Onset of chaos in the redistribution of parametrically excited magnons

A. I. Smirnov

Institute of Physical Problems, Academy of Sciences of the USSR
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The transition from periodic to random redistributions of the magnon density during the parametric excitation of magnons in antiferromagnetic $CsMnF_3$ has been studied. This redistribution sets in at high magnon density, as the result of an instability of the uniform distribution driven by the mutual attraction of magnons. The regions of cyclic and random changes in the plane of the magnetic field and the pump power are determined. At high power levels there is a region in which the onset of chaos is consistent with the Feigenbaum scenario, and the chaos has several of the properties predicted by that scenario. At a low pump power, a transition to chaos takes place as a function of the magnetic field. This transition conforms to the Pomeau-Manneville scenario and involves intermittency of coherent trains.

1. INTRODUCTION

Experiments on the parametric excitation of spin waves in antiferromagnetic $CsMnF_3$ at a high pump power[1] have revealed that at high magnon density the uniform steady-state distribution of magnons over the sample becomes unstable. The density of parametrically excited magnons starts to undergo periodic redistributions. Short-lived condensations far from the boundaries of the sample alternate with comparatively prolonged stages of a uniform distribution. These redistributions are accompanied by changes in the high-frequency susceptibility, which had been discovered earlier.[2]

The physical reason for the disruption of the uniform distribution of parametrically excited magnons may be a nonlinear spectral shift of the magnons, specifically a decrease in the natural frequency of a magnon with a certain wave-vector magnitude when other magnons are excited in the sample. A negative and nonlinear spectral shift has in fact been detected.[3] Such a change in the spectrum is equivalent to the existence of an attraction between magnons. A qualitative theory demonstrating the possibility of a redistribution of parametrically excited magnons as the result of an attraction between magnons under actual experimental conditions and certain predictions were offered in Ref. 1. The present paper supplements that earlier study.

We consider a region G in a sample in which the magnon density n is, for some reason, higher than elsewhere in the sample. This nucleating nonuniformity in the distribution of n might be caused by, for example, damping of magnons at the boundaries of the sample. The attraction between magnons has the consequence that all the magnons which intersect the boundary of G undergo a change in group velocity. This change is directed into G and sets up a flux of magnons into this region; the flux density is proportional to n^2. The rate at which magnons are excited by the pump and the rate at which the magnons are damped are both proportional to n. Consequently, when n becomes high enough the total flux will exceed the dissipation, and an instability will set in. The density increase within G in turn causes the inward flux density to grow, and the region with the relatively high value of n should contract in an avalanche fashion. Rough estimates show that at the magnon density reached in Ref. 1 the flux density caused by the attraction exceeds the dissipation for a region ≈ 0.1 mm in size. The dimensions of the samples are ~ 1 mm; i.e., condensations of parametrically excited magnons can occupy a substantial part of the sample. The wavelength of the magnons excited in Ref. 1 was $\sim 10^{-4}$ cm, and their frequency 10 GHz. So far, theory for the nonuniform distribution of parametrically excited magnons has been developed only for the case of a packet of parametrically excited magnons with wave vectors confined to a narrow angular region.[4]

Periodic repetitions of the condensation of parametrically excited magnons were observed in Ref. 1. The apparent explanation for the cyclic nature of the process is that the growth of n in a condensation region is limited by heating (the relaxation rate of the spin waves increases in accordance with $\alpha + \beta T^7$, where T is the temperature. This heating causes an increase in the relaxation rate, which in turn increases the energy flux into the lattice and causes a heating of the lattice. As a result, the parametrically excited magnons in the condensation region are rapidly damped, and the next condensation process begins after the heated region cools down to the temperature at which the pump is capable of producing a density n of parametrically excited magnons sufficient for the onset of the instability. This picture of events gives a qualitative description of the observed behavior of the period, amplitude, and rise time of the process as functions of the magnetic field and the pump power.[1]

It was also stated in Ref. 1 that in magnetic fields corresponding to small wave vectors of the parametrically excited magnons the periodic condensation process gives way to a random process. The reason for this transition is that in this region of magnetic fields the mean free path becomes significantly smaller than the sample, and the positions of the condensations are not determined by the boundaries. The condensations thus occur first in one place and then in another, disrupting the periodicity. Clearly, as the power is raised there may also be displacements of the regions where the condensations occur, when the magnon density becomes sufficient for a condensation not only in a cold sample but also in a sample which has cooled down after a cycle of condensa-

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tion and "burnup" of magnons. We are interested in the present paper in the transitions to chaos both as the power is raised and as the magnetic field is varied. At a high pump power we can also expect a steady-state absorption of microwave pump power, when the parametric excitation can occur in a region of the sample which has been heated to a great extent.

Substantial progress has recently been achieved in the theory for the transition to chaos in dynamic systems described by systems of differential equations.[5–7] This progress has in turn stimulated interest in experiments on such transitions in a variety of physical systems, particularly systems of large dimensionality and with distributed parameters. The theoretical and experimental research shows that for a wide range of dynamic systems which are not too complicated the transition to chaos occurs in accordance with a few "scenarios."[6] We are accordingly interested in the course of the transition to chaos in this redistribution of magnons, which amounts to a sort of spatial turbulence of spin waves.

2. GENERAL IDEAS REGARDING THE TRANSITION TO CHAOS IN DYNAMIC SYSTEMS

It has proved possible, for a wide range of dynamic systems, to go beyond the analysis of specific differential equations to point out general features in the behavior of the solution by mean of a Poincaré mapping.[5,6] This mapping consists of a set of points at which the phase path of the system intersects some hypersurface in phase space. The initial system of equations can in principle be used to determine the position of each successive intersection from the position of the preceding intersection; i.e., there exists a function of x such that $x_{n+1} = f(x_n)$. The properties of this function largely determine the course of the transition to a chaotic motion. If we choose this hypersurface in such a way that the points on it are recorded at times which are multiples of the fundamental period of motion of the system, then for a periodic situation the Poincaré mapping will consist of a point, while for a random motion the mapping will have a more complex geometry. For dissipative systems the Poincaré mapping approximately reduces to a one-dimensional mapping, and the presence of chaos in the motion is determined by extrema of $f(x)$. Four scenarios for the onset of chaotic motion have now been proposed for dissipative systems.[4,6] Two of them are associated with one-dimensional Poincaré mappings. According to the Feigenbaum scenario,[7] the transition to chaos occurs through successive doublings of the period. This behavior is exhibited by systems for which the extremum of the function $f(x)$ is quadratic. The basic features in the evolution of the motion of a system resulting from variation of a parameter are described by the mapping

$$x_{n+1}=2Cx_n+2x_n^2 \qquad (1)$$

as the parameter C is varied. The infinite sequence of those values of the parameter at which the period doubles converges on the value $C_\infty \approx -0.785$. At $C < C_\infty$, the motion is random. The random motion appears first in an infinite number of narrow zones. As C is reduced further, these zones merge, and the number of zones is halved in each merger (inverse doubling bifurcations). At $C < C_\infty$, there exist small regions in the parameter C which correspond to periodic motion with other periods, including odd integer periods: 7, 6, 5, 3, etc.[8] The periods of high multiplicity correspond to values of C closer to C_∞ than the periods of low multiplicity. Figure 7.22 in Ref. 5 illustrates the intervals of x which are filled by iterative scheme (1) for various values of C. Each of the periodic cycles which occur at $C < C_\infty$ also undergoes doubling bifurcations with decreasing C and a transition to chaos. This road to chaos in the Feigenbaum scenario[7] occurs because the fixed points of mapping (1) lose their stability as C is varied, and they are transformed into two stable fixed points for motion with a doubled value of the period, i.e., fixed points of the mapping $f(f(x))$.

The quadratic mapping

$$x_{n+1}=x_n+x_n^2+\varepsilon \qquad (2)$$

has yet another type of bifurcation, however: a so-called inverse tangential bifurcation. As ε varies from negative to positive values, the stable and unstable fixed points of this mapping merge and disappear. Typical of this bifurcation is another transition to chaos: through intermittency (the Pomeau-Manneville scenario[9]). In this case, intervals of approximately periodic motion are interrupted in a random fashion by following intervals of an irregular motion. The average length of the coherent trains decreases in proportion to $\varepsilon^{-1/2}$. Interestingly, the transition to chaos from the integer cycles corresponding to values $C < C_\infty$ occurs in accordance with the Pomeau-Manneville scenario without a doubling of the period as the parameter C is increased, but not as it is decreased. The intermittency of cycles 3 as we move out of this cyclic regime in the C_∞ direction is illustrated by the result of a numerical simulation with scheme (1) in Fig. 15 in Ref. 6.

In summary, for many nonlinear systems, including systems which are distributed in three dimensions, the transition to chaotic motion may be organized in accordance with some scenario or other. Individual characteristic features of these transitions have been observed experimentally in many experiments in hydrodynamics,[10,11] chemical kinetics,[12] the self-excited oscillations of the magnetization during the parametric excitation of spin waves in ferromagnets,[13,14] etc.

In the present paper we report a study of the course of the transition to chaos in the phenomenon described above: the redistribution of magnon density during the parametric excitation of magnons by a microwave pump.

3. EXPERIMENTAL PROCEDURE

Global characteristics of the overall set of parametrically excited magnons are the number N of these magnons and the phase ψ of the oscillations at the pump frequency of the magnetic moment of the sample which is associated with the parametrically excited magnons. This phase is called the "phase of pairs of parametrically excited magnons."[15] The frequency of the parametrically excited magnons is half the pump frequency, and the wave vector is determined from the condition for a parametric resonance:

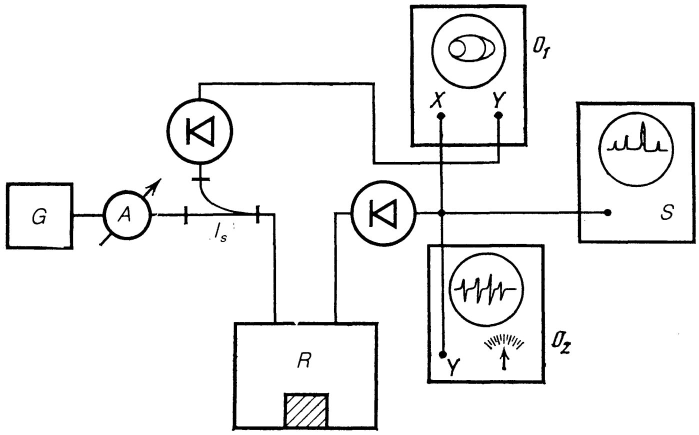

FIG. 1. The experimental layout. G—Microwave pump generator; O_1—oscilloscope for observing phase diagrams; O_2—oscilloscope for observing the actual signal; S—spectrum analyzer; I_s—isolator; A—attenuator; R—resonator with sample.

$$\omega_k = \omega_p/2, \quad (3)$$

where ω_k is the dispersion law for spin waves, and ω_p is the pump frequency (see Refs. 3 and 15 for more details on the parametric excitation of spin waves). The number and phase of the parametrically excited magnons determine the real (χ') and imaginary (χ'') parts of the high-frequency susceptibility and also some experimentally observable quantities: the microwave power transmitted through a resonator and that reflected from the resonator.

To study the excitation of parametrically excited magnons, we analyzed the microwave signal transmitted through a resonator with the sample directly on an oscilloscope screen or by means of a spectrum analyzer. We also studied a (transmitted signal)—(reflected signal) phase diagram of the system. The signals transmitted and reflected by the resonator are expressed in different ways in terms of χ' and χ'', so that this phase diagram is related to a χ', χ'' (or N, ψ) phase diagram by a single-valued transformation. Figure 1 shows a block diagram of the apparatus. We studied a sample of antiferromagnetic $CsMnF_3$ with a diameter of 2 mm and a height of 1 mm at a temperature of 1.4 K and at a pump frequency of 18 GHz. A strip-line resonator is used to produce a microwave field up to 15 Oe at the sample, as in Ref. 1.

4. SUMMARY OF RESULTS

At different values of the pump power and the magnetic field, we observe different types of absorption of the microwave power by the sample. In magnetic fields $H < 2.3$ kOe, an increase in the power is accompanied by a gradual transition to random spikes in the absorbed power through doubling and quadrupling of the period of the spikes in an initial cyclic behavior. Figure 2 shows oscilloscope traces of the actual signal. As the power is raised, the situation with a quadrupled period gives way to chaos. In a power interval corresponding to chaotic motion there are smaller intervals of values of h^2 which correspond to a periodic motion with other periods. We observed cycles with periods which are

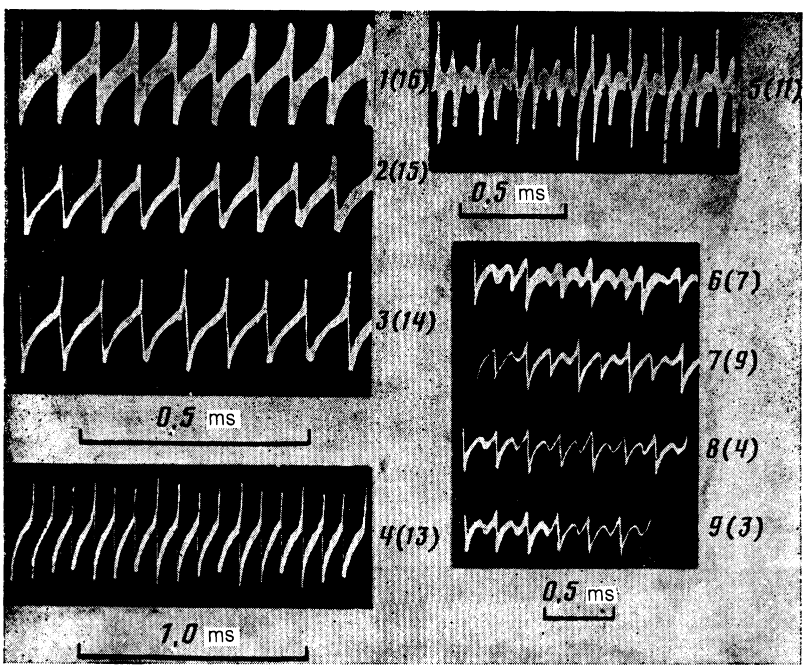

FIG. 2. Oscilloscope traces of the microwave power transmitted through a resonator with a sample at increasing levels of the pump power in a field of 2.0 kOe. Traces 1–3 and 6–9 were obtained during repeated triggering of the sweep; traces 4 and 5 were obtained in a single triggering. The numbers in parentheses specify the case in Fig. 5.

equal to 7, 5, and 3 times the initial period. The oscillation spectrum of the signal in this chaotic situation has a significant intensity over a broad frequency interval, not only near the frequency of the initial periodic behavior but also near the frequencies $f/2$ and $f/4$. We call this situation "chaos 1." With a further increase in the power, there is a transition to chaos through situations with periods of 4 and 2, followed by oscillations with the fundamental value of the period. At sufficiently high power, there is a return to steady-state absorption of microwave power.

Figure 3 is a diagram of the periodic and chaotic states of the signal in the H, h^2 plane, where h is the amplitude of the microwave pump field. If the pump power is maintained at a level too low for doubling of the fundamental period, the transition to chaos occurs when the magnetic field exceeds 2.3 kOe. No period-doubling processes occur, and the evolution of the spectrum consists of a broadening of the fundamental line. This broadening initially occurs only at a level not exceeding 0.1 of the amplitude of the fundamental line (Fig. 4). We will call this region of chaos, b in Fig. 3, "chaos 2."

The spikes in the susceptibility disappear at a field $H_0 = 2.6$ kOe. At $H > H_0$, a parametric excitation of magnons at a low pump power becomes impossible because of the violation of condition (1). At a pump power corresponding to a redistribution of the density of parametrically excited magnons, a slight heating of the sample occurs (0.15 K). Accordingly, the temperature dependence of ω_k causes the

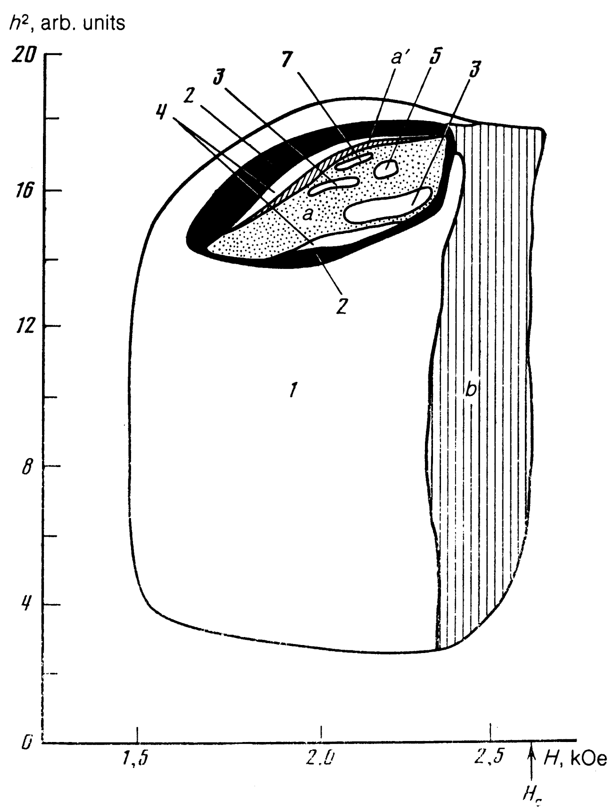

FIG. 3. Diagram of periodic and chaostic regimes 1–5—Periodic situations of the corresponding multiplicity; a—chaos 1 (motion in one and two zones); a'—chaos 1 (motion in four zones); b—chaos 2.

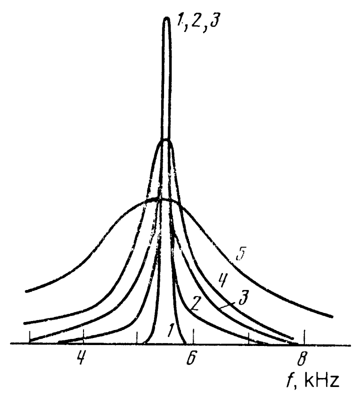

FIG. 4. Change in the spectrum of the amplitude of the signal transmitted through the resonator during the transition to chaos 2. The spectra correspond to different magnetic fields: 1—2226 Oe; 2—2228 Oe; 3—2230 Oe; 4—2233 Oe; 5—2238 Oe.

boundary of the magnetic-field region in which the excitation of parametrically excited magnons is observed to shift slightly to the right. This shift explains the small interval of magnetic fields, about 80 Oe wide, near H_0 in which parametric excitation of magnons occurs, as described in Refs. 2 and 1, and there are no spikes in the susceptibility or redistribution of the magnon density. Parametric excitation actually occurs here only in the slightly heated part of the sample, far from the boundaries, which are cooled by a bath of superfluid helium.

5. FINE STRUCTURE IN THE TRANSITIONS BETWEEN DIFFERENT TYPES OF REDISTRIBUTION OF THE MAGNON DENSITY

The transition from the periodic regime with condensations of parametrically excited magnons to chaos 1 proceeds in accordance with the Feigenbaum scenario: through doublings of the period. As we discussed in the Introduction, an analysis of the quadratic mapping (1) explains many other details of the change in the regime as a function of the parameter C which were found in the present experiments.

Figure 5 shows the temporal sequence of the different types of microwave absorption observed in a field of 2.0 kOe as the pump power was varied; Fig. 6 shows phase diagrams of the system in these cases. The Feigenbaum scenario and other subtleties of the change in situation which are characteristic of mapping (1) can be followed most systematically as the power is varied from high values to low values. Phase diagrams 1–4 in Fig. 6 correspond to the regimes with periods which are 1, 2, and 4 times the fundamental period, described above. Diagram 5 in Fig. 6 corresponds to chaotic motion in four zones; diagram 6 in Fig. 6 to random motion in two zones; and diagrams 8 and 11 in Fig. 6 to chaotic motion in one zone. We can clearly see the process of inverse doubling bifurcations, a merging of the outer zones of the diagram. The spectra of the chaotic motion at the values of the parameter corresponding to these transitions should have a shape common to many systems[16] (see 7.23 in Ref. 5). The spectrum found for the value $C = C_2^*$, which corresponds to the coalescence of four zones into two, has a sharp peak at the frequencies $f/4$ and $f/2$ and rounded, smaller

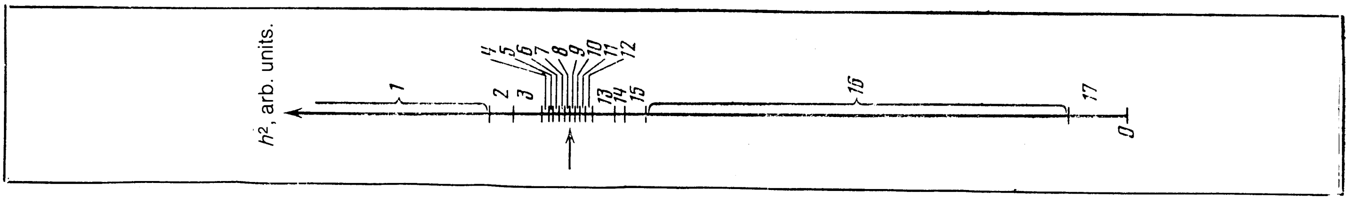

FIG. 5. Sequence of transformations of periodic and chaotic regimes in a field of 2.0 kOe as the power is varied. 1—Steady state; 2—cycle of the fundamental period; 3—cycle 2; 4—cycle 4; 5—chaos of four zones; 6—chaos of two zones; 7—cycle 7; 8—chaos of one zone; 9—cycle 3; 10—cycle $6 = 3\times 2$; 11—chaos of one zone; 12—contraction of zone; 13—chaos of one narrow zone; 14—cycle 4; 15—cycle 2; 16—cycle of the fundamental period; 17—steady state. The horizontal arrow shows the value of h^2 at which the intermittency of cycle 3 is observed.

peaks at these frequencies. At $C = C_1^*$ (the coalescence of two zones into one), there are rounded and sharp peaks at $f/2$. There is a qualitative correspondence between these spectra and those observed in the present experiments. Spectrum 2 in Fig. 7 corresponds to the coalescence of four zones into two, while spectrum 3 corresponds to the coalescence of two zones into one.

How do the transitions from cyclic regimes with periods of 7, 5, and 3 to chaotic regimes occur? Under our experimental conditions we were able to observe transitional regimes (at transitions from cyclic to very random behavior) only for the cycle with a period of 3. As the power is reduced, the period doubles; i.e., a cycle with a period of 6 forms. This process is illustrated by the change in phase diagram between 9 and 10 in Fig. 6 and the change from spectrum 5 to spectrum 6 in Fig. 7. When we go from cycle 3 in the other direction (when we increase the power), the transition occurs differently. The spectral lines corresponding to cycle 3 rapidly become broader, and the actual signal demonstrates and intermittency of trains of cycles 3 and of random intervals (see the oscilloscope traces in Fig. 8). Unfortunately, this intermittency is observed in a very narrow

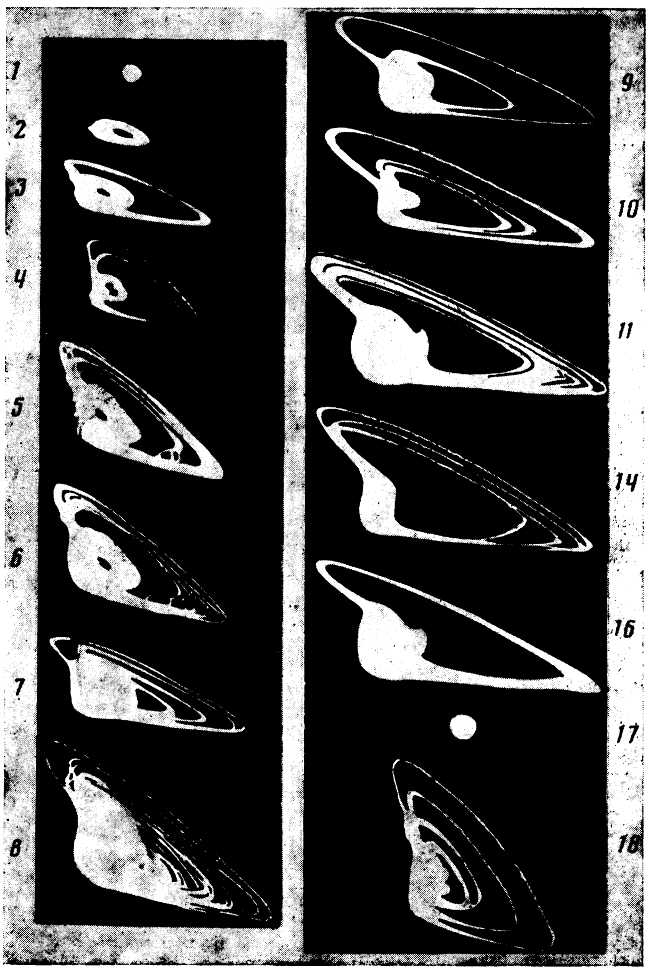

FIG. 6. Phase diagrams for various values of the pump power. Diagrams 1–17 correspond to the regime of the same number in Fig. 5; diagram 18 is the phase diagram for a regime with a period of 5.

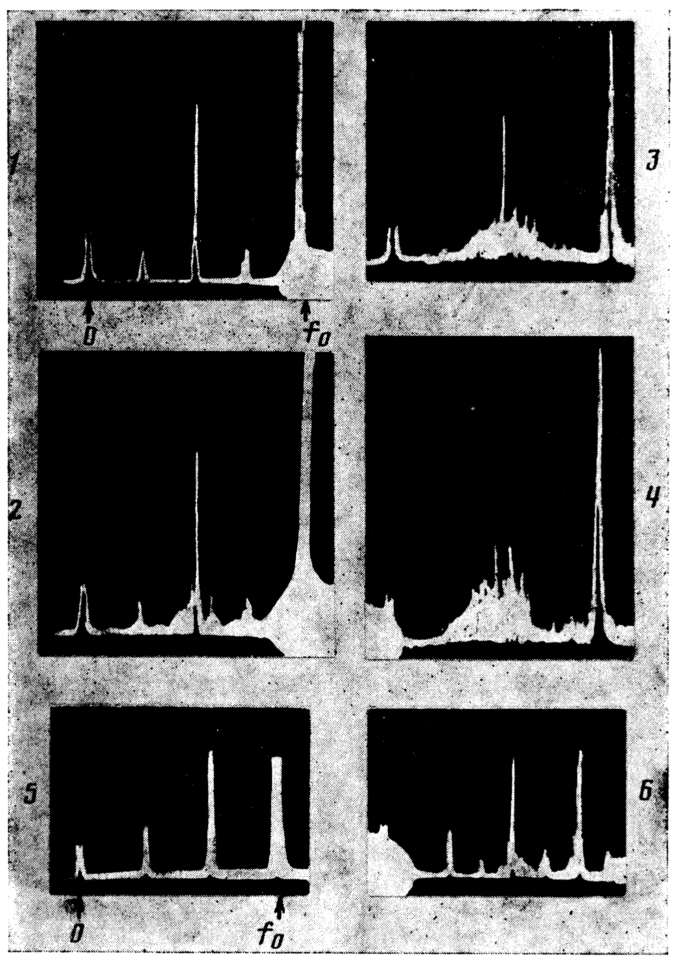

FIG. 7. Amplitude spectra of the signal transmitted through the resonator. 1—Situation with a period which is four times the fundamental period; 2,3—during the coalescence of zones; 4—well-developed chaotic motion (position 8 in Fig. 5); 5,6—double bifurcation of cycle 3.

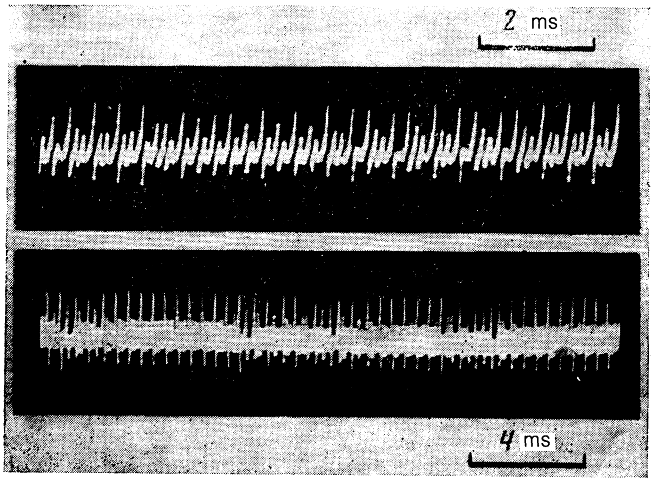

FIG. 8. Oscilloscope trace of the signal transmitted through a resonator during observation of intermittency of cycle 3.

power interval, so that it is not possible to determine the power dependence of the average length of the coherent trains. In this case, however, it can confidently be asserted that the intermittency sets in because stable and unstable cycles coalesce and disappear, since the position for a given change of regime is determined by all the other bifurcations.

Following Lorenz,[17] we construct a particular functional dependence for the chaotic regime at position 8 in Fig. 5, for which an oscilloscope trace of the real signal is shown by trace 5 in Fig. 2. This function is $y_{n+1}(y_n)$, where y_n is the absolute value of the nth minimum on the oscilloscope trace. This function determines the Poincaré mapping for a surface parametrized by the condition that the given coordinate in phase space be maximized. The result of the construction is shown in Fig. 9a; we find a curve with a minimum. According to Feigenbaum's theory, a system with a Poincaré mapping of this sort should undergo a cascade of doubling bifurcations in the transition to chaos. The derivative of the curve is greater than unity in modulus essentially everywhere. This result means[5] that closely spaced phase paths diverge exponentially over time, and the attractor in our system (the phase diagram) is a so-called strange attractor.

The construction of the mapping $y_{n+1}(y_n)$ for a chaotic regime near intermittency of cycle 3 and also for chaos 2 does not lead to anything like a single-valued dependence (Fig. 9b). A similar behavior of the mapping is evidently also observed in a numerical simulation with mapping (1) and with a value for C close to that at which cycle 3 appears when we approach from the side of the point of condensation of doubling bifurcations (Ref. 5 and Fig. 15 in Ref. 5).

The transition from the random motion corresponding to position 12 in Fig. 5 to position 13 is accompanied by a slight hysteresis (1% in terms of the power). Such a hysteresis is usually attributed to the presence of two chaotic attractors in the given region of values of the parameter in the system.

A careful examination of the transition from the cyclic behavior with the fundamental period to chaos 2 shows that

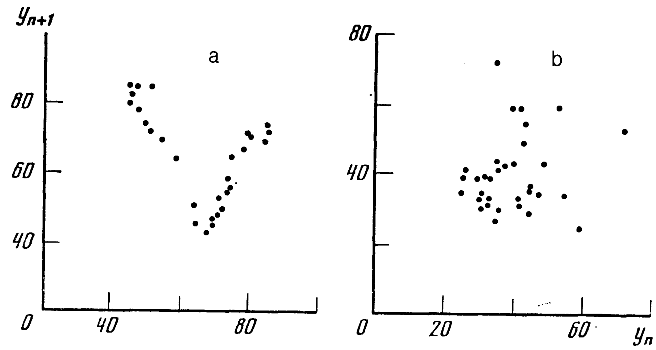

FIG. 9. a—Poincaré mapping for chaotic regime 8 in Fig. 5; b—for chaos 2.

this transition also involves intermittency (see the trace in Fig. 10). The shape of the spectrum at each value of the magnetic field can be used to evaluate the length of the coherent trains, τ: The half-width of the spectrum at a low level (0.1 of the maximum amplitude) is on the order of $1/\tau$. The result found (Fig. 11) does not, however, follow the behavior predicted theoretically for the case without external noise: $\Delta f \propto \varepsilon^{1/2}$.

Chaos 1 is thus more ordered than chaos 2. In chaos 1 there is an order close in time, as is shown by the observed behavior $y_{n+1}(y_n)$ (Fig. 9a) and by the strictly definite hierarchy of transformations between regimes as the power is varied (Fig. 5). The reason is that under conditions such that the mean free path of the parametrically excited magnons is comparable to the dimensions of the sample (in chaos 1) the position of each condensation of the magnon density is well defined, while in the case of a short mean free path (chaos 2) the position of a condensation is random (only the heated region of the preceding density spike is an improbable position for a condensation). This difference in the physical factors which lead to chaos 1 and chaos 2 also explains the change in the amplitude of the absorption spikes. At the transition to chaos 1, the amplitude of the spikes decreases (the condensation occurs in a sample which is not cooled down thoroughly), while in the transition to chaos 2 this amplitude continues to increase, following the general trend of an increase in the intensity of the density redistribution upon a decrease in the wave vector and thus in the group velocity of the parametrically excited magnons.[1]

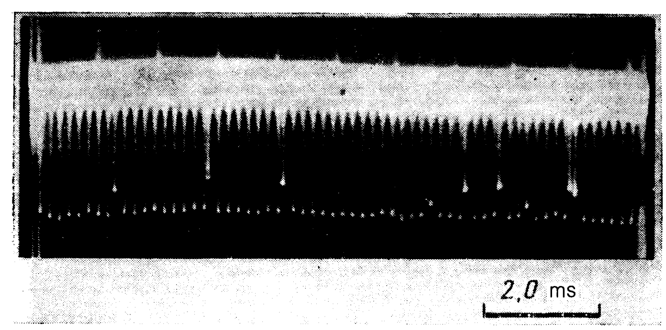

FIG. 10. Oscilloscope trace of the signal transmitted through the resonator during the transition to chaos 2.

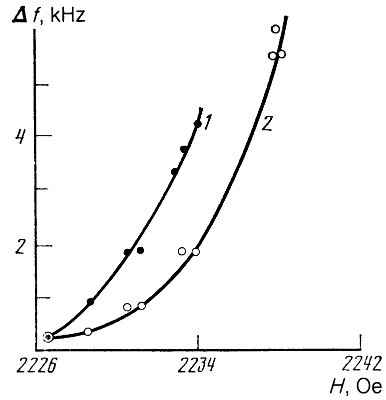

FIG. 11. Width of the spectrum of the signal amplitude at the 0.1 level (●) and at the 0.25 level (○) versus the magnetic field during the transition to chaos 2.

6. OTHER SAMPLES AND OTHER TEMPERATURES

Transitions to chaos were also studied at $T = 1.6$ K in a smaller sample (1.5 mm in diameter and 0.8 mm high). As the temperature is varied from 1.4 to 1.6 K, the nature of the transition from cyclic behavior to chaos 2 remains entirely the same. The picture of the transition to chaos 1 changes. At $T = 1.6$ K, only the regions with periods of 2 and 4 are distinguishable in the initial sample; there is no chaos at fields up to 2.3 kOe. This change in the chaos with the temperature can be explained qualitatively level by saying that at the lower bath temperature (i.e., with more rapid heat removal from the sample) conditions are better for the condensation of the magnons in the sample which has not yet cooled, in a position which does not coincide with the position of the preceding condensation. For this reason, the transition to chaos 1 (i.e., with increasing power) should be better defined at a lower temperature. The transition to chaos 2 involves a change in the mean free path of the sample, and it should not be affected significantly by such a small change in temperature.

In the smaller sample, the general features of the transition to chaos 2 remain the same, but the field interval corresponding to this chaos becomes 15% narrower. This narrowing should be expected since the mean free path of the magnons becomes smaller than the size of the sample at the stronger magnetic fields. At $T = 1.4$ K, the region in the parameters H and h^2 corresponding to chaos 1 is substantially smaller, and it lies at stronger fields and higher power levels than in the initial sample. There is no suppression of the susceptibility oscillations at the highest power level attainable in these experiments.

7. CONCLUSION

In summary, the instability of the uniform distribution of parametrically excited magnons leads to changes over time in the spatial distribution of the spin-wave density. These changes are both periodic and random, and they constitute a specific type of turbulence. The transition to chaos occurs in accordance with the Feigenbaum scenario (through period-doubling) as the pump power is raised or the Pomeau-Manneville scenario (through intermittency of coherent trains) as the magnetic field is increased. The experiments reveal many details of the evolution of the temporal, spectral, and amplitude characteristics of the motion of the system of parametrically excited magnons which are predicted on the basis of an analysis of quadratic mapping (1): periodic motion with periods of 3, 5, and 7; a doubling bifurcation of a cycle with a period of 3; intermittency of cycle 3; and coalescence of random zones.

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