Effective dielectric constant in the calculation of the second field moments in a randomly inhomogeneous medium

V. L. Brekhovskikh

Institute of Physics of the Atmosphere, USSR Academy of Sciences
(Submitted 29 May 1985)
Zh. Eksp. Teor. Fiz. **89**, 2013–2020 (December 1985)

General equations that permit the effective-dielectric-constant method to be extended to the equations for the second moments are derived for an arbitrary randomly inhomogeneous medium. Some attention is paid in the formalism proposed to the intrinsic thermal radiation and to the propagation of a plane wave in an inhomogeneously layered medium.

1. The concept of effective dielectric constant is used to describe the mean field in a randomly inhomogeneous medium.[1] The mean field can be interpreted here as the field in some regular "effective" medium with spatial dispersion. In such a medium, the connection between the electric-induction and electric-field vectors is given by

$$D_i(\mathbf{r})=\int d\mathbf{r}'\,\varepsilon_{ij}^{\text{eff}}(\mathbf{r},\mathbf{r}')E_j(\mathbf{r}'). \tag{1}$$

The problem reduces to a calculation of ε^{eff} on the basis of information concerning the statistical properties of the initial medium. It is impossible, generally speaking, to calculate quadratic functions of the field from the effective dielectric constant. For example, the imaginary part of ε^{eff} determines the mean-field damping constant that contains a term due to phase fluctuations. The latter do not influence the field intensity, so that the energy damping coefficient cannot be expressed in terms of Im ε^{eff}. The analogy with a dispersive medium, however, can be extended also to the second moments of the field. To do this one must generalize in suitable manner the concept of the dielectric constant of a medium with spatial dispersion. We use as our basis the equations for the bilinear quantities

$$\Gamma_{ij}^{E}(\mathbf{r}_1,\ \mathbf{r}_2)=E_i(\mathbf{r}_1)E_j^*(\mathbf{r}_2),$$

which can be obtained by multiplying the wave equations for $E_i(\mathbf{r}_1)$ and $E_j^*(\mathbf{r}_2)$. This generates the functions $\Gamma_{ij}^D(\mathbf{r}_1,\mathbf{r}_2)=D_i(\mathbf{r}_1)D_j^*(\mathbf{r}_2)$, which are determined in an ordinary medium by the products $\varepsilon_{ik}(\mathbf{r}_1)\varepsilon_{jl}^*(\mathbf{r}_2)\Gamma_{kl}^E(\mathbf{r}_1,\mathbf{r}_2)$. It is natural to assume that in spatially dispersive medium this definition should be replaced by a relation of the form

$$\Gamma_{ij}{}^{D}(\mathbf{r}_1,\mathbf{r}_2)=\iint d\mathbf{r}'\,d\mathbf{r}''[B_\varepsilon(\mathbf{r}_1,\mathbf{r}_2;\,\mathbf{r}',\mathbf{r}'')]_{ijkl}\Gamma_{kl}{}^{E}(\mathbf{r}',\mathbf{r}''). \tag{2}$$

The function B_ε, which can be called the correlation function of the dilelectric constant, describes the properties of a dispersive medium with respect to quadratic characteristic field, and is needed, in particular, to calculate energy-dependent quantities. The correlation functions Γ_{ij}^D and Γ_{ij}^E in such a medium break up into the products $D_i(\mathbf{r}_1)D_j^*(\mathbf{r}_2)$ and $E_i(\mathbf{r}_1)E_j^*(\mathbf{r}_2)$ only at $|\mathbf{r}_1-\mathbf{r}_2|\gg l$, where l is the characteristic scale of the dispersion.

Accordingly, the calculation of quadratic quantities in a randomly inhomogeneous medium consists of calculating the "effective correlation tensor" of the dielectric constant. Quantities such as $\langle E_i(\mathbf{r}_1)E_j^*(\mathbf{r}_2)\rangle$ can then be treated as the correlation functions Γ_{ij}^E in an effective medium.

We consider a vector electrodynamic problem that satisfies the wave equation

$$[\hat{L}^0+k_0{}^2\tilde{\varepsilon}(\mathbf{r})]\mathbf{E}(\mathbf{r})=-\frac{4\pi i\omega}{c^2}\mathbf{j}(\mathbf{r})=\mathbf{J}(\mathbf{r}). \tag{3}$$

Here $\tilde{\varepsilon}(\mathbf{r})$ is the fluctuating part of the dielectric constant $\varepsilon(\mathbf{r})=\langle\varepsilon\rangle+\tilde{\varepsilon}(\mathbf{r})$, $j(\mathbf{r})$ is the current density of the extraneous regular sources, $\hat{L}^0$ is a differential operator, with tensor of second rank

$$\hat{L}_{ij}{}^0=\delta_{ij}(\Delta+k_0{}^2\langle\varepsilon\rangle)-\frac{\partial^2}{\partial x_i\partial x_j}\ .$$

A harmonic time dependence of the form $\exp(-i\omega t)$ is assumed for the yield. Equation (3) leads to

$$\mathbf{E}(\mathbf{r})=E^0(\mathbf{r})-k_0{}^2\hat{G}^0[\tilde{\varepsilon}(\mathbf{r})\mathbf{E}(\mathbf{r})], \tag{4}$$

$$\mathbf{E}(\mathbf{r})-\langle\mathbf{E}(\mathbf{r})\rangle=-k_0{}^2\hat{G}^0[\tilde{\varepsilon}(\mathbf{r})\mathbf{E}(\mathbf{r})-\langle\tilde{\varepsilon}(\mathbf{r})\mathbf{E}(\mathbf{r})\rangle], \tag{5}$$

where $\mathbf{E}^0(\mathbf{r})$ is the field in the absence of inhomogeneities, and $\hat{G}^0=[\hat{L}^0]^{-1}$ is an integral operator and is the Green's function of the unperturbed problem. Using a diagram technique[2] or solving directly Eqs. (4) and (5) by iteration, we can obtain the Dyson and Bethe-Salpeter equations for the mean values $\langle\mathbf{E}(\mathbf{r})\rangle$ of the field and for the correlation matrix

$$\langle\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle=\langle\mathbf{E}(\mathbf{r}_1)\otimes\mathbf{E}^*(\mathbf{r}_2)\rangle$$

(the symbol $\otimes$ stands for the tensor product of two vectors:

$$\Gamma_{ij}(\mathbf{r}_1,\ \mathbf{r}_2)=E_i(\mathbf{r}_1)E_j^*(\mathbf{r}_2)).$$

The equation for the mean field is

$$\langle\mathbf{E}(\mathbf{r})\rangle=\mathbf{E}^0(\mathbf{r})+\hat{G}^0\hat{Q}\langle\mathbf{E}(\mathbf{r})\rangle. \tag{6}$$

The mass operator $\hat{Q}$ can be expressed in series form in terms of $\hat{G}^0$ and of the moments of the function $\tilde{\varepsilon}(\mathbf{r})$; just as $\hat{G}^0$, it is a tensor of second rank. The full form of Eq. (6) is thus

$$\langle E_i(\mathbf{r})\rangle=E_i{}^0(\mathbf{r})+\iint d\mathbf{r}'\,d\mathbf{r}''\,G_{ij}{}^0(\mathbf{r},\mathbf{r}')Q_{jk}(\mathbf{r}',\mathbf{r}'')\langle E_k(\mathbf{r}'')\rangle. \tag{7}$$

It can also be written in the integrodifferential form

$$[\hat{L}^0-\hat{Q}]\langle\mathbf{E}(\mathbf{r})\rangle=\hat{D}\langle\mathbf{E}(\mathbf{r})\rangle=\mathbf{J}(\mathbf{r}). \tag{8}$$

 0038-5646/85/121160-04$04.00

The correlation function satisfies the equation

$$\langle\Gamma_{ij}(\mathbf{r}_1,\mathbf{r}_2)\rangle=\langle E_i(\mathbf{r}_1)\rangle\langle E_j^*(\mathbf{r}_2)\rangle + \iint d\mathbf{r}'\,d\mathbf{r}''\,G_{ik}(\mathbf{r}_1,\mathbf{r}')\,G_{jl}^*(\mathbf{r}_2,\mathbf{r}'') \times\iint d\mathbf{r}_3\,d\mathbf{r}_4\,K_{klmn}(\mathbf{r}',\mathbf{r}'',\mathbf{r}_3,\mathbf{r}_4)\langle\Gamma_{mn}(\mathbf{r}_3,\mathbf{r}_4)\rangle \quad (9)$$

or, in shorter notation

$$\langle\Gamma(\mathbf{r}_1,\ \mathbf{r}_2)\rangle=\langle\mathbf{E}(\mathbf{r}_1)\rangle\otimes\langle\mathbf{E}^*(\mathbf{r}_2)\rangle+\hat{G}_1\hat{G}_2^*\hat{K}\langle\Gamma(\mathbf{r}_1,\ \mathbf{r}_2)\rangle. \quad (10)$$

Here $\hat{G}=[\hat{D}]^{-1}$ is the Green's function of Eq. (8) for the mean field, $\hat{K}$ is the intensity operator and is a tensor of fourth rank with integration with respect to two variables. The subscripts 1 and 2 of the operator $\hat{G}$ are the serial numbers of its operand variable.

On the other hand, (4) and (5) lead directly to equations for the mean field and for the correlation functions in terms of the operators of the effective inhomogeneities:

$$\langle\mathbf{E}(\mathbf{r})\rangle=\mathbf{E}^0-k_0^2\hat{G}^0\hat{\tilde{\varepsilon}}^{\text{эфф}}\langle\mathbf{E}(\mathbf{r})\rangle, \quad (11)$$

$$\langle\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle-\langle\mathbf{E}(\mathbf{r}_1)\rangle\otimes\langle\mathbf{E}^*(\mathbf{r}_2)\rangle =k_0^4\hat{G}_1^0\hat{G}_2^{0*}[\hat{B}_\varepsilon^{\text{eff}}\langle\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle-\hat{\tilde{\varepsilon}}_1^{\text{eff}}\,\hat{\tilde{\varepsilon}}_2^{\text{eff}*}\langle\mathbf{E}(\mathbf{r}_1)\rangle\otimes\langle\mathbf{E}^*(\mathbf{r}_2)\rangle], \quad (12)$$

where $\hat{\tilde{\varepsilon}}_{\text{eff}}$ and $\hat{\tilde{B}}^{\text{eff}}$ are defined by the relations

$$\langle\tilde{\varepsilon}(\mathbf{r})\mathbf{E}(\mathbf{r})\rangle=\hat{\tilde{\varepsilon}}^{\text{eff}}\langle\mathbf{E}(\mathbf{r})\rangle, \quad (13)$$

$$\langle\tilde{\varepsilon}(\mathbf{r}_1)\tilde{\varepsilon}^*(\mathbf{r}_2)\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle=\hat{\tilde{B}}_\varepsilon^{\text{eff}}\langle\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle. \quad (14)$$

Comparison of the two systems (11), (12) and (6), (10) allows us to express the operators $\hat{\varepsilon}_{\text{eff}}$ and $\hat{B}^{\text{eff}}$ in terms of $\hat{Q}$ and $\hat{K}$:

$$\hat{\tilde{\varepsilon}}^{\text{eff}}=-\frac{1}{k_0^2}\hat{Q},$$
$$\hat{\tilde{B}}_\varepsilon^{\text{eff}}=\frac{1}{k_0^4}(\hat{K}+\hat{Q}_1\hat{G}_1\hat{K}+\hat{Q}_2^*\hat{G}_2^*\hat{K}+\hat{Q}_1\hat{Q}_2^*]. \quad (15)$$

The operator of the effective correlation function of the inhomogeneities is thus, just as the intensity operator $\hat{K}$, a tensor of fourth rank:

$$\hat{\tilde{B}}_\varepsilon^{\text{eff}}\langle\Gamma_{ij}(\mathbf{r}_1,\mathbf{r}_2)\rangle =\iint d\mathbf{r}'\,d\mathbf{r}''[\tilde{B}_\varepsilon^{\text{eff}}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}',\mathbf{r}'')]_{ijkl}\langle\Gamma_{kl}(\mathbf{r}',\mathbf{r}'')\rangle.$$

To solve actual problems it may be more convenient to introduce the effective-inhomogeneities operator for the second moments, using the equation

$$\langle\tilde{\varepsilon}(\mathbf{r}_1)\,\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle=\hat{\tilde{\mathscr{E}}}_1^{\text{eff}}\langle\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle. \quad (16)$$

We can then use for the correlation function the equation

$$[\hat{L}_1^0+k_0^2\hat{\tilde{\mathscr{E}}}_1^{\text{eff}}]\langle\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle=\mathbf{J}(\mathbf{r}_1)\otimes\langle\mathbf{E}^*(\mathbf{r}_2)\rangle. \quad (17)$$

Applying the operator $\hat{D}_1$ to both sides of the Bethe-Salpeter equation (10) and comparing the result with (17) we obtain

$$\hat{\tilde{\mathscr{E}}}_1^{\text{eff}}=-\frac{1}{k_0^2}[\hat{Q}_1+\hat{G}_2^*\hat{K}]. \quad (18)$$

We can introduce also a definition symmetric to (16):

$$\langle\tilde{\varepsilon}^*(\mathbf{r}_2)\,\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle=\hat{\tilde{\mathscr{E}}}_2^{\text{eff}*}\langle\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle, \quad (19)$$

$$[\tilde{\mathscr{E}}_2^{\text{eff}*}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}',\mathbf{r}'')]_{ijkl}=[\tilde{\mathscr{E}}_1^{\text{eff}*}(r_2,r_1;r'',r')]_{jilk},$$

or

$$\hat{\tilde{\mathscr{E}}}_2^{\text{eff}*}=-k_0^{-2}[\hat{Q}_2^*+\hat{G}_1\hat{K}]. \quad (20)$$

For real fluctuations of the dielectric constant, the obvious equality

$$\langle\tilde{\varepsilon}(\mathbf{r})\Gamma_{ij}(\mathbf{r},\mathbf{r})\rangle=\langle\tilde{\varepsilon}(\mathbf{r})\Gamma_{ji}(\mathbf{r},\mathbf{r})\rangle^*$$

allows us to write down the relation

$$[\hat{\tilde{\mathscr{E}}}_1^{\text{eff}*}(\mathbf{r},\mathbf{r};\mathbf{r}',\mathbf{r}'')]_{ijkl}=[\hat{\tilde{\mathscr{E}}}_1^{\text{eff}*}(\mathbf{r},\mathbf{r};\mathbf{r}'',\mathbf{r}')]_{jilk}, \quad (21)$$

which, rewritten in terms of the operators $\hat{Q}$ and $\hat{K}$, is known as the "optical theorem":

$$Q_{ik}(\mathbf{r},\mathbf{r}')\delta_{jl}\delta(\mathbf{r}-\mathbf{r}'')-Q_{jl}^*(\mathbf{r},\mathbf{r}'')\delta_{ik}\delta(\mathbf{r}-\mathbf{r}') =\int d\mathbf{r}_1[G_{jm}^*(\mathbf{r},\mathbf{r}_1)K_{imkl}(\mathbf{r},\mathbf{r}_1,\mathbf{r}',\mathbf{r}'') -G_{im}(\mathbf{r},\mathbf{r}_1)K_{mjkl}(\mathbf{r}_1,\mathbf{r};\mathbf{r}',\mathbf{r}'')]. \quad (22)$$

Relations (15), (18), and 20 make it possible to determine the dielectric constant $\varepsilon^{\text{eff}}(\mathbf{r},\mathbf{r}')=\langle\varepsilon\rangle+\tilde{\varepsilon}^{\text{eff}}(\mathbf{r},\mathbf{r}')$ of the effective medium and its correlation function

$$B_\varepsilon^{\text{eff}}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}',\mathbf{r}'')=|\langle\varepsilon\rangle|^2+\langle\varepsilon^*\rangle\tilde{\mathscr{E}}_1^{\text{eff}}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}',\mathbf{r}'') +\langle\varepsilon\rangle\tilde{\mathscr{E}}_2^{\text{eff}*}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}',\mathbf{r}'')+\tilde{B}_\varepsilon^{\text{eff}}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}',\mathbf{r}'').$$

2. The presence of singularities in the expressions for $\hat{G}^0$ and $\hat{G}$ may not be convenient in the calculation of the operators $\hat{\tilde{\varepsilon}}^{\text{eff}},\{\hat{\tilde{\mathscr{E}}}^{\text{eff}}\}=\{\hat{\tilde{B}}_\varepsilon^{\text{eff}},\hat{\tilde{\mathscr{E}}}_1^{\text{eff}},\hat{\tilde{\mathscr{E}}}_2^{\text{eff}}\}$. It is useful in this case to introduce the new quantities[3]

$$\mathbf{F}(\mathbf{r})=\frac{\varepsilon(\mathbf{r})+2\varepsilon_0}{3\varepsilon_0}\mathbf{E}(\mathbf{r}),$$
$$\xi(\mathbf{r})=3\frac{\varepsilon(\mathbf{r})-\varepsilon_0}{\varepsilon(\mathbf{r})+2\varepsilon_0}, \quad (23)$$
$$\mathscr{T}(\mathbf{r}_1,\ \mathbf{r}_2)=\mathbf{F}(\mathbf{r}_1)\otimes\mathbf{F}^*(\mathbf{r}_2),$$

where $\varepsilon_0=\text{const}$. The function $\mathbf{F}(\mathbf{r})$ satisfies the equation

$$\mathbf{F}(\mathbf{r})-\langle\mathbf{F}(\mathbf{r})\rangle=-k_0^2\varepsilon_0\hat{G}^{0\prime}[\xi(\mathbf{r})\mathbf{F}(\mathbf{r})-\langle\xi(\mathbf{r})\mathbf{F}(\mathbf{r})\rangle], \quad (24)$$

where $\hat{G}^{0\prime}=\text{P.V.}\hat{G}^0$. The symbol P.V. denotes that the operator $\hat{G}^{0\prime}$ presupposes integration in the sense of principal value, i.e., contains no δ-singularities. Equation (24) for $\mathbf{E}(\mathbf{r})$ is the same as Eq. (5) but with $\hat{G}^0$ and $\tilde{\varepsilon}(\mathbf{r})$ replaced by $\hat{G}^{0\prime}$ and $\varepsilon_0\xi(\mathbf{r})$. It follows therefore that under the condition $\langle\xi(\mathbf{r})\rangle=0$ (from which ε_0 is determined) the same substitution can be carried out also in all the relations that follow, forming the operators $\hat{Q}'$ and $\hat{K}'$ in terms of $\hat{G}^{0\prime}$ and the moments of the function $\varepsilon_0\xi(\mathbf{r})$ in accordance with the same rules as used to express $\hat{Q}$ and $\hat{K}$ in terms of $\hat{G}^0$ and $\varepsilon_0\xi(\mathbf{r})$. A similar procedure is used to introduce the effective-inhomogeneities operators ξ^{eff} for the mean field and $\{\zeta^{\text{eff}}\}=\{\hat{B}_\xi^{\text{eff}},\hat{\zeta}_1^{\text{eff}},\hat{\zeta}_2^{\text{eff}}\}$ for the correlation function. From (23) we can obtain an operator relation that does not contain the auxiliary field $\mathbf{E}(\mathbf{r})$, for example:

$$[\hat{\varepsilon}^{\text{eff}}-\varepsilon_0]\langle E\rangle=\varepsilon_0\hat{\xi}^{\text{eff}}[1-{}^1/_3\hat{\xi}^{\text{eff}}]^{-1}\langle E\rangle,$$

$$[\hat{\mathscr{E}}_1^{\text{eff}}-\varepsilon_0]\langle\Gamma\rangle=\varepsilon_0[\hat{\zeta}_1^{\text{eff}}-{}^1/_3\hat{B}_\xi^{\text{eff}}] \times[1-{}^1/_3(\hat{\zeta}_1^{\text{eff}}+\hat{\zeta}_2^{\text{eff}*})+{}^1/_9\hat{B}_\xi^{\text{eff}}]^{-1}\langle\Gamma\rangle,$$

$$\hat{\mathscr{E}}_1^{\text{eff}}=\langle\varepsilon\rangle+\hat{\tilde{\mathscr{E}}}_1^{\text{eff}}.$$

These relations allow us to exclude the operators ε^{eff} and $\hat{\mathscr{E}}^{\text{eff}}$ from the wave equations for the mean field. A corresponding equation is given in Ref. 4 for the mean field, but the operator $\hat{\xi}^{\text{eff}}$ is considered in a "bilocal" approximation, i.e., accurate to quantities of order $\langle \xi^2 \rangle$. In this approximation, the following equations hold

$$[\hat{L}_0+k_0^2\varepsilon_0\hat{\xi}^{\text{eff}}]\langle \mathbf{E}(\mathbf{r})\rangle=\mathbf{J}(\mathbf{r}), \tag{25}$$

$$[\hat{L}_0+k_0^2(\hat{\xi}_1^{\text{eff}}-{}^1/_3\hat{B}_\xi^{\text{eff}})]\langle\Gamma(\mathbf{r}_1,\mathbf{r}_2)\rangle=\mathbf{J}(\mathbf{r}_1)\otimes\langle\mathbf{E}^*(\mathbf{r}_2)\rangle. \tag{26}$$

The operator $\hat{L}_0$ differs from $\hat{L}^0$ in that it contains the constant ε_0 in place of $\langle\varepsilon\rangle$.

3. The energy characteristics of the field, such as the intensity $I(\mathbf{r}) = |\mathbf{E}(\mathbf{r})|^2$ and the energy flux

$$\mathbf{S}(\mathbf{r})=(c/8\pi)\,\mathrm{Re}[\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})],$$

can be expressed in terms of the correlation-tensor components. Using the equation

$$\mathbf{H}(\mathbf{r})=(ik_0)^{-1}\,\mathrm{rot}\,\mathbf{E}(\mathbf{r}),$$

we can obtain

$$I(\mathbf{r})=\Gamma_{ii}(\mathbf{r},\mathbf{r}),$$

$$S_i(\mathbf{r})=\frac{c}{4\pi k_0}\,\mathrm{Im}\left[\frac{\partial}{\partial(x_2)_j}\Gamma_{ji}(\mathbf{r}_1,\mathbf{r}_2)-\frac{\partial}{\partial(x_2)_i}\Gamma_{jj}(\mathbf{r}_1,\mathbf{r}_2)\right]_{\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}}. \tag{27}$$

In specific cases the mean field intensity in a randomly inhomogeneous medium can be calculated without the use the operators $\{\hat{\mathscr{E}}^{\text{eff}}\}$ (in other words, without using the Bethe-Salpeter equation). This is the situation, for example in the calculation of a fluctuating thermal field in an infinite statistically homogeneous medium with real fluctuations of the dielectric constant: $\varepsilon(\mathbf{r}) = \varepsilon_1 + i\varepsilon_2 + \tilde{\varepsilon}(\mathbf{r})$, $\tilde{\varepsilon}(\mathbf{r}) = \tilde{\varepsilon}^*(\mathbf{r})$. This can be verified by writing down the energy-conservation law

$$\mathrm{div}\,\mathbf{S}=\frac{c}{16\pi}\left[-2k_0\varepsilon_2|\mathbf{E}|^2-\frac{4\pi}{c}\overline{(\mathbf{E}^*\mathbf{j}+\mathbf{E}\mathbf{j}^*)}\right], \tag{28}$$

where the superior bar denotes averaging over the ensemble of realizations of the fluctuating currents $\mathbf{j}(\mathbf{r})$. Since the random thermal sources are uniformly distributed over the volume of the medium, the resultant field is statistically homogeneous. Hence

$$\mathrm{div}\langle\mathbf{S}(\mathbf{r})\rangle=0,$$

or

$$\varepsilon_2\langle|\mathbf{E}(\mathbf{r})|^2\rangle=-\frac{2\pi}{\omega}[\overline{\langle\mathbf{E}^*(\mathbf{r})\rangle\mathbf{j}(\mathbf{r})}+\overline{\langle\mathbf{E}(\mathbf{r})\rangle\mathbf{j}^*(\mathbf{r})}]. \tag{29}$$

Recognizing that the mean field satisfies Eq. (5), and the correlation matrix of the sources is of the form[5]

$$\overline{j_i(\mathbf{r}_1)j_j^*(\mathbf{r}_2)}=\frac{\omega\Theta(\omega,T)}{4\pi^2}\varepsilon_2\delta_{ij}\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{30}$$

we get

$$\langle|\mathbf{E}(\mathbf{r})|^2\rangle=\langle\Gamma_{ii}(\mathbf{r},\mathbf{r})\rangle=-\frac{2i\omega}{c^2}\Theta(\omega,T)\,(G_{ii}(0)-G_{ii}^*(0)), \tag{31}$$

where $G(\mathbf{r})$ is the kernel of the integral operator $\hat{G}(\mathbf{r}_1,\mathbf{r}_2)=\hat{G}(\mathbf{r}_1-\mathbf{r}_2)$.

To calculate the correlation tensor of the thermal field we must use Eqs. (12) or (17) and substitute in them the source correlation function (30). For example, subtracting from (17) the equation symmetric to it and taking into account the statistical homogeneity of the problem, we get

$$2i\varepsilon_2\langle\Gamma_{ij}(\mathbf{r}_1,\mathbf{r}_2)\rangle+\iint d\mathbf{r}'\,d\mathbf{r}''\{[\tilde{\mathscr{E}}_1^{\text{eff}}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}',\mathbf{r}'')]_{ijkl}$$
$$-[\tilde{\mathscr{E}}_1^{\text{eff}*}(\mathbf{r}_2,\mathbf{r}_1;\mathbf{r}'',\mathbf{r}')_{jilk}\}\langle\Gamma_{kl}(\mathbf{r}',\mathbf{r}'')\rangle$$
$$=4\frac{\omega\Theta(\omega,T)}{c^2}[G_{ij}(\mathbf{r}_1,\mathbf{r}_2)-G_{ji}^*(\mathbf{r}_2,\mathbf{r}_1)]. \tag{32}$$

At $\mathbf{r}_1 = \mathbf{r}_2$ (32) leads, in particular, to Eq. (31).

4. By way of illustration of the general relations above we consider, within the framework of perturbation theory, the propagation of a plane wave in a layered randomly inhomogeneous medium. We shall assume that the inhomogeneous medium is located in the region $z<0$ and has a dielectric constant $\varepsilon(z) = \langle\varepsilon\rangle + \tilde{\varepsilon}(z)$, where $\langle\varepsilon\rangle = \varepsilon_1 + i\varepsilon_2$, and $\tilde{\varepsilon}(z)$ is a real random function. We assume that the field in the absence of fluctuations is a plane wave of unit amplitude:

$$E^0(z)=e^{-i\varkappa z},$$

$$\varkappa=k_0\langle\varepsilon\rangle^{1/2}=k+i\gamma.$$

The Green's function of the unperturbed problem is

$$G^0(z,z')=-(i/2\varkappa)e^{i\varkappa|z-z'|}.$$

In the one-dimensional case the system (11) and (12) takes in first-order perturbation theory the form

$$\langle E(z)\rangle=[1-k_0^2\hat{G}^0\hat{\varepsilon}^{\text{eff}}]E^0(z), \tag{33}$$

$$\langle\Gamma(z_1,z_2)\rangle=\langle E(z_1)\rangle\langle E^*(z_2)\rangle+k_0^4\hat{G}_1^0\hat{G}_2^{0*}\hat{B}_\varepsilon^{\text{eff}}E^0(z_1)E^{0*}(z_2). \tag{34}$$

In a statistically homogeneous medium we can transform to difference variables:

$$\tilde{\varepsilon}^{\text{eff}}(z,z')=\tilde{\varepsilon}^{\text{eff}}(z-z'),$$

$$B_\varepsilon^{\text{eff}}(z_1,z_2;z',z'')=B_\varepsilon^{\text{eff}}(z_1-z',z_1-z_2,z_2-z'').$$

If the correlation radius of the inhomogeneities is small enough, the functions $\tilde{\varepsilon}^{\text{eff}}$ and $B_\varepsilon^{\text{eff}}$ can be regarded as rapidly decreasing in each of the variables. It is recongized here that in the limiting case $|z_1-z_2|\to\infty$ the function $B_\varepsilon^{\text{eff}}$ factorizes into $\tilde{\varepsilon}^{\text{eff}}(z_1-z')\tilde{\varepsilon}^{\text{eff}*}(z_2-z'')$ and is consequently a quantity of next order of smallness. Under the foregoing assumptions, the quantity $\hat{\tilde{\varepsilon}}^{\text{eff}}E^0(z)$ in (33) can be approximately calculated in terms of the Fourier component $\tilde{\varepsilon}^{\text{eff}}(\varkappa)$:

$$\int_{-\infty}^{0}\tilde{\varepsilon}^{\text{eff}}(z-z')e^{-i\varkappa z'}\,dz'=e^{-i\varkappa z}\int_{z'}^{\infty}d\tau\,\tilde{\varepsilon}^{\text{eff}}(\tau)e^{i\varkappa\tau}$$
$$\approx e^{-i\varkappa z}\int_{-\infty}^{\infty}d\tau\,\tilde{\varepsilon}^{\text{eff}}(\tau)e^{i\varkappa\tau}=e^{-i\varkappa z}\tilde{\varepsilon}^{\text{eff}}(\varkappa).$$

Similarly, if the correlation function is calculated from (34), we can put

$$\hat{B}_\varepsilon^{\text{eff}} E^0(z_1) E^{0*}(z_2)$$

$$\approx \exp(-i\varkappa z_1 + i\varkappa^* z_2) \iint_{-\infty}^{\infty} d\tau_1\, d\tau_2 \tilde{B}_\varepsilon^{\text{eff}}(\tau_1, z_1 - z_2, \tau_2)$$

$$\times \exp(i\varkappa\tau_1 - i\varkappa^*\tau_2) = \exp(-i\varkappa z_1 + i\varkappa^* z_2) \tilde{B}_\varepsilon^{\text{eff}}(\varkappa, z_1 - z_2, \varkappa^*).$$

As a result we get for the mean field the expression

$$\langle E(z)\rangle = e^{-i\varkappa z}\left[1 - \frac{k_0^2}{4\varkappa^2}\tilde{\varepsilon}^{\text{eff}}(\varkappa) - i\frac{k_0^2}{2\varkappa}\tilde{\varepsilon}^{\text{eff}}(\varkappa) z\right], \quad (35)$$

or

$$\langle E(z)\rangle \approx \exp\left[-i\left(\varkappa + \frac{k_0^2}{2\varkappa}\tilde{\varepsilon}^{\text{eff}}(\varkappa)\right) z\right]$$

$$= \exp[-i(k + \Delta k^{\text{eff}}) z + (\gamma + \Delta\gamma^{\text{eff}}) z]. \quad (36)$$

Account is taken in (36) of the smallness of the corrections to $E^0(z)$ which are contained in (35). A small correction to the constant factor has been left out, since we are interested in the dependence of the mean field on the coordinate z. At $\varepsilon_2 \ll \varepsilon_1$ the effective increments Δk^{eff} and $\Delta\gamma^{\text{eff}}$ take the form

$$\Delta k^{\text{eff}} = \frac{k}{2}\frac{\operatorname{Re}\tilde{\varepsilon}^{\text{eff}}(\varkappa)}{\varepsilon_1}; \quad \Delta\gamma^{\text{eff}} = \frac{k}{2}\operatorname{Im}\frac{\tilde{\varepsilon}^{\text{eff}}(\varkappa)}{\varepsilon_1}. \quad (37)$$

Calculation of the correlation function under the same assumptions yields

$$\langle E(z_1)E^*(z_2)\rangle \approx \exp[-i(k + \Delta k^{\text{eff}})(z_1 - z_2) + (\gamma + \Delta\gamma^{\text{eff}} - \alpha)(z_1 + z_2) - \alpha|z_1 - z_2|], \quad (38)$$

$$\alpha = \frac{k^2}{4}\frac{\tilde{B}_\varepsilon^{\text{eff}}(\varkappa, 0, \varkappa^*)}{\varepsilon_1^2}$$

$$\tilde{B}_\varepsilon^{\text{eff}}(\varkappa, 0, \varkappa^*) = \int_{-\infty}^{\infty} \tilde{B}_\varepsilon^{\text{eff}}(\varkappa, \tau, \varkappa^*)\, d\tau.$$

At $z_1 = 0$ or $z_2 = 0$ Eq. (38) goes over into (36). Putting $z_1 = z_2$, we obtain for the field intensity an expression from which it can be seen that the effective energy damping coefficients differs by $(-\alpha)$ from those of the effective mean-field damping. The reason is that the latter contains "imaginary" damping due to phase fluctuations that do not influence the intensity.

The foregoing examples confirm the conclusion that to calculate the second-order moments of the random function $\mathbf{E}(\mathbf{r})$ we must known the "second-order effective-inhomogeneities operators" such as $\hat{B}^{\text{eff}}$, $\hat{\mathscr{B}}_1^{\text{eff}}$, $\hat{\mathscr{B}}_2^{\text{eff}}$. Of course, equivalent calculations are possible in terms of the mass operator $\hat{Q}$ and of the intensity operator $\hat{K}$. In particular, in first-order perturbation theory $B^{\text{eff}} = k_0^{-4}K$, i.e., the operators ε^{eff} and B^{eff} are equal to $\hat{Q}$ and $\hat{K}$ to within constant coefficients. In the general case, however, the equations that contain the effective-inhomogeneities operators may turn out to be more compact, as is the case, for example, in the formulation of the optical theorem (21), (22). The main advantage of the proposed formalism is that the quantities employed have a lucid physical meaning, being functions of the dielectric constant of an auxiliary effective medium. This permits the results for quantities quadratic in the field in media with spatial dispersion to be used for averaged quadratic quantities in randomly inhomogeneous media, and vice versa.

The author is grateful to V. I. Tatarskiĭ for numerous discussions and valuable remarks.

[1]Yu. A. Ryzhov and V. V. Tamoĭkin, Izv. Vyssh. Ucheb. Zav., Radiofizika **13**, 356 (1980).
[2]S. M. Rytov, Yu. A. Kravtsov, and V. I. Tatarskiĭ, Introduction to Statistical Radiophysics [in Russian], Nauka, 1978, part 2, §48.
[3]Yu. A. Ryzhov, V. V. Tamoĭkin, and V. I. Tatarskiĭ, Zh. Eksp. Teor. Fiz. **48**, 656 (1965) [Sov. Phys. JETP **21**, 433 (1965)].
[4]A. Stogryn, IEEE Transations, **AP-31**, 985 (1983).
[5]See Ref. 2, part 2, §16.

Translated by J. G. Adashko