Relativistic space charge waves in intense neutralized magnetized electron beams

N. V. Kuzelev, A. A. Rukhadze, and G. V. Sanadze

Moscow State University
(Submitted 20 March 1985)
Zh. Exp. Teor. Fiz. **89**, 1591–1602 (November 1985)

It is shown that the theory of space charge waves in intense electron beams can only be relativistic. The dispersion characteristics of linear and nonlinear space-charge waves in these beams are analyzed and, in particular, a strong dependence of the dispersion relation of the slow wave on its amplitude is found. The excitation of the slow wave in an intense beam via the process of a simple decay type of Cherenkov instability is considered. The existence of two saturation mechanisms for the instability is demonstrated. For currents below the vacuum limiting current, the saturation mechanism is trapping and strong turbulence of the beam. At high currents, saturation is due to a nonlinear frequency shift without randomization of the beam. The dependence of the beam radiation efficiency on the current is investigated. The efficiency is found analytically in the ultrarelativistic limit.

1. INTRODUCTION

Space charge waves play an important role in plasma electrodynamics and dense electron beams. These waves are very important in a number of applications, associated with microwave generation, beam transport and methods of collective particle acceleration. At present, the theory of space charge waves for nonrelativistic, i.e., low-current beams, is well developed (see, for example, Ref. 1). For intense beams, however, the situation changes, mainly because intense beams can only be relativistic. It is conventional in the literature, to define an intense beam as one with current greater than the vacuum limiting current.[2] If the beam is magnetized, then the quantitative high current criterion is described by the inequality

$$\omega_b^2\gamma^{-1}/k_{\perp 1}^2u^2 \geqslant 1, \tag{1}$$

where ω_b is the electron plasma frequency of the beam, u is the velocity of the electrons, $\gamma = (1 - u^2/c^2)^{-1/2}$ and $k_{\perp 1}$ is the minimum transverse wave vector (typically of the order of the inverse of the transverse dimension of the drift tube). When (1) is satisfied, the beam can propagate through the drift tube only if its space charge is neutralized.

Nevertheless, there exists a limitation on the current even in the case of a neutralized beam. This is the so called Pierce current.[2] It is known that when the inequality

$$\omega_b^2\gamma^{-3}/k_{\perp 1}^2u^2 \geqslant 1 \tag{2}$$

is satisfied, an aperiodic instability develops in the beam, shutting down the current. Therefore, the domain of existence of an intense neutralized beam is determined by the inequalities

$$\gamma^2 > \omega_b^2\gamma^{-1}/k_{\perp 1}^2u^2 > 1, \tag{3}$$

i.e., this domain is fairly broad only when $\gamma^2 \gg 1$.

When a beam is injected into a drift tube filled with a plasma, there exists an additional, more stringent limitation on the beam current, which is also related to relativity. Neutralization in this beam takes place only if $\omega_p^2 > \omega_b^2$, where ω_p is the electron plasma frequency of the background plasma. Nevertheless, in this case, the Cherenkov beam-plasma instability can develop in the system which, as is well known, does not occur when an upper limit is imposed on the plasma density[3]:

$$\omega_p^2 < k_{\perp 1}^2u^2\gamma^2. \tag{4}$$

This condition, combined with (1), yields the following stability domain for an intense neutralized beam:

$$\gamma > \omega_b^2\gamma^{-1}/k_{\perp 1}^2u^2 > 1. \tag{5}$$

Therefore, in what follows, we will assume $\gamma \gg 1$. The electromagnetic properties of intense, highly relativistic beams are investigated in the present work. The dispersion characteristics of linear and nonlinear space-charge waves in relativistic beams and the interaction of these waves with slow electromagnetic waves in the system are considered. It will be shown that the dispersion characteristics of the space-charge waves in relativistic beams, depend strongly on the amplitude of the waves, in contrast to the nonrelativistic case. Moreover, this dependence is important for dense beams with currents above the vacuum limit. This circumstance is a major factor determining the nonlinear dynamics of the Cherenkov beam instability in the system. For low-current beams, with currents below the vacuum limiting current, electrons are trapped in the field of the wave excited by the beam itself, and strong turbulence develops. For intense beams, in contrast, the instability develops without leading to randomization of the beam and the instability saturates due to a nonlinear shift in the frequency of the slow space-charge wave, as well as detuning of the Cherenkov resonance.

The analysis of the nonlinear dynamics of the beam instability allows us, as shown below, to investigate the efficiency with which the energy of the beam relativistic electrons is transformed electromagnetic radiation as a function of the beam current. In the ultrarelativistic limit, this dependence will be found analytically. It will be shown that as the current increases the electromagnetic component of the mo-

0038-5646/85/110921-07$04.00

mentum of the beam mode grows rapidly, leading to a decrease in the radiation efficiency of the intense beam. The decrease in the efficiency with the current in the long-wavelength limit is much faster than in the short-wavelength limit.

2. DISPERSION OF RELATIVISTIC SPACE CHARGE WAVES IN THE LONG-WAVELENGTH LIMIT.

Consider first the simplest case of oblique waves in an unbounded beam, propagating along an infinitely strong external magnetic field. The corresponding linear dispersion relation is[3]

$$k_{\perp 1}^2+(k_\parallel^2-\omega^2/c^2)[1-\omega_b^2\gamma^{-3}/(\omega-k_\parallel u)^2]=0, \tag{6}$$

where ω is the frequency of the wave and $k_\parallel$ is the longitudinal component of the wave vector along the external magnetic field [Eq. (6) is also valid for a waveguide filled uniformly by the beam]. We seek a solution of Eq. (6) in the form

$$\omega=k_\parallel u(1+\delta x), \tag{7}$$

where $|\delta x|\ll 1$. Then in the long-wavelength limit ($k_\parallel\to 0$), Eq. (6) yields a one-parameter equation

$$\delta^2+\mu\delta-1=0, \tag{8}$$

where

$$\mu=2(\gamma^2-1)a\approx(I_b/I_0)^{1/2},\quad \delta=\delta x/a, \tag{9}$$

$$a=(\omega_b^2\gamma^{-5}/k_{\perp 1}^2u^2)^{1/2}.$$

Here I_b is the beam current and I_0 is the limiting vacuum current (see Eq. (1)).

The roots of Eq. (8) are

$$\delta_{1,2}=-\mu/2\pm(\mu^2/4+1)^{1/2}, \tag{10}$$

where the plus sign is associated with the fast positive-energy wave, while the minus sign describes the slow negative-energy wave. In the low-current case $\mu\ll 1$ and the roots have the usual form $\delta_{1,2}=\pm 1$. For $\mu\gg 1$, in contrast, we have

$$\delta_1=1/\mu,\ \delta_2\approx-\mu. \tag{11}$$

Note that the root $\delta_1=1/\mu$, as could be expected, corresponds to the dispersion relation $\omega=k_\parallel c$.

By using Eqs. (9) and (10), we can convert the inequality $\delta x\ll 1$ into an explicit form, which determines the maximum value of the function δ:

$$|\delta|_{max}a\approx I_b/I_p\ll 1, \tag{12}$$

where I_p is the Pierce limiting current (see Ref. 2). Furthermore, as can be seen, the transition to the limit $k_\parallel\to 0$ means that the displacement current is small compared to the high-frequency beam current, i.e., $k_\parallel^2\ll k_{\perp 1}^2\gamma^2$. Note, also, that the physical meaning of the term $\mu\delta$ in Eq. (8) is related to the electromagnetic nature of the space charge waves, characteristic of relativistic beams only.

Let us proceed now to nonlinear space charge waves in the long-wavelength limit. Equations describing the nonlinear waves are conveniently written by using the Lagrangian coordinates $z(t,z_0)$ and $v(t,z_0)$ of the beam electrons where z_0 and v are the position and velocity of the electron which was at the point z_0 at time $t\to-\infty$, namely, i.e., the amplitude of the wave was zero. By introducing dimensionless time and coordinates

$$\tau=k_\parallel ua,\quad y=k_\parallel(z-ut),$$
$$\eta=\frac{v-u}{u}\frac{1}{a},\quad y_0=k_\parallel z_0 \tag{13}$$

and expressing the longitudinal component of the electric field of the wave via the space charge perturbation in the beam, we can write the following system of relativistic equations of motion

$$\frac{dy}{d\tau}=\eta,\quad \rho=\frac{1}{\pi}\int_0^{2\pi}e^{-iy}dy_0,$$
$$\frac{d\eta}{d\tau}=-{}^1/_2 i(1-\mu\eta)^{3/2}[e^{iy}(1-i\mu d/d\tau)\rho-\text{c.c.}], \tag{14}$$

where ρ is the amplitude of the space charge perturbation in the beam divided by the density of the unperturbed beam.

The relativistic properties of system (14) are determined by the parameter μ and manifest themselves in two ways. Firstly, there exists a term containing $\mu dg/d\tau$, and related to the electromagnetic nature of the waves. Secondly, the multiplier $(1-\mu\eta)^{3/2}$ is associated with the dependence of the electron mass on the velocity perturbations, which is absent in the linear theory.

The first integral of Eq. (14) is

$$P=\frac{1}{2\pi}\int_0^{2\pi}(1-\mu\eta)^{-1/2}dy_0+\frac{1}{8}\mu^2|\rho|^2=\text{const}, \tag{15}$$

reflecting momentum conservation in the space charge wave. The first term in (15) is the mechanical momentum and the second term describes the momentum of the electromagnetic field, which is substantial only in the high-current case. It should be emphasized that the electromagnetic momentum is not the free-radiation momentum.

For stationary space-charge waves $|\rho|=\text{const}$. Nevertheless, even in this case, Eq. (14) can be solved analytically only in the limit $\mu\gg 1$, i.e., when the beam current is much higher than the vacuum limit. Only this case will be considered here. Let us introduce an electron "momentum" $p=(1-\mu\eta)^{-1/2}$ and rewrite the first two equations of the system (14) in the form

$$\frac{dp}{d\tau}=-\frac{i\mu}{4}\left[e^{iy}\left(1-i\mu\frac{d}{d\tau}\right)\rho-\text{c.c}\right],\quad \frac{dy}{d\tau}=\frac{1}{\mu}\frac{p^2-1}{p^2}. \tag{16}$$

Next, we write the coordinate y of the electron as a sum $y_0+\tilde{y}$, where $\tilde{y}$ is a perturbation which is small ($\tilde{y}\ll 1$) when $\mu\gg 1$, as follows from the second equation in (16). This allows us to linearize with respect to $\tilde{y}$. Furthermore, according to the first equation in (16), the electron momentum can be written

$$p=\langle p\rangle+{}^1/_2[\alpha(\tau)e^{iy_0}+\text{c.c}], \tag{17}$$

where $\alpha(\tau)$ depends on time only. With no loss of generality, we can assume $\langle p \rangle = 1$, which is achieved by an appropriate choise of the constant in Eq. (15).

By substituting (17) into Eq. (16) and linearizing with respect to $\tilde{y}$, we obtain the following nonlinear equations for the amplitudes of the space-charge waves in the ultrarelativistic limit:

$$\frac{d\alpha}{d\tau} = -\frac{i\mu}{2}\rho - \frac{\mu^2}{2}\frac{d\rho}{d\tau}, \quad \frac{d\rho}{d\tau} = -\frac{2i}{\mu}\alpha(1-|\alpha|^2)^{-3/2}. \qquad (18)$$

For a stationary wave

$$|\alpha| = A_0 = \text{const}, \quad \alpha = A_0 e^{-i\delta\tau} \qquad (19)$$

and Eq. (18) yields the dispersion relation:

$$\delta^2 + (\mu\delta - 1)(1 - A_0^2)^{-3/2} = 0, \qquad (20)$$

the solution of which for $\mu \gg 1$ is

$$\delta_1 = 1/\mu, \quad \delta_2 = -\mu(1 - A_0^2)^{-3/2}. \qquad (21)$$

The spectrum of the fast wave, as expected, does not depend on the amplitude A_0 of the momentum oscillations. For the slow wave this dependence is strong and causes a considerable decrease of the phase velocity when A_0 decreases. It will be shown later that precisely this effect determines the dynamics of the development of the beam instability in the system.

By using Eqs. (18) and (20), we get an expression for the amplitude of the space-charge perturbation in a stationary wave:

$$|\rho| = |2A_0/\mu\delta(1 - A_0^2)^{3/2}|. \qquad (22)$$

This and (21) yield, for the slow wave,

$$|\rho| = (2/\mu^2)|A_0| \ll 1, \qquad (23)$$

i.e., the modulation of the density in the slow wave for $\mu \gg 1$ is always small. In the fast wave, in contrast,

$$|\rho| = 2|A_0|(1 - A_0^2)^{-1/2}. \qquad (24)$$

In this case, $|\rho| \ll 1$ only if $|A_0| \ll 1$. Note, also, that consistently with (17), and since $P > 0$, inequality $|A_0| < 1$ is satisfied.

3. EXCITATION OF SLOW SPACE CHARGE WAVES IN INTENSE ELECTRON BEAMS

Equation (14) describes the properties of the space-charge waves propagating in the electron beam. In order to investigate the excitation of these space charge waves, one should consider the interaction of the beam with a system supporting electromagnetic waves with phase velocities less than the velocity of light. As a result of this interaction, the space charge and electromagnetic waves in the system are coupled.[1] Mathematically, the existence of the coupling gives rise to an additional equation in (14) describing the amplitude of the electromagnetic wave $|\varepsilon|$, and to an additional force acting on the electrons in the beam:

$$\frac{d\varepsilon}{d\tau} = -\nu\rho\exp(i\eta_0\tau), \quad \frac{dy}{d\tau} = \eta,$$
$$\frac{d\eta}{d\tau} = (1-\mu\eta)^{3/2}\left\{-\frac{i}{2}\left[\exp(iy)\left(1 - i\mu\frac{d}{d\tau}\right)\rho - \text{c.c.}\right] + \frac{\nu}{2}\left[\varepsilon\exp(iy - i\eta_0\tau) + \text{c.c.}\right]\right\}. \qquad (25)$$

Here, ν is a parameter defined by the specific nature of the coupling and η_0 is the detuning. In terms of the dimensional variables t and z, the detuning equals $\omega_0 - k_0 u$, where ω_0 and k_0 are the frequency and wave vector of the electromagnetic wave, respectively.

It is important to emphasize that for $\nu \gg 1$, the excitation of the beam modes proceeds in the so called Compton or single-particle regime, i.e., so rapidly that the properties of the beam modes do not have time to manifest themselves. Therefore in the following we will assume that the coupling is weak, i.e., $\nu \ll 1$. This excitation regime is usually referred as the collective, or Raman regime (in the theory of parametric instabilities the single-particle regime is called a modified decay instability and the collective regime is referred as a resonant simple decay instability.[4])

Initial conditions for Eqs. (25) follow from Eq. (13) (at $\tau = 0$ the current is unperturbed) and can be written

$$y|_{\tau=0} = y_0 \in [0, 2\pi], \quad \eta|_{\tau=0} = 0, \quad \varepsilon|_{\tau=0} = \varepsilon_0. \qquad (26)$$

Let us consider the excitation of a slow wave in the linear approximation.

For this purpose, let

$$\varepsilon \propto \exp(i\eta_0\tau - i\delta\tau), \quad \rho \propto \exp(-i\delta\tau)$$

and linearize Eq. (25) by assuming $\tilde{y} \ll 1$. As a result, the following dispersion relation is obtained:

$$(\delta - \eta_0)(\delta - \delta_1)(\delta - \delta_2) = \nu^2, \qquad (27)$$

where $\delta_{1,2}$ are defined in (10). It follows from Eq. (27) that the maximum value of δ is obtained at the resonance:

$$\eta_0 = \delta_2 = -\mu/2 - (\mu^2/4 + 1)^{1/2}, \qquad (28)$$

where

$$\delta = \delta_2 + i\delta_0(1 + \mu^2/4)^{-1/4}, \qquad (29)$$

where $\delta_0 = \nu/\sqrt{2}$ is the growth rate in the nonrelativistic theory. It is seen that the relativistic correction to the growth rate is always small for $\nu \ll 1$. Nevertheless, this correction strongly affects the synchronism condition between the beam and the wave and determines all the nonlinear dynamics of the instabilities. It is also easy to see that solution (29) is valid, provided that

$$\nu \ll 2^{3/2}(1 + \mu^2/4)^{1/4}, \qquad (30)$$

in agreement with the condition $\nu \ll 1$.

Let us now use Eqs. (25) to investigate the nonlinear dynamics of the slow space-charge waves in intense beams. This question has a fundamental importance for microwave electronics, since it affects the generation of intense electromagnetic waves. It is in this context that this question will be addressed.

Let us introduce several auxiliary quantities. We exploit Eq. (26) to find the first integral of Eqs. (25),

$$P - 1 = -\tfrac{1}{8}\mu(|\varepsilon|^2 - |\varepsilon_0|^2), \qquad (31)$$

where P is the momentum of the beam mode given by Eq. (15). Expression (31) reflects the conservation of the

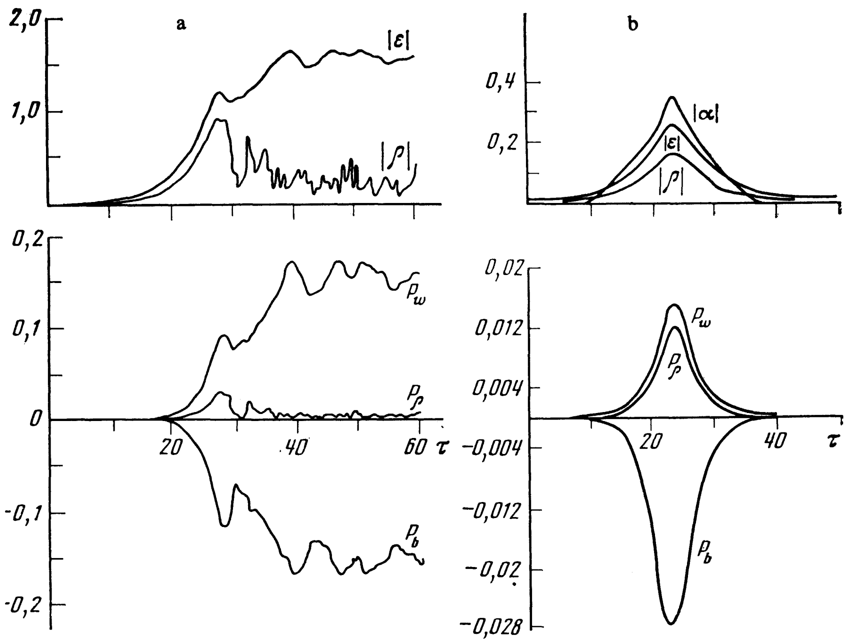

FIG. 1. The time dependence of the momenta P_b, P_ρ, the radiation efficiency P_w and the wave amplitudes for cases (a) $\mu = 0.5$ and (b) $\mu = 2$.

total momentum of the beam-electromagnetic wave system. It can be seen that a change in the mechanical momentum of the beam

$$P_b = \frac{1}{2\pi}\int_0^{2\pi} (1-\mu\eta)^{-1/2} dy_0 - 1 \tag{32}$$

is split between the momentum of the radiation field

$$P_w = {}^1/_8\mu(|\varepsilon|^2 - |\varepsilon_0|^2) \tag{33}$$

and the electromagnetic momentum of the beam mode

$$P_\rho = \mu^2\rho^2/8, \tag{34}$$

where $|\rho|$ is the amplitude of the space-charge wave. During the excitation of the slow wave, when the detuning is given by Eq. (28), $P_b < 0$, $P_w > 0$, i.e., the beam is losing its mechanical momentum in favor of the electromagnetic wave.

In the nonrelativistic microwave electronics the excitation efficiency of an electromagnetic wave is characterized by an electronic efficiency coefficient equal to the negative of Eq. (32). In the case considered, however, the efficiency is given by the quantity (33), because the mechanical part of the momentum is wasted in electromagnetic momentum P_ρ of the beam mode. It will be shown below that the drastic decrease in the radiation efficiency of intense beams is directly related to the latter quantity.

Eqs. (25) have been solved numerically for $\varepsilon_0 = 0.01$ $\nu = 0.3$ and different values of μ. Fig. 1 shows the τ dependence of the amplitude of the electromagnetic wave $|\varepsilon|$ as well as of the amplitude of the space-charge wave $|\rho|$, the mechanical momentum of the beam mode P_b, the electromagnetic part of the momentum P_ρ and the radiation efficiency P_w. Figure 2 shows the phase planes (η, y) of the electron beam.

The saturation mechanism for the instability when $\mu < 1$ is self-trapping, due to the reflection of the beam from

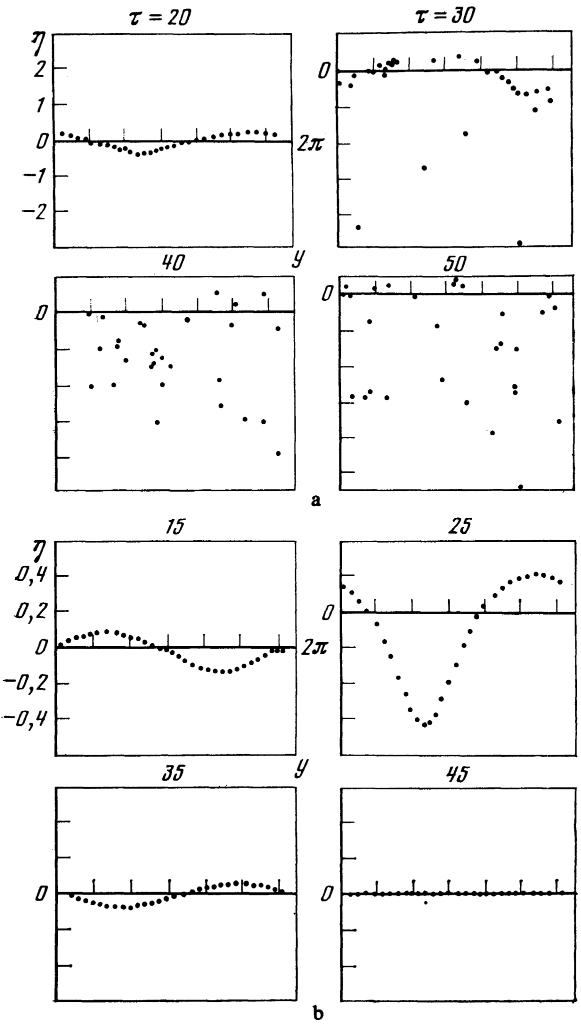

FIG. 2. The phase planes of the electron beam for (a) $\mu = 0.5$ and (b) $\mu = 2$ at different times τ.

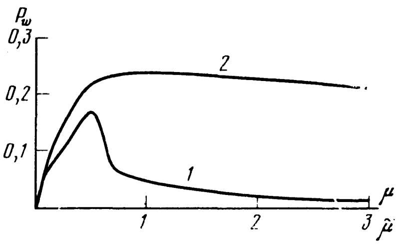

FIG. 3. The dependence of the beam radiation efficiency on the high-current parameters μ (the long-wavelength limit, curve 1) and $\tilde{\mu}$ (the short-wavelength limit, curve 2).

the humps of the potential of the slow wave. The beam becomes highly turbulent. This is a nonrelativistic process.[1] The relativistic nature of the beam electrons and the electromagnetic character of the beam mode for $\mu < 1$, are manifested only via quantitative changes. As an example of the self-trapping and beam turbulence, we show in Fig. 1a the results of the calculations for $\mu = 0.5$.

Qualitative changes occur for $\mu > 1$. Fig. 1b, corresponding to $\mu = 2$, shows that the saturation of the instability takes place at low density modulation ($|\rho| \ll 1$) of the beam, when there is no self-trapping. Earlier we showed that the frequency of the slow wave depends strongly on the amplitude for large values of μ [see Eq. (21)]. Therefore, the saturation of the instability for large μ is always related to a nonlinear frequency shift. We will consider this process analytically in the following.

A salient feature of the numerical solutions for large values of μ is their ideal periodicity. It can be seen in Fig. 1b that after the development and saturation of the instability, the system returns to its initial state and then the whole process is repeated again and again. The phase densities are especially informative; they show (see Fig. 2b) that, as time goes on, the beam returns to the initial, unperturbed state. This behavior, for systems of type (25), is unusual, since even in cases when electron trapping is absent, the beam thermalizes gradually.

Figure 1b also shows the quantity $|\alpha|$ [see Eq. (17)]. Its values were computed by using the expression

$$|\alpha|=\frac{1}{\pi}\int_0^{2\pi}(1-\mu\eta)^{-1/2}e^{iy}dy_0. \tag{35}$$

One can see in Fig. 1b that to good accuracy $|\rho| = 2|\alpha|/\mu^2$, i.e., the approximate formula (23) is valid.

Figure 3 (curve 1) shows the maximum radiation efficiency P_w as a function of μ. The relativistic effects result in a rapid decrease of the radiation efficiency at high currents. We will derive below an analytic formula for P_w for the case $\mu \gg 1$.

4. ANALYTIC THEORY OF EXCITATION OF SLOW RELATIVISTIC WAVES IN AN INTENSE BEAM

Analytic solutions are obtained by assuming the validity of the following inequalities:

$$\nu \ll 1, \quad \mu \gg 1. \tag{36}$$

By introducing, as before, representation (17), substituting it into system (25) and linearizing with respect to $\tilde{y}$, we have

$$\begin{aligned}
&\frac{dA}{d\tau}+\frac{1}{2}\left(i\rho+\mu\frac{d\rho}{d\tau}\right)=\frac{\nu\varepsilon}{2}\exp(-i\eta_0\tau),\\
&\frac{d\varepsilon}{d\tau}=-\nu\rho\exp(i\eta_0\tau),\\
&\frac{d\rho}{d\tau}=-2i[\langle p\rangle^2-\mu^2|A|^2]^{-1/2}A,\\
&\langle p\rangle=1-{}^1/_8\mu|\varepsilon|^2-{}^1/_8\mu^2|\rho|^2.
\end{aligned} \tag{37}$$

Here $A = \alpha/\mu$ and in addition it is assumed that $\varepsilon_0 = 0$, i.e., the inclusion of the field is assumed to be adiabatic.

For $\mu \gg 1$, at the resonance $\eta_0 = -\mu$ [cf. Eq. (28)]. This allows us to write the solution of system (37) in the form

$$\rho=\rho_1(\tau)e^{i\mu\tau}, \quad A=A_1(\tau)e^{i\mu\tau}, \tag{38}$$

where ρ_1 and A_1 are the amplitudes of the slow wave. Suppose also that $\mu|\varepsilon|^2 \sim \mu^2|\rho|^2 \ll \mu^2|A_1|^2 \ll 1$. Then by substituting (38) into Eqs. (37), we obtain

$$\frac{d\varepsilon}{d\tau}=-\nu\rho_1, \quad \frac{d\rho_1}{d\tau}-\frac{3}{8}i\mu^5|\rho_1|^2\rho_1=-\frac{\nu}{\mu}\varepsilon, \tag{39}$$

and [cf. Eq. (23)]

$$A_1=-{}^1/_2\mu\rho_1. \tag{40}$$

The resulting equations contain the nonlinear detuning, which stabilizes the instability.

The first integral of system (39), for adiabatic initial conditions, is

$$|\varepsilon|^2=\mu|\rho_1|^2. \tag{41}$$

The solution of Eqs. (39), can be easily found now and has the following form

$$\begin{aligned}
&|\rho_1(\tau)|^2=\frac{32}{3}\tilde{\delta}\mu^{-5}\left\{1-\left[\frac{1-\exp(4\tilde{\delta}\tau)}{1+\exp(4\tilde{\delta}\tau)}\right]^2\right\}^{1/2},\\
&\qquad -\infty<\tau<\infty,\\
&\qquad \tilde{\delta}=\delta_0(1+\mu^2/4)^{-1/4},
\end{aligned} \tag{42}$$

where $\tilde{\delta}$ is the relativistic increment [see Eq. (29)]. Equation (42) yields the following expressions for the maximum wave amplitudes

$$\begin{aligned}
&|\rho_{max}|^2=\frac{\tilde{\delta}}{3}\left(\frac{2}{\mu}\right)^5,\\
&|\varepsilon_{max}|^2=\frac{2\tilde{\delta}}{3}\left(\frac{2}{\mu}\right)^4, \quad |A_{max}|^2=\frac{\tilde{\delta}}{3}\left(\frac{2}{\mu}\right)^3.
\end{aligned} \tag{43}$$

The validity of the assumed inequalities for $\mu \gg 1$ also follows from these last equations.

Essentially, Fig. 1b shows the solutions of Eq. (43) for nonadiabatic initial conditions. Complete agreement is not obtained, of course, since for $\mu = 2$, the above inequalities are marginally valid. Numerical solutions of Eqs. (37), in contrast, are in good agreement with the solutions of system (25) until $\mu = 1$.

By using Eqs. (43), one can obtain an expression for the decreasing branch of $P_w(\mu)$ (curve 1 in Fig. 3):

$$P_w(\mu)|_{\mu>1}={}^4/{}_3\tilde{\delta}/\mu^3. \quad (44)$$

The radiation power is given by $\mu^2 P_w \sim \tilde{\delta}/\mu$, and hence decreases as $I_b^{-1/2}$ in the high current regime.

5. SPACE CHARGE WAVES IN "THIN" ELECTRON BEAMS

Till now, "oblique" waves have been considered in the case of a magnetized electron beam, filling the entire space. In practice, however, beam propagation in waveguides is of interest. All the previous results remain valid, as far as the linear theory is concerned provided the beam is uniform across the waveguide. Nevertheless, Eqs. (25), in this case, are not applicable, since they do not take into account the transverse stratification of the beam. The study of the stratification of the beam is difficult and we will limit our discussion to the important case when it is absent.

Consider a "thin" beam, namely the beam with a density profile given by

$$p_b(\mathbf{r}_\perp)=S_b\delta(\mathbf{r}_\perp-\mathbf{r}_b), \quad (45)$$

where S_b is the area of the cross section of the beam. $\mathbf{r}_\perp$ is the transverse coordinate in the cross section of the waveguide and $\mathbf{r}_b$ is the transverse coordinate of the beam. The dispersion relation in the linear approximation, in this case, can be written in the form[5]

$$(\omega-k_\parallel u)^2=\omega_b^2\gamma^{-3}\sum_{n=1}^{\infty}\frac{k_\parallel^2-\omega^2/c^2}{k_{\perp n}^2+k_\parallel^2-\omega^2/c^2}\frac{S_b\varphi_n^2(\mathbf{r}_b)}{\|\varphi_n\|^2}, \quad (46)$$

where φ_n is the eigenfunction corresponding to the eigenvalue $k_{\perp n}$ and $\|\varphi_n\|^2$ is the square of the norm.

If we use representation (7) in the long-wavelength limit ($k_\parallel \to 0$), Eq. (46) yields

$$(\delta x)^2-\left(1-2\gamma^2\frac{u^2}{c^2}\delta x\right)a^2G=0,$$
$$G=\sum_{n=1}^{\infty}\frac{k_{\perp 1}^2}{k_{\perp n}^2}\frac{S_b\varphi_n^2(\mathbf{r}_b)}{\|\varphi_n\|^2}, \quad (47)$$

where G is a geometric factor. It follows from (47) that the linear theory of the space-charge waves in thin beams reduces to the previously described theory, provided the parameter μ is associated with the quantity

$$\mu=\left(4\frac{\omega_b^2\gamma^{-1}}{k_{\perp 1}^2u^2}G\right)^{1/2}. \quad (48)$$

It is easy to show, that the nonlinear equations for a thin beam also reduce to the form (25) with the parameter μ defined in Eq. (48). Furthermore, with respect to the limiting vacuum and Pierce currents, the parameter (48) behaves like the parameter (9) with respect to the limiting current of the unbounded beam.

6. SPACE CHARGE WAVES IN THE SHORT-WAVELENGTH LIMIT

The short-wavelength limit is characterized by $k_\parallel^2 \gg k_{\perp 1}^2\gamma^2$ or, for a thin beam, $k_\parallel^2 S_b \gg \gamma^2$. The field of the beam mode in these conditions is confined inside the beam, i.e., there exists a potential describing this field (the sum of the polarization and displacement currents is zero). Therefore, independently of the thickness of the beam, the linear spectra of the space charge waves are described by equation

$$(\omega-k_\parallel u)^2=\omega_b^2\gamma^{-3}. \quad (49)$$

Thus the linear theory of the space-charge waves in the short wavelength limit is nonrelativistic, which is natural, since the field of these waves is, to a good approximation, a potential field.

It is convenient in the nonlinear theory to use dimensionless variables different from (13), namely

$$\tau=(\omega_b^2\gamma^{-3})^{1/2}t,\quad \eta=k_\parallel u(\omega_b^2\gamma^{-3})^{-1/2}(v-u)/u, \quad (50)$$

in terms of which the analog of Eqs. (25) is written

$$\frac{d\varepsilon}{d\tau}=-\nu\exp(i\eta_0\tau),\quad \frac{dy}{d\tau}=\eta,$$
$$\frac{d\eta}{d\tau}=(1-\tilde{\mu}\eta)^{3/2}\left\{-\frac{i}{2}[\rho\exp(iy)-\text{c.c.}]+\frac{\nu}{2}[\varepsilon\exp(iy-i\eta_0\tau)+\text{c.c.}]\right\}. \quad (51)$$

The relativistic properties of system (51) are described by the parameter

$$\tilde{\mu}=2(\gamma^2-1)\left(\frac{\omega_b^2\gamma^{-3}}{k_\parallel^2u^2}\right)^{1/2}\approx\left(\frac{k_{\perp 1}^2\gamma^2}{k_\parallel^2}\right)^{1/2}\left(4\frac{\omega_b^2\gamma^{-1}}{k_{\perp 1}^2u^2}\right)^{1/2}, \quad (52)$$

which differs from (9) by a factor much less than unity. Therefore, in the short-wavelength limit relativistic effects manifest themselves at higher currents than in the long-wavelength limit. Furthermore, because of (40) and the definition of τ, the detuning, at the resonant excitation of the slow wave, is $\eta_0 = -1$.

Equations (51) are similar to Eqs. (25), and therefore their numerical solutions will not be discussed here. We will only exhibit the dependence of the radiation efficiency on parameter $\tilde{\mu}$ in the short-wavelength limit (curve 2 in Fig. 3). It can be seen that in the short-wavelength limit the radiation efficiency is much higher than in the long-wavelength limit.

Analytic solutions of system (51) can be found, provided

$$\nu\ll 1,\quad \tilde{\mu}\gg 1, \quad (53)$$

i.e., in the case similar to (36). The following system, in this case, is analogous to Eqs. (37)

$$\frac{d\varepsilon}{d\tau}=-\nu\rho\exp(i\eta_0\tau),\quad \frac{dA}{d\tau}+\frac{i}{2}\rho=\frac{\nu}{2}\varepsilon\exp(-i\eta_0\tau),$$
$$\frac{d\rho}{d\tau}=-2i[\langle p\rangle^2-\tilde{\mu}^2|A|^2]^{-3/4}A,\quad \langle p\rangle=1-{}^1/{}_8\tilde{\mu}|\varepsilon|^2, \quad (54)$$

where, as was mentioned earlier, $\eta_0 = -1$.

Equations (54) are solved similarly to Eqs. (37). By omitting the intermediate steps, we can write the final result as

$$|\rho|^2=\frac{64}{3}\delta_0\tilde{\mu}^{-2}\left\{1-\left(\frac{1-\exp(4\delta_0\tau)}{1+\exp(4\delta_0\tau)}\right)^2\right\}^{1/2}, \qquad |\varepsilon|^2=2|\rho|^2. \tag{55}$$

Equation (53) yields the maximum amplitudes of the waves

$$|\rho_{max}|^2={}^{64}/_3\delta_0\tilde{\mu}^{-2}, \quad |\varepsilon_{max}|^2={}^{128}/_3\delta_0\tilde{\mu}^{-2}, \quad |A_{max}|^2={}^{16}/_3\delta_0\tilde{\mu}^{-2}, \tag{56}$$

and the maximum radiation efficiency

$$P_w(\tilde{\mu})|_{\tilde{\mu}>1}={}^{16}/_3\,\delta_0\tilde{\mu}^{-1}. \tag{57}$$

Equation (57) describes the decreasing branch of curve 2 in Fig. 3. The radiation power is proportional to $\tilde{\mu}$, i.e., it increases as the square root of the current in the high-current regime. The latter result makes the short-wavelength limit advantageous as compared to the long-wavelength limit.

[1)] The role of such electrodynamical systems in vacuum electronics is played by waveguides with corrugated walls, undulators, dielectric waveguides, etc. (For details, see Ref. 3.)

[1] A. I. Akhiezer, I. A. Akhiezer, P. V. Polovin *et al.* "Elektrodinamika Plazmy" (Plasma Electrodynamics) Moscow, Nauka, 1974, Ch. 8. [Engl. transl. publ. by Pergamon Press, Oxford (1975)].

[2] A. A. Rukhadze, L. S. Bogdankevich, S. E. Rosinskiĭ, and V. G. Rukhlin "Fizika Sil'notochnykh Relativistskikh Elektronnykh Puchkov" (Physics of Intense Relativistic Electron Beams) Moscow, Energoizdat, 1980, Chs. 1, 3.

[3] L. S. Bogdankevich, M. V. Kuzelev, and A. A. Rukhadze "Plazmennaya SVCh Elektronika" (Microwave Plasma Electronics) Usp. Fiz. Nauk **137**, 3 (1981) [Sov. Phys. Uspekhi **24**, 1 (1981)].

[4] V. P. Silin "Parametricheskoe Vozdeistvie Izlucheniya Bolshoĭ Moshchnosti na Plazmu" (Parametric Interaction of High-Power Radiation with Plasmas) Moscow, Nauka, 1973.

[5] M. V. Kuzelev, G. V. Sanadze, and A. G. Shkvarunets "Predel'nye Toki Pirsa dlya Sistem so Slozhnoi Geometrieĭ" (Pierce Limiting Currents for Systems with Complex Geometries) Kratkie Soobshcheniya Po Fizike **11**, 8 (1983).

Translated by L. Friedland