Analysis of the twinning-plane superconductivity in tin and niobium

A. I. Buzdin and N. A. Khvorikov

M. V. Lomonosov State University, Moscow; Computing Center of the Academy of Sciences of the USSR

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Numerical methods are used to solve the nonlinear equations of the Ginzburg-Landau type describing the superconductivity of the twinning plane in tin and niobium. The shape of the field-temperature phase diagram is determined and is found to be in good agreement with the available experimental data. The nature of the change in the phase diagram is tracked as the Ginzburg-Landau parameter is increased. The basic properties are determined for superconductivity localized at the twinning plane.

1. INTRODUCTION

It has been established by Khaĭkin and Khlyustikov[1,2] that the presence of a twinning plane in crystals of tin, indium, rhenium, and thallium gives rise to superconductivity localized around the twinning plane. This twinning-plane superconductivity (TPS) arises at a temperature T_c above the critical temperature T_{c0} of the bulk metal. All of the superconductors mentioned belong to type I. Recently, however, superconductivity of an isolated twinning plane has been detected and studied in niobium, a type-II superconductor.[3] There are also results[4] which indicate that the critical temperature of niobium increases when a set of twins is produced by plastic deformation. The field-temperature (H,T) phase diagram for the TPS in niobium has a fundamentally different character from that of tin (the most thoroughly studied superconductor exhibiting TPS). In niobium the TPS exists[3] at all temperatures $0<T<T_c$ in fields somewhat in excess of the upper critical field of the bulk metal, while in tin the existence region of the TPS is limited[2,5] to a narrow temperature interval near T_{c0}.

In all cases, however, the critical temperature for TPS is only slightly higher than the critical temperature of the bulk metal, i.e., $\tau_0=(T_c-T_{c0})/T_c \ll 1$ ($\sim 10^{-2}$ in tin and niobium). This circumstance is due to the proximity effect of the normal, nonsuperconducting (at $T>T_{c0}$) bulk metal. However, even though $\tau_0 \ll 1$, the change in the properties of the superconductor near the twinning plane can be extremely important. This has been demonstrated most clearly in the experiments of Khlyustikov and Khaĭkin,[6] who observed that the critical temperature more than doubled in tin samples having a high concentration of twins. Analogous results were also obtained in microscopic particles of tin containing twinning planes.[6] An estimate shows[5] that a still greater increase in T_c can be achieved by optimizing the size of the small twinned particles of tin.

The mechanism for the enhancement of superconductivity near the twinning plane has not been explained. It has been suggested[1,2] that the mechanism may involve a softening of the phonons and the appearance of new phonon modes near the twinning plane or perhaps the appearance of an external group of electrons (absent in the ordinary three-dimensional crystal) moving along the twinning plane. A preliminary theoretical analysis[7] has shown that the presence of a lattice-dilatation wave at the twinning plane should lead to an increase in the electron-phonon interaction constant.

In any case the "enhancement" of the superconductivity occurs in a thin layer near the twinning plane, with a thickness on the order of a few interatomic spacings, i.e., the layer is thin compared to the correlation length $\xi_0=0.18v_F/T_{c0}$ of the superconductor. Therefore, to describe the TPS a model of the superconductor has been proposed[8,9] in which the dimensionless electron-phonon interaction constant λ has a maximum near the twinning plane in a layer of thickness $d \ll \xi_0$. A similar model was considered independently in Refs. 10–12.

Near the critical temperature T_{c0} of the bulk metal the TPS can be described in the framework of a modified Ginzburg-Landau theory in which the free-energy-density functional contains an additional δ-function term[8,9] $-\gamma\delta(x)|\psi(\mathbf{r})|^2$; here $\psi(\mathbf{r})$ is the order parameter for the superconductor, and the twinning plane is the plane $x=0$.

An analogous approach has been used previously to describe surface magnetism[13] and to treat structural phase transitions in systems with defects.[14] In our case there is actually only one free parameter, γ, which is related to the increase in the critical temperature for TPS in comparison with T_{c0}, i.e., to τ_0 (the relationship of γ to the increase in the constant λ near the twinning plane has been found[8,10] from the solution of the integral equation which determines T_c). Using the experimental values of τ_0 and the familiar properties of the superconductivity of the bulk metal, one can obtain a complete description of the twinning-plane superconductivity and, in particular, its behavior in a magnetic field. It should be emphasized that in a type-I superconductor the modified Ginzburg-Landau theory describes the TPS at all temperatures where it exists, since this region is limited to the neighborhood of T_{c0}.[5]

In Refs. 8, 9, and 5 the critical fields of the transition and the phase diagram for TPS were found for an extreme type-I superconductor (having a Ginzburg-Landau parameter $\varkappa \to 0$), and the upper critical field and the magnetic moment near T_c were determined. The results were in qualitative agreement with the experimental data.[1,2,5] In principle, however, the theory permits an exact quantitative description, and so one wonders what causes the quantitative disagreement between the experimental and theoretical results. We have therefore carried out in this study a numerical solution of the nonlinear Ginzburg-Landau equations de-

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scribing the TPS and have found the shape of the phase diagram, the magnetic moment in a field, and the dependence of the order parameter on the coordinate in a field for the case of the TPS in tin, for which $\varkappa = 0.13$; for completeness we also give some of the results from a previous brief communication.[5] We have also determined the properties of the TPS for superconductors with other values of $\varkappa$; in particular, we have made a detailed analysis of the case $\varkappa = 1$, which corresponds to the TPS in niobium[3] (a type-II superconductor). In addition, we discuss the diamagnetic susceptibility for TPS and develop a method for calculating the upper critical field throughout the entire temperature interval. We briefly discuss the possibility of a Kosterlitz-Thouless transition in the case of an inhomogeneous twinning plane.

2. DESCRIPTION OF THE TWINNING-PLANE SUPERCONDUCTIVITY

When the critical temperature T_c for superconductivity localized near the twinning plane is slightly higher than the corresponding value T_{c0} in the bulk metal,[1)] the characteristic length scale $\xi(\tau_0) = \xi_0[T_{c0}/(T_c - T_{c0})]^{1/2}$ for the TPS turns out to be large compared to ξ_0, i.e., $\xi(\tau_0) \gg \xi_0$. As a result, one can describe the TPS at temperature near T_{c0} in a modified Ginzburg-Landau theory with a free-energy-density functional[8,9]

$$F = \frac{(\mathbf{B}-\mathbf{H})^2}{8\pi} + \frac{1}{4m}\left|\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\psi\right|^2$$

$$+a|\psi|^2+b|\psi|^4-\gamma\delta(x')|\psi|^2. \qquad (1)$$

Here $\mathbf{H}$ is the external field, m is the electron mass, and we have used the standard notation for the coefficients of the functional (see, e.g., Ref. 15):

$$a=(T-T_{c0})/\eta T_{c0}, \quad b=1/N\eta,$$

where N is the electron density and $\eta = 0.12\varepsilon_F/T_{c0}{}^2$ in the case of a clean superconductor. The functional (1) differs from the usual functional by the last, δ-function (on the ξ_0 scale) term, which estimates the enhancement of the Cooper pairing near the twinning plane $x' = 0$. The constant γ is related to the temperature at which localized superconductivity arises by the relation

$$\gamma=[(T_c-T_{c0})/\eta mT_{c0}]^{1/2}=(\tau_0/\eta m)^{1/2}.$$

The characteristic scale for the reduced temperature in a treatment of the TPS is $T_c - T_{c0}$; the field scale is

$$H_0=H_c(\tau_0)=(2\tau_0/\eta)(\pi/b)^{1/2};$$

the length scale is

$$\xi(\tau_0)=1/2m\gamma;$$

and the scale for the order parameter is

$$\psi_0=\psi(\tau=-\tau_0)=(\tau_0/\eta b)^{1/2}.$$

It is therefore convenient to use dimensionless variables for the temperature $t = (T_c - T)/(T_c - T_{c0})$, field $h = H/H_0$, coordinate $x = x'/\xi(\tau_0)$, and order parameter $\varphi = \psi/\psi_0$.

In this notation the equations for the order parameter and field distribution from functional (1) have the form [for a field parallel to the twinning plane and for $A(x) \equiv A_y(x)$]

$$-\varphi'' + \frac{\varphi}{2}\tilde{A}^2+t\varphi+\varphi^3=0, \quad \left.\frac{d\varphi}{dx}\right|_{x=0} = -\varphi(0), \qquad (2)$$

$$\tilde{A}'' = \frac{\varphi^2}{\varkappa^2}\tilde{A}, \quad \tilde{A}(x\to\infty)=\frac{hx}{\varkappa}, \quad \tilde{A}(0)=0, \qquad (3)$$

where

$$\tilde{A}=A/A_0, \quad A_0=(\tau_0 mc^2/2\eta e^2)^{1/2}, \quad \varkappa^2=(\lambda/\xi)^2=m^2c^2b/2\pi e^2,$$

and $\varkappa$ is the Ginzburg-Landau parameter.

In the absence of magnetic field the solution of Eq. (2) is of the form

$$\varphi(x)=\frac{(2t)^{1/2}}{\sinh(xt^{1/2}+p)}, \quad p=\frac{1}{2}\ln\frac{1+t^{1/2}}{1-t^{1/2}}, \qquad (4)$$

and the twinning-plane superconductivity exists, of course, only in the temperature interval $T_{c0} < T < T_c$, i.e., $0 < t < 1$.

3. TWINNING-PLANE SUPERCONDUCTIVITY IN A TYPE-I SUPERCONDUCTOR (TIN)

To determine the transition field and the phase diagram for TPS in a type-I superconductor we must solve the system of nonlinear equations in (2) and (3). In the limiting case of a type-I superconductor we can neglect the penetration of the fields into the TPS region, and then we can write the solution of the equation for φ in quadratures.[9] The field h_c of the first-order transition is now determined by the system of equations[5]

$$\int_0^1 [1-t(1-y^2)-0{,}5z(y^4-1)]^{1/2}dy=1,$$

$$h_c=[2(1-t)z-z^2]^{1/2}, \qquad (5)$$

giving the TPS phase diagram shown in Fig. 3 for $\varkappa = 0$.

The accuracy of this calculation, however, is not good; it is of order $\varkappa^{1/2}$, as in the treatment of the energy of the boundary between the superconducting and normal phases in a field (see, e.g., Ref. 15). It is this circumstance that explains the disagreement between the experimental data for the field h_c in tin[2,5] and the function in Eq. (5).

By numerically solving Eqs. (2) and (3) by a finite-element method, we have found the TPS phase diagram (Fig. 1) for the case $\varkappa = 0.13$, which corresponds to tin.[2)] As we see from Fig. 1, there is very good quantitative agreement with experiment.

Twinning-plane superconductivity in tin can be ob-

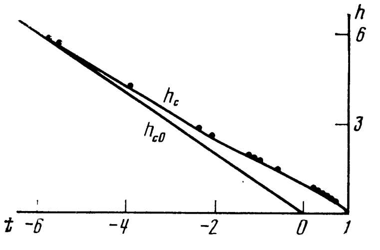

FIG. 1. TPS phase diagram for tin ($\varkappa = 0.13$) in a parallel field; h_c is the field of the first-order transition to the TPS state, and $h_{c0} = |t|$ is the field of the first-order transition of the bulk metal. The points are the experimental data.[2,5]

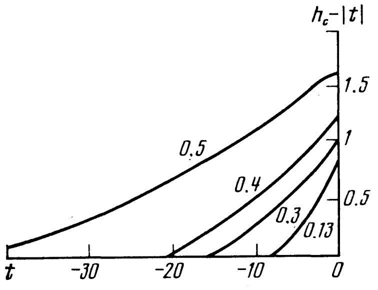

FIG. 2. Increase in the critical field for TPS with respect to the transition field in the bulk metal ($h_{c0} = |t|$) for various values of the parameter $\varkappa$.

served only in the temperature range $-(6\text{–}7) < t < 1$, i.e., for $T_{c0} - 0.25\,K < T < T_{c0} + 0.04$ K. The presence of a lower bound on the TPS existence temperatures is specific to type-I superconductors. Interestingly, this circumstance permits a complete description of the TPS in these superconductors in the framework of the Ginzburg-Landau approach.

As $\varkappa$ increases, the $h_c(t)$ curve becomes steeper and the TPS existence region becomes wider, as is seen in Fig. 2, which shows the temperature dependence of $h_c - |t|$ (i.e., the difference of the critical fields for the TPS and bulk metal) for various values of the parameter $\varkappa$.

However, immediately below the temperature $t = 1$ at which the TPS arises there is always a second-order transition in a field.[9] The reason for this is that the effective screening depth $\lambda_{\rm eff}$ goes as φ^{-1}, and $\lambda_{\rm eff} \to \infty$ as $t \to 1$, whereas the correlation length $\xi(\tau_0) \sim \xi_0 \tau_0^{-1/2}$ here depends only weakly on the temperature.

The phase diagram for the TPS of a type-I superconductor always has a tricritical point $(h_c(t^*), t^*)$ at which the character of the transition changes; in the case $\varkappa^{1/2} \ll 1$ this point is located[9] at $t^* \approx 1 - 1.5\varkappa^2$. The phase diagrams for the TPS near $t = 0$ for various $\varkappa$ are shown in Fig. 3, with the tricritical points indicated by stars.

In pure tin the region of the second-order phase transition is very narrow, $1 - t^* \lesssim 0.01$, and one can scarcely hope to observe experimentally the change in the character of the transition. It would be very interesting to try to follow the change in the slope of the $h_c(t)$ curve in tin when the parameter $\varkappa$ changes; the latter could be increased, for example, by introducing an impurity into the sample or by irradiating it. Here the temperature at which the TPS arises should increase (in the "dirty" limit $l \ll \xi_0$, where l is the electron mean free path, the value of τ_0 increases by a factor of ξ_0/l),[8] and it might also be possible to observe the change in the character of the transition at the tricritical point (as we see in Fig. 3, the region of the second-order phase transition expands rapidly with increasing $\varkappa$).

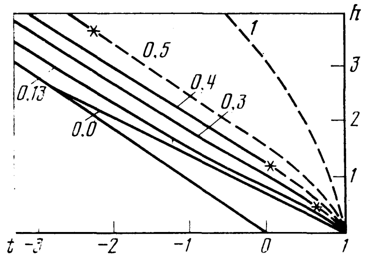

FIG. 3. Appearance of the TPS phase diagram for various values of $\varkappa$ in a parallel field near the temperature $t = 0$. The dashed curves show the line of second-order transitions, and the stars denote the tricritical points, at which there is a change in the character of the transition.

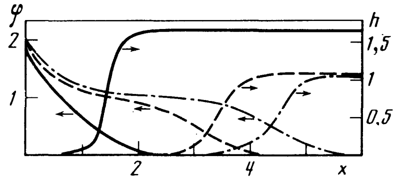

FIG. 4. Spatial profiles of the order parameter and of the screening of a field parallel to the twinning plane ($x = 0$) in tin at a temperature $t = -1$. The solid curve corresponds to an external field $h = 1.62$, the dashed curve to $h = 1.05$, and the dot-and-dash curve to $h = 1.011$.

Impurities and irradiation can also cause defects of the twinning plane, which lower the critical temperature for TPS. Apparently the dependence of the critical temperature on the impurity concentration or radiation dose can be nonmonotonic.

The behavior of the order parameter and screening of the field as functions of the coordinate in tin is illustrated in Fig. 4. Interestingly, in the region of negative t, where bulk superconductivity can exist in smaller fields, a plateau appears on the $\varphi(x)$ curve as the field approaches $h_{c0} = |t|$; on this plateau the order parameter is practically constant and equal to its value in the bulk metal at the given temperature.

Figure 5 shows the field dependence of the magnetic moment calculated at various temperatures (for the case $\varkappa = 0.13$ in the bulk metal) under the assumption that a continuous layer of superconductor arises. At the transition of the layer in a field from the superconducting to the normal state there is a jump in the magnetic moment, and heat should be absorbed. The quantity of heat q is related to the jump in the magnetic moment and the derivative dH_c/dT by the usual relation

$$q = -TM(H_c)\,(dH_c/dT)$$

and can easily be found from the $M(H)$ and $H_c(T)$ curves obtained in this study (see Figs. 2 and 5). No calorimetric studies of the TPS have been done, however.

A type-I superconductor in a perpendicular magnetic field should exhibit an intermediate state. The problem of finding the structure of the intermediate state of the TPS in a perpendicular field differs from the standard formulation for a thin superconductor film and is yet to be completely

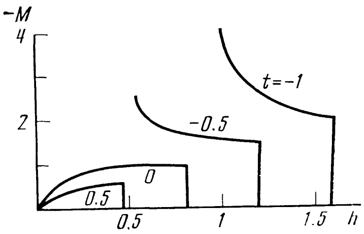

FIG. 5. Field dependence of the magnetic moment (per unit area) of the TPS in tin at various temperatures. The moment is measured in units of $M_0 = [\tau_0/(\eta m\pi b)]^{1/2} = H_0\xi(\tau_0)/\pi$.

solved. The perpendicular critical field for the first-order transition is of the same order of magnitude as the critical field parallel to the twinning plane. The former, however, is apparently somewhat smaller than the latter, since the structure of the intermediate state which arises should have a nonzero demagnetizing factor. Herein lies the distinction between TPS and the ordinary situation for a film of a type-I superconductor, where the intermediate state arises at the same field H_{c0} at which the transition occurs in a parallel field (because the superconducting layer arising in the sample has a demagnetizing factor of zero, since the width of this layer in the direction perpendicular to the field is much less than the thickness of the film). In the case of TPS the thickness of the superconducting region localized near the twinning plane is equal in order of magnitude to $\xi(\tau_0)$. This means that the characteristic transverse dimension of the intermediate state will also be of the same order of magnitude. The superconducting nuclei in this case most likely are in the shape of strips having a thickness of the order of $\xi(\tau_0)$.

4. TWINNING-PLANE SUPERCONDUCTIVITY IN A TYPE-II SUPERCONDUCTOR (NIOBIUM)

The results of experiments[3] in which TPS was observed in niobium permit comparison of the features of the phenomenon in type-II (niobium) and type-I (tin) superconductors. In niobium, where the Ginzburg-Landau parameter $\varkappa$ is equal to 1,[17] the transition to the TPS state is always second order and the TPS can exist all the way to the lowest temperatures in fields somewhat in excess of the critical field H_{c2}^0 of the bulk metal. The field of the second-order transition to the TPS state was calculated in Ref. 8 for temperatures in the neighborhood of T_{c0} and in Ref. 11 for low temperatures.

The simplest situation is one in which the magnetic field is perpendicular to the twinning plane. In this case the $H_{c2}^*(T)$ curve in the neighborhood of T_{c0} differs from the usual case only in that the field $H_{c2}^*(T)$ goes to zero at the point T_c rather than at T_{c0}. Thus $H_{c2}^{*\perp}(T)$ is a straight line passing through the point T_c and parallel to $H_{c2}^0(T)$. As to the behavior of the TPS in a perpendicular field, we note that it is completely analogous to the behavior of a thin superconducting slab of type-II superconductor, for which the effective London penetration depth is given by the expression

$$\lambda_{\rm eff}^{-2} = \frac{16\pi e^2}{mc}\int \psi^2(x)\,dx.$$

If we allow for this change, all the results of Ref. 18 will apply to TPS in a perpendicular field.

We now consider the case when the field is parallel to the twinning plane; we obtain an expression for $H_{c2}^{*\parallel}$ in a convenient form for numerical calculation and find the temperature dependence of the critical field in the neighborhood of T_{c0}, i.e., in the region where the Ginzburg-Landau theory applies.

The field of the second-order transition, as usual, is determined from the condition that a solution exist for the linearized equation for the order parameter:

$$-\varphi''(x)+\tilde h^2x^2\varphi(x)+t\varphi(x)-2\delta(x)\varphi(x)=0, \qquad (6)$$

where

$$\tilde h = H_{c2}^{*\parallel}/H_{c2}^0(t=-1) = t/\sqrt{2}\varkappa.$$

The eigenfunctions $\varphi_n(x)$ of the above equation without the δ function (i.e., a linear oscillator) with the boundary conditions $\varphi\to 0$ at $x\to\pm\infty$ are well known[19]: they form a complete orthonormal basis. Let us expand solution (6) in these eigenfunctions:

$$\varphi(x) = \sum_n c_n\varphi_n(x). \qquad (7)$$

Substituting (7) into (6) and using the orthonormality of the functions φ_n, we find the expansion coefficients

$$c_n = 2\varphi(0)\varphi_n(0)/(t+\varepsilon_n), \qquad (8)$$

where $\varepsilon_n = 2\tilde h(n+1/2)$ is the eigenenergy corresponding to eigenfunction φ_n. Using the "self-consistency" condition

$$\varphi(0) = \sum_n c_n\varphi_n(0),$$

and the explicit form of $\varphi_n(x)$, we find the solution of our problem in the form of a convergent series:

$$\tilde h^{1/2} = \frac{2}{\pi^{1/2}}\sum_{k=0}^{\infty}\frac{(2k)!}{(k!)^2 2^{2k}[(t/\tilde h)+4k+1]}. \qquad (9)$$

Numerical summation of (9) leads to the $\tilde h(t)$ curve shown in Fig. 6.

In the limit $t\to 1$ the $\hat h(t)$ curve has a square-root character, $h(t)\propto(1-t)^{1/2}$, and can easily be found[8] by treating the magnetic field in (6) as a perturbation and using the explicit form (4) of the wave function $\varphi(x)$ at $H=0$.

In the limit $|t|\gg 1$ the critical field differs little from the corresponding bulk value $\tilde h_0 = -t$; in this case we only need to keep the first ($k=0$) term in the sum (9), since it is much larger than the rest. As a result we find

$$(\tilde h-\tilde h_0)/\tilde h_0 = 2(\pi|t|)^{-1/2} \approx 1{,}1|t|^{-1/2}, \qquad (10)$$

i.e., $\tilde h-\tilde h_0 = (2/\sqrt{\pi})|t|^{1/2}$. This case corresponds to taking the δ-function potential in (6) into account by perturbation theory.

To treat the behavior of the TPS at $|t|\gg 1$ in fields that are somewhat smaller than $\tilde h(t)$, one can use the usual method used to study the mixed state for $(H_{c2}-H)/H_{c2}\ll 1$ (see, e.g., Ref. 20). The mixed state here is characterized by the parameter $\beta_A = \overline{\varphi^4}/(\overline{\varphi^2})^2$. In our case this parameter is easily found by substituting for φ the lowest-energy eigen-

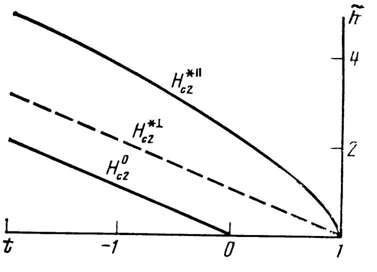

FIG. 6. Upper critical field for the transition to the TPS state.

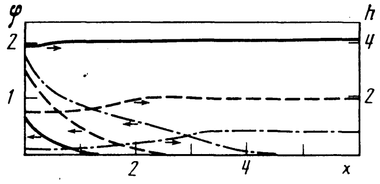

FIG. 7. Spatial profiles of the order parameter and of the screening of a field parallel to the twinning plane in niobium ($\varkappa = 1$) at a temperature $t = -0.5$. The solid curve corresponds to a field $h = \tilde{h}/\sqrt{2} = 4.2$, the dashed curve to $h = 1.9$, and the dot-and-dash curve to $h = 0.71$.

function of the linear oscillator [Eq. (6) without the δ function]. As a result we get

$$\beta_A = (|t|/2\pi)^{1/2}/\xi(\tau_0)$$

and the magnetic moment of the TPS per unit area of the twinning plane is

$$M = -\frac{H_{c2}^{*\parallel} - H}{4\pi\beta_A(2\varkappa^2-1)} = -\frac{(H_{c2}^{*\parallel}-H)\,\xi(\tau_0)}{2(2\pi|t|)^{1/2}(2\varkappa^2-1)}. \qquad (11)$$

This expression is valid for $H_{c2}^{*\parallel} > H > H_{c2}^{0}$ and $t \gg 1$. It is important to note that as the field approaches H_{c2}^{0}, analysis of the equations describing the mixed state in our case shows that modulation of the solution along the twinning plane does not arise at fields below $H = H_{c2}^{0}$, and the assumption[12] that a chain of vortex filaments parallel to the field and twinning plane arises near H_{c2}^{0} is thus not justified. Interestingly, Eqs. (10) and (11) imply that the magnetic moment of the TPS in a field $H = H_{c2}^{0}$ (i.e., when the bulk superconductivity arises) is independent of the temperature.

For temperatures $t \approx 0$, the behavior of the TPS in a field can be analyzed only on the basis of a numerical solution of the equations for the order parameter and magnetic field.

Figure 7 illustrates the results of the numerical calculations for the case $\varkappa = 1$, corresponding to niobium, and for a temperature $t = -0.5$. The field dependence of the magnetic moment for $\varkappa = 1$ (niobium) at various temperatures is shown in Fig. 8. Interestingly, for $t > 0$ this field dependence is bell-shaped. The initial growth in the magnetization is due to the growth of the magnetic field; then, at stronger fields, the superconductivity begins to be suppressed, which causes the moment to fall off as the field approaches H_{c2}^{*}.

On the whole, the theoretical scheme gives a qualitatively correct description of the observed behavior[3] of the TPS in niobium. However, the quantitative agreement here is somewhat poorer than in the case of tin. For example, according to the experimental data[3] the value of dH_{c2}^{*}/dT at $T = T_{c0}$ exceeds dH_{c2}^{0}/dT by a factor of about 1.2, while the theoretical value of this ratio is about 1.5 (see Fig. 6). This discrepancy may be due to the fact that detection of the magnetic moment of the TPS at a second-order phase transition becomes a very complicated experimental problem, and the measurements of Ref. 3 give only a lower bound on the field H_{c2}^{*}. Another important factor is whether the field was strictly parallel to the twinning plane in the experiment, since the increase in the perpendicular critical field for TPS with respect to the bulk value at $|t| \gg 1$ is substantially smaller than for the case of a parallel field.

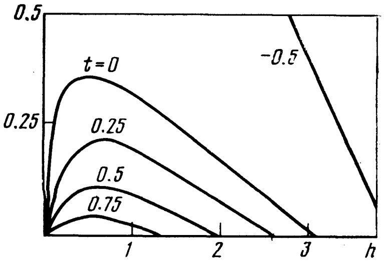

FIG. 8. Magnetic moment (per unit area) of the TPS versus the field $h = \tilde{h}/\sqrt{2}$ in niobium ($\varkappa = 1$) at various temperatures. The moment is measured in the same units as in Fig. 5.

5. DIAMAGNETIC SUSCEPTIBILITY OF THE TWINNING-PLANE SUPERCONDUCTIVITY

In the region $t > 0$, where the TPS exists all the way down to the smallest fields, the problem of finding the diamagnetic susceptibility of the TPS [the initial slope of the $M(H)$ curve] can in certain cases be solved analytically. Let us consider the screening of a weak magnetic field parallel to the twinning plane. In the equation for the field distribution here we should take for $\varphi(x)$ the expression (4) for the order parameter in the absence of field. The corresponding equation for the vector potential is of the form

$$\tilde{A}'' = 2t\tilde{A}/\varkappa^2 \sinh^2(xt^{1/2}+p). \qquad (12)$$

Making the change of variables $\xi = \coth(xt^{1/2}+p)$ and going over from Eq. (12) to an equation for the field B, we find that the distribution of the magnetic induction is determined by the equation

$$(\xi^2-1)\frac{d^2B}{d\xi^2} = \frac{2}{\varkappa^2}B. \qquad (13)$$

Transforming to a new function $y(\xi)$ by means of the relation $B = (\xi^2-1)^{1/2}y(\xi)$, we obtain for this new function the Legendre equation

$$(\xi^2-1)y'' + 2\xi y' - \left(\frac{2}{\varkappa^2} + \frac{1}{\xi^2-1}\right)y = 0. \qquad (14)$$

The magnetic-field distribution is thus described by the Legendre functions $Q_{v_{1,2}}^{(-1)}$ with parameter $v_{1,2} = \frac{1}{2}[-1 \pm (1+8/\varkappa^2)^{1/2}]$ or, in the corresponding representation in terms of the hypergeometric functions,

$$B = A_1\xi^{-\nu_1}F\left(\frac{1}{2}+\frac{\nu_1}{2}, \frac{\nu_1}{2}, \nu_1+\frac{3}{2}, \frac{1}{\xi^2}\right) + A_2\xi^{-\nu_2}F\left(\frac{1}{2}+\frac{\nu_2}{2}, \frac{\nu_2}{2}, \nu_2+\frac{3}{2}, \frac{1}{\xi^2}\right). \qquad (15)$$

In the general case it is a difficult problem to find the explicit expression for the constants A_1 and A_2 from the boundary conditions, and we shall therefore analyze a number of limiting cases.

For an extreme type-II superconductor ($\varkappa \gg 1$) the screening of the field turns out to be weak over practically the entire temperature interval $0 < t < 1$, i.e., $B(x) = H - b(x)$, where $b(x) \ll H$. Equation (13) can then be written to first order in the parameter $\varkappa^{-2}$ as

$$d^2b/d\xi^2 = 2H/\varkappa^2(\xi^2-1), \quad (16)$$

an equation which admits an exact solution. Integrating (16) with the boundary conditions $b \to 0$ at $\xi \to 0$ and $b'(r) = 0$, we find the magnetic-field distribution

$$B = H\left\{1 - \frac{1}{\varkappa^2}\left[\xi \ln\left(\frac{\xi+1}{\xi-1}\frac{r-1}{r+1}\right) + \ln\frac{(\xi^2-1)(r+1)}{4(r-1)}\right]\right\},$$
$$r = \coth\left(\frac{1}{2}\ln\frac{1+t^{1/2}}{1-t^{1/2}}\right). \quad (17)$$

The magnetic field right at the twinning plane,

$$B(r) = H\left[1 - \frac{2}{\varkappa^2}\ln\frac{r+1}{2}\right],$$

differs little from the external field H. The total magnetic moment of the TPS per unit area of the twinning plane is

$$M = \xi(\tau_0)\int_{-\infty}^{\infty}\frac{B(x)-H}{4\pi}dx = -\frac{\xi(\tau_0)}{2\pi\varkappa^2 t^{1/2}}\left\{\ln\left(\frac{r+1}{2}\right)\ln(r^2-1)\right.$$
$$-\ln(r+1)\ln\left(4\frac{r-1}{r+1}\right) + \ln^2 2 + 2\left[\left(1-\frac{2}{r+1}\right)\right.$$
$$\left.\left.+\frac{1}{2^2}\left(1-\left(\frac{2}{r+1}\right)^2\right) + \frac{1}{3^2}\left(1-\left(\frac{2}{r+1}\right)^3\right) + \ldots\right]\right\}. \quad (18)$$

For $t \to 0$, i.e., $T \to T_{c0}$, the specific (per unit area) magnetic moment behaves as

$$M \approx -\frac{H\xi(\tau_0)}{\pi\varkappa^2 t^{1/2}}\ln^2\frac{1}{t^{1/2}}, \quad (19)$$

which implies that the diamagnetic susceptibility diverges approximately as $t^{-1/2}$. In a type-II superconductor the applicability condition for relations (16)–(19) is $t \gg \exp(-\varkappa^2)$, i.e., they are valid practically over the entire temperature interval $0 < t < 1$.

Near the critical temperature for TPS we have $t \to 1$, $r \to 1 + (1-t)/2$, and our previous discussion also applies to a type-I superconductor under the condition $1 - t \ll \varkappa^2$. We note that this case has been considered previously.[8] If the opposite inequality holds, the magnetic field around the twinning plane is found to be strongly screened, and for $\varkappa \ll 1$ the field is practically zero in a layer of thickness $x \gg 1$ around the twinning plane. Effective screening thus occurs at distances $x \gg 1$, i.e., the region of interest in Eq. (13) is $\xi - 1 \ll 1$, where this equation can be written in the form

$$(\xi-1)\,d^2B/d\xi^2 = \varkappa^{-2}B. \quad (20)$$

The solution of (20) which falls off as it approaches the twinning plane is

$$B = (\xi-1)^{1/2}\frac{2H}{\varkappa}K_1\left(\frac{2}{\varkappa}(\xi-1)^{1/2}\right), \quad (21)$$

where K_1 is the modified Bessel function [in the case $\varkappa \ll 1$ the second solution of (20), the function I_1, must be discarded].

The magnetic field at the twinning plane in this case is exponentially small, and the magnetic moment per unit area is given by

$$M = \xi(\tau_0)\int_{-\infty}^{\infty}\frac{B(x)-H}{4\pi}dx = -\frac{H}{2\pi}\xi(\tau_0)\frac{\ln(2/\varkappa)}{t^{1/2}}. \quad (22)$$

The resulting temperature dependence of the diamagnetic susceptibility agrees qualitatively with the available experimental data,[1–3] but the inhomogeneous character of the twinning plane (see below) evidently causes substantial deviations from the corresponding theoretical expressions.

6. UPPER CRITICAL MAGNETIC FIELD FOR TWINNING-PLANE SUPERCONDUCTIVITY AT LOW TEMPERATURES

In this section we consider the critical field at low temperatures, where the Ginzburg-Landau approximation does not apply. A calculation of the low-temperature critical fields in the dirty limit is given in Ref. 11. Here we shall give a somewhat different method of finding the critical field that lends itself to use in both the dirty and clean limits.

Let us consider the most interesting case, where the magnetic field is parallel to the twinning plane. The general form of the integral equation is

$$\psi(\mathbf{r}) = \int K_{H,T}(\mathbf{r};\mathbf{r}')\lambda(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'; \quad (23)$$

we have chosen a gauge such that the kernel depends explicitly only on the coordinate x (see, e.g., Ref. 21). We note that the integral equation is written for the quantity $\psi(\mathbf{r}) = \Delta(\mathbf{r})/\lambda(\mathbf{r})$, which is proportional to the anomalous Gor'kov function at coincident points and is a slowly varying function of $\mathbf{r}$ (on an atomic scale). This circumstance evidently causes our results to differ from those of Nabutovskii and Shapiro,[10–12] who considered the equation for the discontinuous function $\Delta(\mathbf{r})$. We note in this regard that it is difficult to interpret the result of Refs. 10–12 that the region of TPS existence temperatures grows as the constant $g(\mathbf{r}) \propto \lambda(\mathbf{r})$ decreases near a certain plane.

Assuming that the increase in λ is localized near the twinning plane and introducing the quantity $\Lambda = \int \delta\lambda(x)dx$, we write Eq. (23) in the form

$$\psi(x) = \lambda_0\int K(x;x',\rho')\psi(x')\,dx'\,d\rho' + \psi(0)\Lambda\int K(x;0,\rho')\,d\rho'; \quad (24)$$

here ρ is the coordinate in the $x = 0$ plane. Since the critical magnetic field for TPS differs little from the corresponding value H^0_{c2} for the bulk metal, i.e., $\Delta H^{\parallel}_{c2} = H^{*\parallel}_{c2} - H^0_{c2} \ll H^0_{c2}$, the function $\psi(x)$ is close to the known eigenfunction $\psi_0(x)$ (corresponding to the lowest eigenvalue) of the kernel for a homogeneous superconductor.[21] Multiplying Eq. (24) by $\psi_0(x)$, integrating, and recalling[21] that in the clean limit the kernel K depends only on the ratio $H^{1/2}/T$, we find for a clean superconductor

$$\frac{\Delta H_{c2}^{\parallel}}{H_{c2}^0} = 2\left(1 - \frac{T}{2H}\frac{\partial H_{c2}^0}{\partial T}\right)\frac{\Lambda}{\lambda_0^2}\left(\frac{2eH_{c2}^0}{\pi}\right)^{1/2}. \quad (25)$$

Since the quantity $\tau_0 = 0.45\Lambda^2/\lambda_0^4\xi_0^2$ (Ref. 8) involves the same parameters, we can find a relation between τ_0 and the increase in the critical field for TPS at $T = 0$:

$$\Delta H_{c2}^{\parallel}/H_{c2}^0(T=0) \approx 1.7\tau_0^{1/2}. \quad (26)$$

For niobium the data of Ref. 3 give $\tau_0 = 10^{-2}$, and so the theoretical value of the increase is $\Delta H^{\parallel}_{c2}/H^0_{c2}(T=0) \approx 0.17$, while the results of Ref. 3 imply that $\Delta H^{\parallel}_{c2}/H^0_{c2} = 0.11$ at $T = 3$ K. In view of the fact that it is actually a lower bound (in terms of the field) on the TPS existence region that was determined in Ref. 3, the agreement with experiment can be regarded as satisfactory.

For dirty superconductors, in view of the fact[21] that the kernel K depends on the ratio H/T, we find

$$\frac{\Delta H_{c2}^{\parallel}}{H_{c2}^{0}}=\left(1-\frac{T}{H}\frac{\partial H_{c2}^{0}}{\partial T}\right)\frac{\Lambda}{\lambda_0^2}\left(\frac{2eH_{c2}^{0}}{\pi}\right), \qquad (27)$$

and at $T=0$

$$\Delta H_{c2}^{\parallel}/H_{c1}^{0}(T=0)\approx 0.9\tau_0^{1/2}.$$

It is important to note the pronounced anisotropy of the quantity $\Delta H_{c2}=H_{c2}^{*}-H_{c2}^{0}$. When the field is perpendicular to the twinning plane, we have in order of magnitude

$$\Delta H_{c2}^{\perp}/H_{c2}^{0}\sim(\Delta H_{c2}^{\parallel}/H_{c2}^{0})^2,$$

i.e., $\Delta H_{c2}^{\perp}$ is much smaller than $\Delta H_{c2}^{\parallel}$, and the anisotropy is rather sharp. The increase ΔH should fall from $\Delta H_{c2}^{\parallel}$ to a value of the order of $\Delta H_{c2}^{\perp}$ (i.e., to a value on the order of that found in niobium) as the angular deviation of the field from an orientation parallel to the twinning plane increases to $\Delta\theta\sim\tau_0^{1/2}\sim 0.1$ rad. The angular dependence of the critical field for TPS was examined in detail in Refs. 11 and 12.

7. CONCLUSION

In conclusion, there is good qualitative agreement between the theoretical description of TPS and the experimental data. For tin, where the transition in a field to the TPS state is first-order, numerical calculations (with the value of the Ginzburg-Landau parameter for tin) give good quantitative agreement between the experimental and theoretical phase diagrams. The phase diagrams (in normalized units) obtained for various values of $\varkappa$ are essentially universal. An important feature of TPS is that the field of the first-order transition depends on the parameter $\varkappa$, a radical departure from the corresponding behavior of bulk type-I superconductors.

At the same time, however, there are discrepancies with the experimental data as to the character of the temperature dependence of the magnetization and the value of the magnetic moment. These discrepancies are apparently due to the inhomogeneity of the real twinning plane. If the regions of the twinning plane with elevated values of λ have a finite size, then each such region will have its own value (smaller than τ_0) of the temperature at which local superconductivity sets in.

The presence of small-scale inhomogeneities in the structure of the twinning plane can also bring about a situation in which the different regions of the twinning plane are characterized by a TPS transition temperature $T_c(\rho)$ that depends on the coordinate $\boldsymbol{\rho}$ in the twinning plane. Here the characteristic dimension of the regions is large compared to $\xi(\tau_0)$, and their transition temperature is determined by the concentration of the inhomogeneities. In fact, here we also arrive at the concept of a twinning plane having regions with different temperatures.

The inhomogeneous character of the TPS does not interfere with the comparison of the theoretical and experimental curves for $h_c(t)$, however, since the experiment[1,2] detects the maximum field, which corresponds to the most perfect regions of the twinning plane. It can be assumed that the measured field is the same as the field for an ideal twinning plane, and so a direct comparison with the theory can be made. In contrast, the diamagnetic moment of the TPS is determined to a large extent by the nature of the inhomogeneity of the twinning plane, and although the theoretical field dependence of the moment agrees qualitatively with experiment, a quantitative comparison of the theory and experiment would require information on the distribution of the regions with different superconducting properties on the twinning plane.

In the TPS of niobium[3] there is some temperature T_k ($T_{c0}<T_k<T_c$) below which the flow of microscopic currents due to flux trapping is observed. In the experiments on TPS in tin,[1-3] however, no such currents have been detected at an isolated twinning plane. This circumstance may be due to the onset of an intermediate state or to the inhomogeneous character of the twinning plane (different in tin and niobium). An alternative possibility is a transition of the Kosterlitz-Thouless type[22,23] for the TPS, as was conjectured in Ref. 3. Such a transition can occur because of the specific two-dimensional character of the superconducting vortices. In fact, though, the superconductivity extends from the twinning plane a distance $\xi(\tau_0)$, which is substantially greater than the interatomic distance, and the condition for the formation of self-induced superconducting vortices is satisfied only in a narrow region near the temperature T_c. Nevertheless, there is an alternative possibility for a Kosterlitz-Thouless transition, which can occur in a system of superconducting granules at the two-dimensional twinning plane. In fact, the weak Josephson interaction of the individual superconducting regions (in which the phase of the superconducting order parameter is constant) causes an inhomogeneous superconducting twinning plane to resemble a classical system of plane rotators, for which a Kosterlitz-Thouless transition has been predicted.[22,23] Here the temperature T_K of the Kosterlitz-Thouless transition agrees in order of magnitude with the energy J of the Josephson interaction of adjacent superconducting regions. In the case of two such regions of dimension R_0 separated by a distance $R\gg R_0$, $\xi(T)$, we have the following energy estimate:

$$J\sim\tau_0^2(T_c^2/\varepsilon_F)N(R_0^3\xi(\tau_0)/R)\exp(-R/\xi(T)).$$

The sharp exponential dependence of J on the distance means that in practice the transition temperature T_K is determined by the condition $\xi(T)\sim R$. Using the value of T_K found for niobium in Ref. 3, we obtain the estimate $R\sim 10^4$ Å. We emphasize that in this interpretation it is the inhomogeneity of the twinning plane that makes the topological transition possible.

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[1)]An estimate for τ_0 is easily obtained from the following considerations:

$\tau_0 = (T_c - T_{c0})/T_{c0} \sim (\delta\lambda/\lambda_0^2)[d/\xi(T_c)]$, where $\delta\lambda$ is the increase in the Cooper pairing constant (in comparison with its value λ_0 in the bulk metal) in a layer with a thickness of the order of d near the twinning plane. Recalling that the correlation length at the point T_c is given by $\xi(T_c) \sim \xi_0\tau_0^{-1/2}$, we find $\tau_0 \sim (\delta\lambda d/\lambda_0^2\xi_0)^2$, in agreement with the results of a rigorous treatment.[8,10]

[2)]This value of $\varkappa$ for tin was obtained in Ref. 16 from measurements of the fluctuational diamagnetism using the same samples as in Refs. 1, 2, and 5 and a similar experimental technique.

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