\Big] = \frac{2}{3^{1/2}}K_{1/3}\Big(\frac{2\lambda}{3}\Big),$$
$$\int_{-\infty}^{\infty} d\tau\, \tau \exp\Big[-i\lambda\Big(\tau+\frac{\tau^3}{3}\Big)\Big] = -\frac{2i}{3^{1/2}}K_{2/3}\Big(\frac{2\lambda}{3}\Big), \tag{3.5}$$

and the recurrence relations for the modified Bessel functions $K_\nu(z)$. Carrying out this calculation, we obtain for the pair production probability in the limit $\vartheta_0 \ll V_0/m$

$$W_e^F = \frac{\alpha m^2}{3^{1/2}\pi\omega^2}\int^{\omega} d\varepsilon \int\frac{d^2\rho_0}{S}\Big\{\varphi(\varepsilon)K_{2/3}(\lambda) + \int_\lambda^\infty dy\, K_{1/3}(y) - \frac{1}{3}\frac{\mathbf{b}(\nabla\mathbf{n})^2\mathbf{b}}{|\mathbf{b}|^4}\varphi(\varepsilon)\Big[K_{2/3}(\lambda) - \frac{2}{3\lambda}K_{1/3}(\lambda)\Big]$$
$$- \frac{\lambda}{30|\mathbf{b}|^4}[((\mathbf{n}\nabla)\mathbf{b})^2+3\mathbf{b}(\mathbf{n}\nabla)^2\mathbf{b}] \times\Big[K_{1/3}(\lambda) - \frac{4}{3\lambda}K_{2/3}(\lambda) - \varphi(\varepsilon)\Big(\frac{4}{\lambda}K_{2/3}(\lambda) - K_{1/3}(\lambda) - \frac{16}{9\lambda^2}K_{1/3}(\lambda)\Big)\Big]\Big\},$$
$$\varphi(\varepsilon) = \varepsilon/\varepsilon' + \varepsilon'/\varepsilon, \quad \lambda = 2m^2\omega/3\varepsilon\varepsilon'|\mathbf{b}|. \tag{3.6}$$

Since the expression for $W_e^F$ does not depend on $z_0$, we have

$$\int d^3r_0/V \to \int d^2\rho_0/S,$$

where $S$ is the area of the cross section of the crystal perpendicular to the selected axis. In view of the periodicity of the crystal, the average over $\rho_0$ can be carried out over the region of the cross section associated with a single axis; then $S = S_0$ is the area of this region. The first two terms in (3.6), which do not depend on $\mathbf{n}$, are the pair production probabilities assuming a constant (over the pair-formation length) field which were found previously in Refs. 4 and 5, and the remaining terms are a correction $\propto \vartheta_0^2$.

Further simplifications of Eq. (3.6) are possible if the potential of the axis can be assumed to be axisymmetric. In some cases, for example, for the ⟨111⟩ axis in crystal structures of the fcc(d) and bcc types, the potential is very accurately axisymmetric about the distinguished axis over the entire region $S_0$ asociated with this axis. However, even if the potential is not axisymmetric over the entire region $S_0$, it in any case has these properties for $|\boldsymbol{\rho}_0| \lesssim a_s$, where $a_s$ is the effective screening radius of the potential of the string and $\rho_0$ is measured from the axis. The intensity of the electric field in which pair production occurs is maximal just in the region $|\boldsymbol{\rho}_0| \lesssim a_s$. Assuming that $U = U(\rho_0^2)$, we can carry out the integration over the angles of the vector $\boldsymbol{\rho}_0$ in Eq. (3.6):

$$W_e^F = \frac{\alpha m^2}{3^{1/2}\pi\omega^2}\int_0^{x_0}\frac{dx}{x_0}\int_0^{\omega} d\varepsilon\Big\{\varphi(\varepsilon)K_{2/3}(\lambda) + \int_\lambda^\infty dy\, K_{1/3}(y) - \frac{1}{6}\Big(\frac{m\vartheta_0}{V_0}\Big)^2\Big[\varphi(\varepsilon)\frac{xg''+2g'}{xg^3}\Big(K_{2/3}(\lambda) - \frac{2}{3\lambda}K_{1/3}(\lambda)\Big)$$

$$+\frac{1}{20g^4x^2}(2x^2g'^2+g^2+14gg'x+6x^2gg'')\Big(\lambda K_{1/3}(\lambda)-\frac{4}{3}K_{2/3}(\lambda)$$

$$-\varphi(\varepsilon)\Big(4K_{2/3}(\lambda)-\Big(\lambda+\frac{16}{9\lambda}\Big)K_{1/3}(\lambda)\Big)\Big)\Big]\Big\}, \qquad (3.7)$$

where we have gone over to the variable $x=\rho_0^2/a_s^2$; $x_0^{-1}=\pi a_s^2 nd$; $d$ is the average distance between atoms of the string forming the axis, $n$ is the density of atoms in the crystal, and we have used the notation $U'(x)=V_0 g(x)$; then we have

$$\lambda=\frac{m^3\omega a_s}{3\varepsilon\varepsilon' V_0|x^{1/2}g|}=\frac{1}{3}\frac{\omega^2}{\varepsilon\varepsilon'}\frac{1}{\varkappa(a_s)}\frac{1}{|x^{1/2}g|}.$$

Eq. (3.7) is applicable for an arbitrary shape of an axisymmetric lattice potential. To obtain specific predictions we shall use an explicit form of the axis potential which has been found to be completely adequate in the radiation problem[12] and has been used in Refs. 4, 5, and 13:

$$U(x)=V_0\left[\ln\left(1+\frac{1}{x+\eta}\right)-\ln\left(1+\frac{1}{x_0+\eta}\right)\right]. \qquad (3.8)$$

For estimates we can assume that $V_0\approx Ze^2/d$, $\eta\approx 2u_1^2/a_s^2$, where $u_1$ is the amplitude of thermal vibrations. Actually the potential parameters were determined by means of a fitting procedure, of which the details can be found in Ref. 12, using a model of the lattice potential based on a Moliere potential for an isolated atom (the parameters are given in the table). For the potential (3.8) the value of the electric field and the parameter $\varkappa$ are

$$eE=-\frac{\partial U}{\partial\rho}=\frac{2V_0}{a_s}x^{1/2}g(x),$$

$$g(x)=\frac{1}{x+\eta}-\frac{1}{x+\eta+1},\quad \varkappa(x)=\frac{2V_0\omega}{a_s m^3}x^{1/2}g(x). \qquad (3.9)$$

The electric field (3.9) vanishes at $x=0$ as a result of thermal vibrations, reaches a maximum at

$$x=x_m=\tfrac{1}{6}\{[1+16\eta(\eta+1)]^{1/2}-1-2\eta\},$$

and then falls off. In what follows we shall use

$$\varkappa_m\equiv\varkappa(x_m),\qquad \varkappa_s=V_0\omega/a_s m^3.$$

We note that the spectrum $dW_e^F/d\varepsilon$ of one of the particles of the produced pairs follows from Eq. (3.7) if we omit integration over $\varepsilon$.

Equations (3.6) and (3.7) were obtained from the probability (2.8) in the region where the latter is applicable ($\vartheta_0\gg\vartheta_c$), but as the angle $\vartheta_0$ decreases further the probability changes insignificantly, which obviously follows from (3.7), so that Eqs. (3.6) and (3.7) and therefore also Eq. (3.8) are valid down to $\vartheta_0=0$.

The behavior of the probability in a constant field is illustrated in Figs. 1 and 2. In Fig. 1 we have shown the pair production probability as a function of photon energy for several materials. In Fig. 2 we have shown the spectrum of the electrons or positrons produced for various photon energies for the $\langle 111\rangle$ axis in tungsten. It can be seen that the peak at $\varepsilon\approx\omega/2$ at comparatively low energies gradually goes over into a broad plateau as the energy increases, and in the middle of the plateau a dip then appears, so that the spectrum shape becomes rather similar to the Bethe-Heitler spectrum.

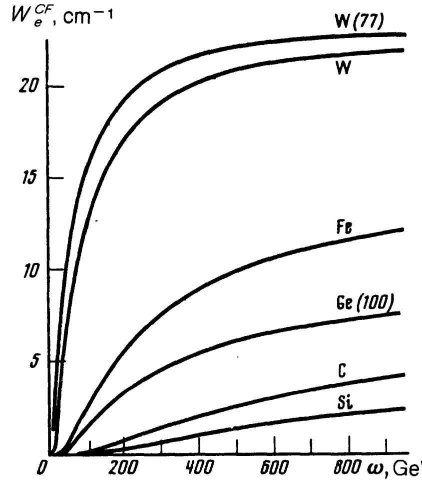


FIG. 1. Probability of production of a pair by a photon for an entry angle $\vartheta_0=0$ relative to the $\langle 111\rangle$ axis (for Ge the $\langle 110\rangle$ axis). The numbers in parentheses denote the temperature of the crystal; where the temperature is not given it is $T=993$ K.

There is an analogy between Eq. (3.7), which we shall give in the form

$$W_e^F=F_1+(m\vartheta_0/V_0)^2F_2, \qquad (3.10)$$

and the probability of radiation in quasiperiodic motion $W_r$, when the principal parameter which determines the properties of the radiation is $\rho=2\gamma^2[\langle \mathbf{v}^2\rangle-\langle\mathbf{v}\rangle^2]\gg 1$. Here the angle brackets indicate averaging over time. For $\rho\ll 1$ the radiation is dipole, and for $\rho\gg 1$ we have the limit of synchrotron radiation. In the latter case the quantity $W_r$ can be represented in the form $W_r=f_1+f_2/\rho$ [compare with Eq. (4.16) in Ref. 11]. Thus, the quantity $(V_0/m\vartheta_0)^2$ plays the role of the parameter $\rho$. The ratio $F_2/F_1$ of the functions in Eq. (3.10) is shown in Fig. 3 as a function of energy for various materials. At those values of $\omega$ for which $F_2>0$ the probability (3.10) has a minimum at $\vartheta_0=0$. For $\varkappa_s\approx(5.1–5.3)\eta^{1/2}$ the function $F_2$ changes sign and at large values of $\varkappa_s$ the probability $W_e^F$ has a maximum at $\vartheta_0=0$. As $\omega(\varkappa_s)$ continues to increase the height of this maximum increases and the width begins to narrow, since for $\varkappa_s\gg 1$ $F_2$ increases faster than $F_1$.

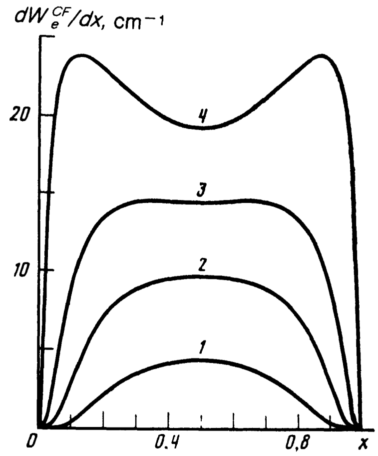


FIG. 2. Distribution in energy $\varepsilon$ of one of the particles of the produced pair ($x=\varepsilon/\omega$) for the $\langle 111\rangle$ axis in tungsten, $T=293$ K, $\vartheta_0=0$; for $\omega=25$ GeV (1), 50 GeV (2), 100 GeV (3), and 500 GeV (4).

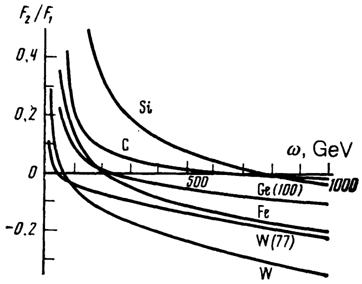


FIG. 3. The ratio $F_2/F_1$ in Eq. (3.10), which characterizes the magnitude of the correction to the constant-field limit. The notation is the same as in Fig. 1.

The maximum value of $W_e^F$ from (3.7) and (3.10) at $\vartheta_0 = 0$ is reached for $\varkappa_s \gg 1$. In this case we can obtain an explicit asymptotic expression[3)] for $W_e$ which is very convenient for our analysis. The region of integration in (3.7) can be broken up into three parts: 1) $0 \leqslant x \leqslant x_1 \ll 1$; 2) $x_1 \leqslant x \leqslant x_2$, where $x_2 \gg 1$; 3) $x_2 \leqslant x \leqslant x_0$, assuming $x_0 \gg 1$, with $\lambda(x_1) \ll 1, \lambda(x_2) \ll 1$. It is necessary also to take into account that the integral over energy gets its main contribution from the region in which $\omega^2/4\varepsilon\varepsilon' \sim 1$. Then in regions 1 and 3 we can replace $x^{1/2}g(x)$ by approximate expressions for small and large $x$ and go over to integration over $\lambda$, while in region 2 we can use the expansion of the functions $K_\nu(\lambda)$ for $\lambda \ll 1$, extending the integration to infinity. As the result of rather involved calculations we obtain

$$W_e^F(\varkappa_s \gg 1) \approx \frac{\alpha}{3^{1/2}\pi}\frac{V_0}{m x_0 a_s}\Big\{c_1\varkappa_s^{-1/3}[\ln\varkappa_s + B_1(\eta)] - \left(\frac{m\vartheta_0}{V_0}\right)^2 (1+2\eta)c_2\varkappa_s^{1/3}[\ln\varkappa_s + B_2(\eta)]\Big\}, \quad (3.11)$$

where

$$c_1 = \frac{45}{7}\frac{\Gamma^3(2/3)}{\Gamma(1/3)}\left(\frac{4}{3}\right)^{1/3} \approx 6.558,$$

$$c_2 = \frac{7\Gamma^3(1/3)}{15\cdot 495\Gamma(2/3)}\left(\frac{3}{4}\right)^{1/3} \approx 0.097,$$

$$B_1(\eta) = B_1(0) - \frac{3}{2}\int_0^\infty \frac{dx}{x}\{(1+x)^{-2/3} - (1+\eta x)^{-2/3}[1+(1+\eta)x]^{-2/3}\},$$

$$B_2(\eta) = B_2(0) - \frac{3}{2}\int_0^\infty \frac{dx}{x}\{(1+x)^{-1/3} - (1+\eta x)^{-1/3}[1+(1+\eta)x]^{-1/3}\}$$

$$-\frac{3}{16(1+2\eta)}\int_0^\infty dx(1+\eta x)^{-4/3}[1+(1+\eta)x]^{-4/3} + \frac{9}{16}, \quad (3.12)$$

$$B_1(0) = 3^{1/2}\pi/2 + {}^5/_2 \ln 3 + \ln 2 - C + {}^3/_{70} - 6 \approx -0.374,$$

$$B_2(0) = -3^{1/2}\pi/2 + {}^5/_2 \ln 3 + \ln 2 - C + 4 + {}^3/_7 - {}^6/_{11} - {}^{13}/_{40} \approx 3.70.$$

Here $C = 0.577$ ... is Euler's constant. The integrals in $B_1$ and $B_2$ are expressed in terms of hypergeometric functions. Taking into account that usually $\eta \ll 1$, we give the following approximate expressions:

$$B_1(\eta) = -0.374 - 3.975\beta^{2/3}(1 + {}^8/_{15}\beta + {}^7/_{18}\beta^2) + \beta({}^3/_2 + {}^9/_8\beta + {}^{13}/_{14}\beta^2),$$

$$B_2(\eta) = 3.70 - 2.31\beta^{1/3}(1 + {}^2/_3\beta - {}^{19}/_{63}\beta^2) - {}^1/_{160}\beta(30 + 144\beta - 29\beta^2), \quad (3.13)$$

where $\beta = \eta/(1+\eta)$.

In deriving (3.11) the integration over $x$ was carried out between infinite limits. Since the main (logarithmic) contribution to (3.11) is from the region $x \sim \varkappa_s^{2/3}$ (region 3), it is clear that the asymptotic behavior (3.11) is valid only for $x_s^{2/3} < x_0$, i.e., for photon energies less than a few TeV. At such energies when $\varkappa_s^{2/3} \sim x_0$, the main contribution is from large distances from the axis, where the cylindrical symmetry of the potential can be destroyed and it is necessary to take into account the influence of several axes. Then the probability of pair production at small entry angles is given by Eq. (3.6), in which it is necessary to substitute the potential (3.2).

The function $F_1(x_s \gg 1) \equiv F_1^{as}$ in (3.10) and (3.11) has a maximum at the point $\varkappa_s^0 = \exp[3 - B_1(\eta)]$, near which the function $F_1^{as}$ changes very smoothly. Therefore the energy value $\omega^0$ obtained from $\varkappa_s^0$ is only an order of magnitude estimate. However, the value $W_e^{F\max} = F_1^{as}(\varkappa_s^0)$ is given by Eq. (3.11) quite satisfactorily and is in good agreement with the results of a numerical calculation using the exact expression (3.7):

$$W_e^{F\max} \approx k\frac{\alpha}{3^{1/2}\pi}\frac{V_0}{m a_s x_0}, \quad k = c_1 \exp\left[\frac{1}{3}B_1(\eta) - 1\right], \quad (3.14)$$

where the coefficient $k$ depends on the material and, for the materials usually used, varies in the range 1.93–1.59 as $\eta$ varies from 0.025 (diamond) to 0.15 (silicon). Equation (6) of Ref. 5 used the rounded value $k = 2$.

It is of interest to determine the maximum value of the increase of the probability for pair production in the field of an axis over the pair production probability in the corresponding amorphous medium (the Bethe-Heitler mechanism). Taking (3.14) and $W_{BH}$ (see for example Refs. 3 and 9), we obtain

$$r^{\max} \equiv W_e^{F\max}/W_{BH} \approx \frac{3^{1/2}k}{28}\frac{V_0 m a_s d}{Z^2\alpha^2\ln(183Z^{-1/3})} \sim \frac{a_s m}{3Z\alpha\ln(183Z^{-1/3})}, \quad (3.15)$$

where the second (simplified) estimate is relatively crude. It can be seen from this that $r^{\max}$ is greater, the lower is $Z$ and the larger is $a_s$. The greatest gain (among the materials used) is achieved in diamond ($r^{\max} \sim 160$); values of $r^{\max}$ are given in Table I. Here it is appropriate to mention that the effect of pair production in the field of an axis becomes appreciable at $\varkappa_m \sim 1$ [see Eq. (4.9)], and a rough estimate of the corresponding energy[4] is

$$\omega \sim m^3 u_1 d/Z\alpha, \quad (3.16)$$

from which it is evident that the effect shows up primarily for those materials for which the gain in $r^{\max}$ is minimal.

## IV. PROBABILITY OF THE PROCESS FOR $\vartheta_0 > V_0/m$; MODIFICATION OF THE THEORY OF COHERENT PRODUCTION OF PAIRS

Let us consider now the region of large photon angles of incidence. We shall show first that in this region the results of the standard theory of coherent pair production follow from (2.8). From the estimates made at the beginning of Section III it follows that the order of magnitude of the double sum in $A_3$ (2.8) is $(V_0/m\vartheta_0)^2$. Therefore for $\vartheta_0 \gg V_0/m$ the term with the double sum in $A_3$ is small and $\exp(iA_3)$ can be expanded accordingly. As a result the probability (2.8) takes the form

$$W_e^{coh} = \frac{i\alpha m^2}{2\pi\omega^2}\int^{\omega} d\varepsilon \int_{-\infty}^{\infty} \frac{d\tau}{\tau+i0}\exp\left(i\frac{m^2\omega\tau}{\varepsilon\varepsilon'}\right) \times \sum_{\mathbf{q},\mathbf{q}'} \frac{G(\mathbf{q})G(\mathbf{q}')}{m^2 q_\parallel q_\parallel'}\mathbf{q}_\perp\mathbf{q}_\perp' \times\left\{\frac{\varepsilon'^2+\varepsilon^2}{\varepsilon\varepsilon'}\sin q_\parallel\tau \sin q_\parallel'\tau + \frac{im^2\omega\tau}{\varepsilon\varepsilon'}\left[\frac{\sin(q_\parallel+q_\parallel')\tau}{(q_\parallel+q_\parallel')\tau} - \frac{\sin q_\parallel\tau}{q_\parallel\tau}\frac{\sin q_\parallel'\tau}{q_\parallel'\tau}\right]\right\} \times \int \frac{d^3r_0}{V}\exp[-i(\mathbf{q}+\mathbf{q}')\mathbf{r}_0]. \tag{4.1}$$

Integration over $r_0$ in (4.1) is elementary and gives $\delta_{\mathbf{q}+\mathbf{q}',0}$ ($\mathbf{q}$ is a discrete variable), after which the sum over $\mathbf{q}'$ is calculated and the integrals over $\tau$ are easily evaluated by means of the theory of residues. Finally we obtain

$$W_e^{coh} = \frac{\alpha}{2\omega^2}\int^{\omega} d\varepsilon \sum_{\mathbf{q}} |G(\mathbf{q})|^2\frac{\mathbf{q}_\perp^2}{q_\parallel^2} \times\left\{\frac{\varepsilon^2+\varepsilon'^2}{\varepsilon\varepsilon'} + \frac{2\omega m^2}{q_\parallel^2\varepsilon\varepsilon'}\left(q_\parallel - \frac{\omega m^2}{2\varepsilon\varepsilon'}\right)\right\} \times\theta\left(q_\parallel - \frac{\omega m^2}{2\varepsilon\varepsilon'}\right). \tag{4.2}$$

Carrying out the integration over $\varepsilon$ in Eq. (4.2), we obtain

$$W_e^{coh} = \frac{\alpha}{\omega}\sum_{\mathbf{q}} |G(\mathbf{q})|^2\frac{\mathbf{q}_\perp^2}{q_\parallel^2} f\left(\frac{2m^2}{\omega q_\parallel}\right)\theta\left(q_\parallel - \frac{2m^2}{\omega}\right), \tag{4.3}$$

where

$$f(x) = \left(1 + x - \frac{x^2}{2}\right)\ln\frac{1+(1-x)^{1/2}}{1-(1-x)^{1/2}} - (1+x)(1-x)^{1/2}. \tag{4.4}$$

The probabilities (4.2) and (4.3) coincide with the results of the standard theory of coherent pair production (see for example Refs. 1 and 2). We recall (see Ref. 1) that the well-known expression for the cross section for production of a pair of particles by two photons leads directly to Eq. (4.3) by means of the method of equivalent photons (see for example page 414 of Ref. 9):

$$\sigma_{\gamma\gamma} = \pi(\alpha^2/m^2)yf(y), \quad y = 2m^2/kq.$$

Then for a photon density $|\mathbf{J}_\mathbf{q}|$ given by Eq. (1.3) we have

$$W_e = \sum_{\mathbf{q}} \sigma_{\gamma\gamma}|\mathbf{J}_\mathbf{q}|\theta(1-y),$$

where the $\theta$ function corresponds to the threshold of the reaction $\gamma + \gamma \to e^+ + e^-$.

For $\varkappa_s \gg 1$ one can obtain from the general expression (2.8) approximate expressions for the probabilities, the region of applicability of which is considerably broader than that of the standard theory of coherent production. For this purpose it is necessary to take into account that the second term in the double sum in the function $A_3$ in (2.8) has an order of magnitude $(V_0/m\vartheta_0)/\varkappa_s^2$, while the first term contains contributions of two types: for $q_\parallel + q_\parallel' = 0$ of order $(V_0/m\vartheta_0)^2$, and for $q_\parallel + q_\parallel' \neq 0$ of order $(V_0/m\vartheta_0)^3/\varkappa_s$. Contributions of order $(V_0/m\vartheta_0)^3/\varkappa_s$ [and even more so those of order $(V_0/m\vartheta_0)^4/\varkappa_s^2$] can turn out to be small even for angles $\vartheta_0 \lesssim V_0/m$. Therefore we shall assume that these contributions are small and shall carry out the corresponding expansions in $\exp iA_3$, while the terms with $q_\parallel + q_\parallel' = 0$ in the first term in the double sum in $A_3$ will be retained in the exponential. As a result we obtain an expression which coincides in form with (4.1) except that in it we must make the substitution

$$\exp(im^2\omega\tau/\varepsilon\varepsilon') \to \exp(im_*^2\omega\tau/\varepsilon\varepsilon'), \quad m_*^2 = m^2(1+\rho/2) \tag{4.5}$$

and in the first term in square brackets it is necessary to assume $q_\parallel + q_\parallel' \neq 0$. Here

$$\frac{\rho}{2} = \gamma^2[\langle \mathbf{v}^2\rangle - \langle \mathbf{v}\rangle^2] = \frac{1}{m^2}\sum_{\mathbf{q},\mathbf{q}'}\frac{G(\mathbf{q})G(\mathbf{q}')\mathbf{q}_\perp\mathbf{q}_\perp'}{q_\parallel q_\parallel'}[\delta_{q_\parallel+q_\parallel',0} - \delta_{q_\parallel,0}\delta_{q_\parallel',0}] = \sum_{\mathbf{q},q_\parallel\neq 0}\frac{|G(\mathbf{q})|^2\mathbf{q}_\perp^2}{m^2 q_\parallel^2}. \tag{4.6}$$

The transition $m \to m_*$ corresponds to the transition to the effective mass in the field of a wave, or in other language we have taken into account the parameter $\rho \neq 0$ [see the discussion following Eq. (3.10)].

The remaining calculations are carried out in the same way as in the transition from (4.1) to (4.2). We finally obtain

$$W_e^{m\,coh} = \frac{\alpha}{2\omega^2}\int^{\omega} d\varepsilon \sum_{\mathbf{q}} |G(\mathbf{q})|^2\frac{\mathbf{q}_\perp^2}{q_\parallel^2} \times\left[\frac{\varepsilon^2+\varepsilon'^2}{\varepsilon\varepsilon'} + \frac{2\omega m^2}{q_\parallel^2\varepsilon\varepsilon'}\left(q_\parallel - \frac{\omega m_*^2}{2\varepsilon\varepsilon'}\right)\right] \times\theta\left(q_\parallel - \frac{\omega m_*^2}{2\varepsilon\varepsilon'}\right), \tag{4.7}$$

and after integration over $\varepsilon$ we have

$$W_e^{m\,coh} = \frac{\alpha}{\omega}\sum_{\mathbf{q}} |G(\mathbf{q})|^2\frac{\mathbf{q}_\perp^2}{q_\parallel^2}\theta\left(q_\parallel - \frac{2m_*^2}{\omega}\right)\tilde{f}(\tilde{x}), \tag{4.8}$$

where

$$\tilde{f}(z) = \left(1 + \frac{z - z^2/2}{1+\rho/2}\right)\ln\frac{1+(1-z)^{1/2}}{1-(1-z)^{1/2}} - \left(1 + \frac{z}{1+\rho/2}\right)(1-z)^{1/2}, \quad \tilde{x} = \frac{2m_*^2}{\omega q_\parallel}.$$

Equations (4.7) and (4.8) are no more complicated than (4.2) and (4.3) but have a significantly broader range of applicability. For $\vartheta_0 \gg V_0/m$ ($\rho \ll 1$) Eqs. (4.7) and (4.8) go over respectively into (4.2) and (4.3).

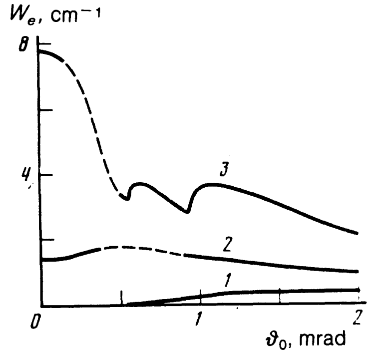


FIG. 4. Orientation dependence of the probability of pair production by a photon in Ge at $T = 100$ K for $\omega = 30$ GeV (1), 100 GeV (2), and 1000 GeV (3). The photon entry angle $\vartheta_0$ is measured from the $\langle 110\rangle$ axis.

## V. ORIENTATION DEPENDENCE OF THE PAIR PRODUCTION PROBABILITY; DISCUSSION OF RESULTS

As follows from our analysis, the orientation dependence of the probability $W_e(\vartheta_0)$ can serve as a good test of the mechanism of pair production. For $\vartheta_0 \ll V_0/m$ the mechanism of production in a constant field is dominant, for $\vartheta_0 \gg V_0/m$ the usual mechanism of coherent production is dominant, and in the intermediate region it is necessary to use a general approach. As an illustration we shall give here the probability $W_e(\vartheta_0)$ for cooled germanium ($\langle 110\rangle$ axis, $T = 100$ K, which corresponds to the conditions of the experiment of Ref. 14; see Fig. 4) and for tungsten ($\langle 111\rangle$ axis, Fig. 5). In Fig. 4 the orientation dependence of $W_e$ is given for the following energies: $\omega = 30$ GeV ($\varkappa_m \approx 0.6$)—curve 1; $\omega = 100$ GeV ($\varkappa_m \approx 2$)—curve 2; $\omega = 1000$ GeV ($\varkappa_m \approx 20$)—curve 3. In Fig. 5 curves 1, 2, and 3 are given respectively for energies $\omega = 5$ GeV ($\varkappa_m \approx 0.35$), $\omega = 35$ GeV ($\varkappa_m \approx 2.4$), and $\omega = 300$ GeV ($\varkappa_m \approx 21$). The deviation from the $\langle 111\rangle$ axis lies in the plane containing this axis and forming an angle $\varphi_0$ with the $(\bar{1}10)$ plane. In order to avoid the vector $\mathbf{n}$ falling in the principal planes of the crystal, we selected a value $\operatorname{tg}\varphi_0 = \sqrt{3}/6\pi \approx 0.092$.

The quantities $G(\mathbf{q})$ entering into the formulas for $W_e(\vartheta_0)$ [see Eq. (1.2)] we shall write in the form

$$G(\mathbf{q}) = l^{-3}\varphi(\mathbf{q})S_{mnk}, \tag{5.1}$$

where $S_{mnk}$ is the structure factor, $l$ is the lattice constant, $q = (2\pi/l)\times(m, n, k)$, and $m$, $n$, and $k$ are integers over which the summation is carried out in (1.2). For an fcc(d) lattice of the diamond type which occurs in crystals of germanium we have

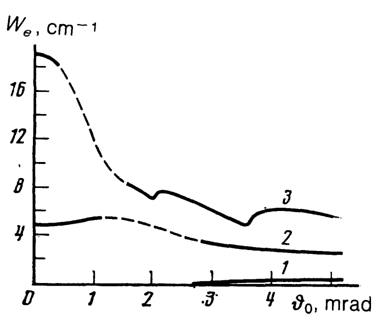


FIG. 5. The same as in Fig. 4 but for tungsten, $T = 293$ K, the $\langle 111\rangle$ axis, and $\omega = 5$ GeV (1), 35 GeV (2), and 300 GeV (3).

$$S_{mnk}^{(d)} = \{1 + \exp[{}^1/_2 i\pi(m+n+k)]\} \times(\cos\pi k + \cos\pi m)(\cos\pi m + \cos\pi n), \tag{5.2}$$

and for the bcc lattice (W, Fe)

$$S_{mnk}^{(b)} = 1 + \cos\pi(m+n+k). \tag{5.3}$$

The quantity $\varphi(\mathbf{q})$ in (5.1) is the Fourier component of the potential of an individual atom averaged to include thermal vibrations. We used for the atom the Moliere potential, and in that case

$$\varphi(\mathbf{q}) = 4\pi Ze^2 \exp(-u_1^2 q^2/2)\sum_i \frac{\alpha_i}{q^2 + b_i^2}, \tag{5.4}$$

where $u_1$ is the amplitude of thermal vibration and $\alpha_i$ and $b_i$ are the parameters of the potential (see for example Ref. 3). Use of a potential averaged over the thermal vibrations actually excludes from consideration the Bethe-Heitler mechanism. The latter also changes in a crystal in comparison with an amorphous material (see for example Ref. 1 in the theory of coherent production). The parameter $\rho/2$ which enters into Eqs. (4.7) and (4.8) was calculated in accordance with (4.6). Here it is necessary to take care that the contributions of planes of high order are adequately treated. Under the conditions given above, the calculation yielded $\rho/2 \approx 1.87(V_0/m\vartheta_0)^2$ for Ge and $\rho/2 \approx 1.04(V_0/m\vartheta_0)^2$ for W.

The curves labeled 1 in Figs. 4 and 5 show the situation in which $\varkappa_m < 1$; here the effects of the constant field are small and the specific features of the crystal appear in the action of the coherent pair production mechanism. The effect is nonvanishing for those $\vartheta_0$ for which $\rho/2 \ll 1$, so that the probabilities are described by the standard formulas (4.2) and (4.3). The curves 2 in Figs. 4 and 5 correspond to the case in which with increasing energy the mechanism of pair production in a constant field is already included. In accordance with the results of Section III, the probability $W_e(\vartheta_0)$ has a minimum at $\vartheta_0 = 0(\varkappa_m < 4)$. With further increase of the energy the minimum in the probability at $\vartheta_0 = 0$ goes over to a maximum, and then with increase of the energy the width of this maximum narrows. This case corresponds to the curves labeled 3 in Figs. 4 and 5. We note that the quantity $W_e(0) = W_e^{CF}$ continues to increase as long as $\varkappa_s < 25$, and the maximum value $W_{e,\max}^{CF}$ is given with high accuracy by Eq. (3.14).

It must be kept in mind that it is comparatively simple to calculate the probability by using the formulas for a constant field with the corrections (3.7) and the modified formula for coherent pair production (4.8), while the calculation according to the complete formula (2.8) is a rather complicated computational problem. Therefore in the present work we did not carry out calculations using the formula (2.8), but limited ourselves to a calculation with Eqs. (3.7) and (4.8) and used interpolation. We made use of the fact that for $\varkappa \gg 1$ the interval separating the regions of applicability of Eqs. (3.7) and (4.8) is comparatively narrow, and we checked the accuracy of the interpolation on the basis of a model of pair production in the field of a plane wave, where it was possible to carry out exact calculations also. For $\varkappa \sim 1$ this interval becomes broader, but in return all of the dependences become smoother. In Figs. 4 and 5 the dashed part of curves 2 and 3 was obtained by means of interpolation.

A recent experiment[14] reported an enhancement of pair production by photons with energy $\omega = 50$–$110$ GeV incident along the $\langle 110 \rangle$ axis of a germanium crystal cooled to temperature $T = 100$ K. However, the observed probability turned out to be about a factor of three smaller than the theoretical prediction (see Ref. 13), and the orientation dependence of the radiation was studied quite inadequately. To clarify this situation, further experiments are necessary, in particular a detailed study of the orientation dependence.

The situation described above, in which depending on the photon entrance angle the pair production is determined by different mechanisms, exists also in the radiation problem. At small entry angles, high-energy particles radiate, moving in the fields of the axes or planes (in the channeling mode or in superbarrier motion), and at comparatively large angles $\vartheta_0 \gg V_0/m$ coherent bremsstrahlung occurs. For the case of motion in the field of planes this question was analyzed in Ref. 15.

The present work has been devoted to pair production by photons. It is evident that any processes occurring in an external field can be discussed in an analogous fashion. Such processes include, for example, pair production by a charged particle, production of neutrino pairs ($e \to e\nu\bar{\nu}$), and photon splitting.

[1)]In this work we use the system of units $\hbar = c = 1$.

[2)]For simplicity in what follows we shall assume in this work that the vector **n** does not lie near crystallographic planes. The question of production of pairs in the planar mode will be considered separately.

[3)]Simple explicit formulas for the probability $W_e^{CF}$ for the case $\varkappa_m \lesssim 1$ are given in Ref. 4.


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