Low-temperature properties of Heisenberg ferromagnets with arbitrary spin in the Holstein-Primakoff formalism

A. V. Chubukov

Institute of Physics Problems, Academy of Sciences of the USSR
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The Holstein-Primakoff formalism, based on the introduction of Bose quasiparticles, is used to examine the low-temperature properties of easy-plane Heisenberg ferromagnets with arbitrary spin. A complete calculation is made of the leading quantum and temperature corrections to the magnetic properties of the model in three-dimensional space and in spaces of lowered dimensionality. The critical properties are studied for easy-plane ferromagnets near the low-temperature phase transitions.

1. INTRODUCTION

The traditional method of studying magnetic phenomena in crystals at low temperature is based on the replacement of the spin operators by creation and annihilation operators for quasiparticles (magnons) obeying Bose-Einstein statistics and on the use of the standard diagram techniques for Bose operators to study the Hamiltonian of the Bose gas.[1-3] This approach does not lead to any difficulties if the site spin S of the magnetic system under study is large, $S \gg 1$. In this case the transition from spin operators to bosons is accomplished through the simple (but approximate) relations

$$S^z = S - a^+a; \quad S^+ = (2S)^{1/2}a; \quad S^- = (2S)^{1/2}a^+, \tag{1}$$

and all the anharmonic terms in the Bose Hamiltonian are small (containing the parameter $1/S$), so that only the lowest orders of perturbation theory need to be kept.[2,4]

For magnetic systems with an arbitrary values of the spin S, however, an approach based on the introduction of quasiparticles encounters serious difficulties. For one thing, the commutation relations for spin operators are much more complicated than for Bose operators, so that each of the spin projection operators can be represented only in the form of an infinite series in powers of the boson creation and annihilation operators[4,5] (see below).[1)] In addition, for $S \sim 1$ the anharmonic terms of exchange origin do not have a small parameter.

As a result, the problem of studying the magnetic properties of a spin system with arbitrary S is formulated in the language of quasiparticles as one of studying a Bose gas with an infinite number of potentials describing strong interactions between magnons. Such as formulation makes sense only if the density of quasiparticles is small, i.e., if the Bose gas satisfies the weak-nonideality criterion.[1,3]

For collinear ferromagnets, in which the ground state is the state of maximum magnetization and so the oscillations of the magnetic moment are solely of thermal origin, the weak-nonideality criterion is clearly satisfied if the temperature T of the crystal is small compared to the Curie temperature T_c. In this case allowance for the strong exchange interaction between magnons reduces to the summation of a series of ladder diagrams over the number of two-particle exchanges. Such a summation was first done explicitly by Dyson.[7]

For many ferromagnets, however, the state with maximum projection of each of the spins along the quantization axis is not an eigenstate of the spin Hamiltonian.[8] Structures of this kind are called noncollinear. They have the distinctive property that even at $T = 0$ the magnetization along the quantization axis is less than the nominal value. The absence of complete ferromagnetic ordering at absolute zero (i.e., the instability of the classical ground state with respect to the spontaneous creation of magnons) means that, because of quantum effects, noncollinear systems have a finite density of quasiparticles even at $T = 0$. Therefore, to satisfy the weak-nonideality criterion for the Bose gas it is necessary to impose certain conditions directly on the parameters of the spin Hamiltonian in addition to requiring a small ratio T/T_c. For example, in Heisenberg-exchange systems for which the deviation from nominal magnetization at $T = 0$ is due to the presence of easy-plane anisotropy, the requirement of a low quasiparticle density means that the ratio of the anisotropy constant to the exchange constant must be small (see below). The study of quantum effects in such systems is the subject of this paper.

Let us start from the spin Hamiltonian

$$H = -\frac{J}{2}\sum_{\mathbf{l},\Delta} \mathbf{S}_\mathbf{l}\mathbf{S}_{\mathbf{l}+\Delta} + \alpha\sum_{\mathbf{l}} (S_\mathbf{l}^z)^2 - 2h\sum_{\mathbf{l}} S_\mathbf{l}^z. \tag{2}$$

Here $\mathbf{S}_\mathbf{l}$ is the spin operator at site $\mathbf{l}$, Δ is the vector joining nearest neighbors in the lattice, and J is the exchange integral.

The type of spin structure depends on the value of the anisotropy constant α and the magnitude of the magnetic field h. If the anisotropy constant is negative, the ground state is characterized by a maximum magnetization along the Z axis for any value of the external field. A similar situation also occurs for positive values of α but in the region of strong magnetic fields[9]: $|h| > h_c = \alpha S(1 - 1/2S)$.

In this paper we consider the case in which $\alpha > 0$ and $|h| < h_c$. In such fields the quantization axis deviates from the Z direction, and a long-range order in the XY plane arises in the system.[2)] However, because of the noncommutivity of Hamiltonian (2) and the operators for the transverse spin

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components S^x and S^y, the state with maximum magnetization along the rotated (with respect to Z) quantization axis is not an eigenstate, and allowance for the quantum nature of the spins should lead, even at $T = 0$, to a change in the values of the magnetic properties in comparison with the results of a classical treatment.

One quantum effect is well known: the renormalization of the single-ion anisotropy constant α, which appears in the answers only in the "quantum combination"[10-12]

$$\tilde{\alpha}=\alpha(1-1/2S). \tag{3}$$

Renormalization (3) is natural, since there is no single-ion anisotropy for $S = 1/2$.

It is of interest to study quantum effects which do not reduce solely to the renormalization $\alpha\to\tilde{\alpha}$. We shall show that for this it is necessary to calculate the properties of the Bose gas to the first two orders in α/J. Similar calculations have been done by a number of authors.[13-19] However, most of these authors made the additional assumption that the site spin of the system was large. For example, in Refs. 15 and 16 the coefficients of the terms of second order in α/J were found to first and second order (respectively) in $1/S$. In my recent paper[17] the quantum corrections to second order in α/J were calculated for arbitrary spin S. The general formalism used in Ref. 17 included diagonalizing the quadratic form in the Bose Hamiltonian with the aid of a generalized UV transformation.[2,12,20] After this the influence of the anharmonicity was taken into account by perturbation theory.

However, this (rather standard) approach leads to serious difficulties, since the total number of anharmonic terms is increased considerably after the UV transformation. For this reason I was unable to calculate the anharmonic corrections to the necessary accuracy[17] and had to determine them with the aid of the *a priori* assumption that the easy-plane ferromagnet had a Goldstone spectrum, which is in general not completely correct, since the assumption of a Goldstone spectrum cannot be verified in the framework of this method.

In the present paper the low-temperature properties of Heisenberg ferromagnet (2) are calculated using the diagram technique for Bose systems with condensation. The advantage of this method over the UV transformation is that no new interaction potentials arise in the course of solution. By a systematic summation of the diagrams (as is shown in Sec. 2) one can correctly determine the leading nontrivial quantum and temperature corrections determine the leading nontrivial quantum and temperature corrections to the magnetic properties of a three-dimensional easy-plane Heisenberg ferromagnet with arbitrary spin. The problem of determining the lifetime of the elementary excitations in this system is also discussed. In Sec. 3 the magnetic properties of an easy-plane ferromagnet are considered for spatial dimensions $D = 2$ and $D = 1$. In Sec. 4 an analysis is made of the role of quantum fluctuations near order-disorder phase transitions, which occur in an easy-plane Heisenberg ferromagnet (2) when the single-ion anisotropy constant is comparable in size to the exchange constant.

Before turning to the calculation, let us briefly discuss the choice of the transformation relating the spin projection operators to bosons, which enables one to write the Bose equivalent of the spin Hamiltonian.

For the present problem I believe it is most convenient to use Goldhirch's exact transformation,[5] which associates with each of the spin projection operators an infinite series in powers of the normal products of Bose operators a^+ and a:

$$S^z=\sum_{n=0}^{\infty} C_n(a^+)^n a^n, \tag{4}$$

$$S^+=(2S)^{1/2}\sum_{n=0}^{\infty} b_n(a^+)^n a^{n+1}; \qquad S^-=(S^+)^+ \tag{5}$$

(the Z axis is taken along the quantization axis). Expressions for the coefficients b_n and C_n are given in Ref. 5.

The advantage of the Goldhirch transformation over the other known methods of relating the spin and Bose operators[4,6,21-23] is that calculations can be done using the resulting Bose Hamiltonian without introducing an additional projection operator. Of course, this transformation is not one-to-one. According Ref. 5, to every value S_Z ($|S_Z|\leqslant S$) there corresponds an infinite number of eigenvalues of the operator $\hat{N}=a^+a$, differing by a multiple of $2S + 1$, i.e., the Bose space is partitioned into closed blocks, each of which is isomorphic to the spin space. Therefore, the problem of the projection operator is actually replaced by the problem of guaranteeing convergence of the partition function on summation over the infinite sequence of identical blocks. It is important, however, that the weight factors that must be assigned to each block to guarantee convergence of the partition function are independent of the parameters of the system, and therefore the magnetic properties, which are derivatives of the free energy, do not depend on this factor (see Refs. 5 and 24 for details). On the other hand, the presence of an infinite number of interaction potentials in the Bose Hamiltonian does not complicate the calculations in the present case, since processes involving a large number of magnons are ineffective at low quasiparticle densities, so that rather than working with the total Bose Hamiltonian one can truncate it, keeping only the lowest anharmonic terms.

The Goldhirch transformation is a modification of the familiar Holstein-Primakoff transformation[4] to permit automatic exclusion of the nonphysical states in the calculation of the partition function. As was shown in Ref. 5, this modification affects only the state containing at least $2S + 1$ elementary excitations. The first $2S + 1$ terms in (4) are the same as the corresponding terms in the series expansion of the radicals in the Holstein-Primakoff formulas in powers of the normal products of the operators a^+ and a.[25] Therefore, in problems where only the first $2S + 1$ terms in (4) are important, the calculation can actually be done using the Holstein-Primakoff transformation, treating it (for an arbitrary value of the spin!) as exact. The analysis carried out below shows that for our purposes the difference between the Goldhirch and Holstein-Primakoff transformations is unimportant.

2. HEISENBERG FERROMAGNET WITH EASY-PLANE ANISOTROPY

Let us consider ferromagnet (2) in the field region $|h| < h_c$ when the quantization axis is at an angle $\theta \neq 0$ to the Z axis, i.e., when the spin structure is noncollinear. Transforming from the spin projection operators to the Bose operators by means of formulas (4), we rewrite Hamiltonian (2) in the form of a series of powers of the normal products of the operators a^+ and a:

$$H = H_0 + H_1 + H_2 + H_a^{rel} + H_a^{ex}. \tag{6}$$

Here H_0 is the classical ground-state energy but with allowance for the quantum renormalization of the anisotropy constant:

$$H_0 = -SN(\tfrac{1}{2}JZS + 2h\cos\theta - \tilde{\alpha}S\cos^2\theta + \tfrac{1}{2}\alpha), \tag{7}$$

and H_1 and H_2 are, respectively, the linear and quadratic (in the Bose operators) forms:

$$H_1 = -i(2NS)^{1/2}\sin\theta\,(h - \tilde{\alpha}S\cos\theta)\,a_0^+ + \text{H.c.} \tag{8}$$

$$H_2 = \sum_{\mathbf{k}} A_{\mathbf{k}} a_{\mathbf{k}}^+ a_{\mathbf{k}} + \frac{B_{\mathbf{k}}}{2}(a_{\mathbf{k}} a_{-\mathbf{k}} + a_{\mathbf{k}}^+ a_{-\mathbf{k}}^+). \tag{9}$$

The coefficients of the quadratic form are

$$A_{\mathbf{k}} = JZS(1-\nu_{\mathbf{k}}) + 2h\cos\theta + \tilde{\alpha}S(1-3\cos^2\theta), \tag{10}$$

$$B_{\mathbf{k}} = -\alpha S(1-1/2S)^{1/2}\sin^2\theta. \tag{11}$$

In formulas (7)–(11) we have used the following notation: Z is the number of nearest neighbors, $\alpha = \alpha(1 - 1/2S)$, and

$$\nu_{\mathbf{k}} = \frac{1}{Z}\sum_{\Delta} e^{i\mathbf{k}\Delta} \quad \left(\nu_0 = 1;\ \sum_{\mathbf{k}} \nu_{\mathbf{k}} = 0\right).$$

The anharmonic potentials in (6) can be of both exchange and relativistic origin. Of the exchange anharmonicities we shall need the terms describing two- and three-particle scattering:

$$H_a^{ex} = \frac{1}{N}\sum \psi_4^{ex} a_1^+ a_2^+ a_3 a_4 + \frac{1}{N^2}\sum \psi_6^{(ex)}\, a_1^+ a_2^+ a_3^+ a_4 a_5 a_6. \tag{12}$$

The amplitudes ψ_4^{ex} and ψ_6^{ex} are written explicitly as

$$\psi_4^{ex} = -\frac{JZ}{8}[\nu_{1-3} + \nu_{2-3} + \nu_{1-4} + \nu_{2-4} - q(\nu_1 + \nu_2 + \nu_3 + \nu_4)], \tag{13}$$

$$\psi_6^{ex} = -\frac{JZS}{18}\left[\left(\frac{q}{4S}\right)^2\left(\sum_{\substack{i=1,2,3\\ p,m=4,5,6;\ p\neq m}} \nu_{1-p-m}\right) - 3\left(1-(1-1/S)^{1/2} - \frac{q}{2S}\right)\left(\sum_{i=1}^{6}\nu_i\right)\right], \tag{14}$$

where we have used the notation $q = 4S[1 - (1 - 1/2S)^{1/2}]$. In the limiting case of large spins ($S \gg 1$), q is close to unity.

We note that strictly speaking, formula (14) as written is valid only for $S > 1$. For the cases $S = 1/2$ and $S = 1$ the analysis must be carried out separately, since for these spin values the exact expansion for S^z [see Ref. 5 and Eq. (4)] contains terms of sixth order in the Bose operators, giving an additional, wave-vector independent contribution to amplitude (14). Analysis shows, however, that the presence of an additional constant term in the bare amplitude of the three-particle process for $S = 1$ does not affect the final results. And the $S = 1/2$ case is uninteresting, since for $S = 1/2$ there is no single-ion anisotropy and, hence, no noncollinearity in the structure.

The anharmonic terms of relativistic origin describe processes involving both even and odd numbers of Bose operators:

$$H_a^{rel} = \frac{1}{N^{3/2}}\sum \psi_3^{rel} a^+ a^+ a + \frac{1}{N}\sum (\psi_4^{rel} a^+ a^+ a a + \tilde{\psi}_4^{rel} a^+ a^+ a^+ a) + \text{H.c.} \tag{15}$$

The higher anharmonicities will not be needed.

The amplitudes ψ_i^{rel} appearing in (15) are given by the expressions

$$\psi_3^{rel} = -i(2S)^{1/2}\sin\theta\left[\alpha(1-1/2S)^{1/2}\cos\theta - \frac{q}{4S}(h - \tilde{\alpha}S\cos\theta)\right], \tag{16}$$

$$\psi_4^{rel} = -\frac{\alpha}{4}(3\sin^2\theta - 2); \qquad \tilde{\psi}_4^{rel} = \frac{\tilde{\alpha}S}{2}\frac{1-(1-1/S)^{1/2}}{(1-1/2S)^{1/2}}\sin^2\theta. \tag{17}$$

Terms (8) and (16) exist only in nonzero field and stem from a deviation of the quantization axis from both the Z axis and the XY plane.

2a. QUASIPARTICLE SPECTRUM AND THE MAGNETIZATION

Let us study Hamiltonian (6) using the diagram technique for Bose systems with condensation,[1,3] introducing normal $G^{+-}(\mathbf{k}, \omega)$ and anomalous $G^{++}(\mathbf{k}, \omega)$ and $G^{--}(\mathbf{k}, \omega)$ Green functions. The formal solution of the system of Dyson equations for these function is

$$G^{+-}(\mathbf{k}, \omega) = [i\omega + A_{\mathbf{k}} + \Sigma_S(\mathbf{k}, \omega) - \Sigma_A(\mathbf{k}, \omega)]D^{-1}(\mathbf{k}, \omega), \tag{18}$$

$$G^{++}(\mathbf{k}, \omega) = G^{--}(\mathbf{k}, \omega) = -[B_{\mathbf{k}} + \Sigma^{++}(\mathbf{k}, \omega)]D^{-1}(\mathbf{k}, \omega), \tag{19}$$

where

$$\Sigma_{S,A}(\mathbf{k}, \omega) = \tfrac{1}{2}[\Sigma^{+-}(\mathbf{k}, \omega) \pm \Sigma^{+-}(-\mathbf{k}, -\omega)], \tag{20}$$

$$D(\mathbf{k}, \omega) = [\omega + i\Sigma_A(\mathbf{k}, \omega)]^2 + [A_{\mathbf{k}} + \Sigma_S(\mathbf{k}, \omega)]^2 - [B_{\mathbf{k}} + \Sigma^{++}(\mathbf{k}, \omega)]^2. \tag{21}$$

The poles of the Green functions correspond to zeros of $D(\mathbf{k}, -i\varepsilon_{\mathbf{k}})$.

Our problem is to evaluate the self-energy parts Σ^{+-} and Σ^{++} with accuracy to terms of order

$$(\alpha/J)^2 \text{ and } \left(\frac{\alpha}{J}\frac{1}{N}\sum_{\mathbf{p}}\frac{A_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}} n_{\mathbf{p}}\right)$$

inclusive. This will enable us to determine quantum effects which do not reduce solely to a renormalization of the anisotropy constant and to find the leading temperature cor-

rections to the magnetization and to the spectum of elementary excitations.

The diagram for Σ^{+-} and Σ^{++} are given in Figs. 1a and 1b. We have used the following notation:

$$\longrightarrow = G^{+-}, \qquad \longleftrightarrow = G^{++}, \qquad \longleftrightarrow = G^{--};$$
$$\longrightarrow = \frac{1}{A_{\mathbf{k}}-i\omega}, \qquad \longleftrightarrow = \longleftrightarrow = -\frac{B_{\mathbf{k}}}{D(\mathbf{k},\omega)}. \tag{22}$$

In formulas (18)–(21) the angle of rotation θ of the quantization axis relative to the Z axis enters as a parameter. This angle is determined from the condition[26]

$$\langle\psi_0|S^+|\psi_0\rangle=\langle\psi_0|S^-|\psi_0\rangle=0, \tag{23}$$

where ψ_0 is the ground-state wave function. To the adopted accuracy condition (23) is the requirement that all the diagrams having one external line go to zero. This condition is depicted graphically in Fig. 1c.

The graphs in Fig. 1 contain loops consisting of normal (Π_1) and anomalous (Π_2) Green functions. To the adopted accuracy we can write analytical expressions for the loop diagrams in the form

$$\Pi_1=\frac{1}{N}\sum_{\mathbf{k}}\frac{A_{\mathbf{k}}-\varepsilon_{\mathbf{k}}^{(0)}}{2\varepsilon_{\mathbf{k}}^{(0)}}+\frac{A_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}^{(0)}}n_{\mathbf{k}};$$
$$\Pi_2=-\sum_{\mathbf{k}}\frac{B_{\mathbf{k}}}{2\varepsilon_{\mathbf{k}}^{(0)}}(1+2n_{\mathbf{k}}), \tag{24}$$

where

$$\tilde{B}_{\mathbf{k}}=B_{\mathbf{k}}+\Sigma_{(1)}^{++}(\mathbf{k},-i\varepsilon_{\mathbf{k}}^{(0)}); \qquad (\varepsilon_{\mathbf{k}}^{(0)})^2=(A_{\mathbf{k}})^2-(\tilde{B}_{\mathbf{k}})^2. \tag{25}$$

For $T=0$ the loop graphs consisting of normal Green functions can be neglected, since

$$\Pi_2\sim(\tilde{\alpha}/J), \text{ and } \Pi_1\sim(\tilde{\alpha}/J)^{3/2}\ll\Pi_2.$$

The difference between Π_1 and Π_2 is due to the fact that in three-dimensional systems the integral Π_1 is determined by the region of small wave vectors $k\sim k_{\text{char}}\sim a^{-1}(\alpha/J)^{1/2}\ll a^{-1}$, while integral Π_2 is determined by large wave vectors $k\sim k_{\text{char}}\sim a^{-1}$. The fact that the integrals for which $k_{\text{char}}\ll a^{-1}$ are small compared to the integrals determined by values $k_{\text{char}}\sim a^{-1}$ also allows us to drop the terms $\Sigma_{(2),b,c,d}^{+-}$ and $\Sigma_{(3)}^{++}$, at $T=0$ and to replace the total Green functions G^{+-} in the remaining diagrams by the bare Green functions. We also note that for our purposes the quantities $\Sigma_{(1),(2),(3),(4)}^{+-}$ need be determined only to leading (second) order in $\tilde{\alpha}/J$ and for $k=0$, while for $\Sigma_{(1)}^{++}$ we must know the first two (the linear and quadratic) terms in the expansion in $\tilde{\alpha}/J$, and the k dependence of the linear term is important.

The temperature parts of the loop diagrams Π_1^T and Π_2^T behave differently: in the region $0<T\ll\tilde{\alpha}S\sin^2\theta$, for the characteristic wave vectors $\mathbf{k}$ [such that $\varepsilon_{\mathbf{k}}^{(0)}\sim T$] the spectrum is linear, $A_{\mathbf{k}}\approx B_{\mathbf{k}}$ and, consequently, $\Pi_1^T\approx\Pi_2^T$, while in the region $T\gg\tilde{\alpha}S\sin^2\theta$ the quasiparticle spectrum for the characteristic values of k is practically the same as in a purely exchange ferromagnet, i.e., $A_{\mathbf{k}}\gg B_{\mathbf{k}}$ and $\Pi_1^T\gg\Pi_2^T$.

It is seen that for $T\ll\tilde{\alpha}S\sin^2\theta$, where the characteristic wave vectors are very small, the contribution of the relativistic terms to the temperature parts of the diagrams Σ^{+-} and Σ^{++} is important. For calculations in this temperature region the total Green function cannot be replaced by the bare function. Furthermore, for $T\ll\tilde{\alpha}S\sin^2\theta$ all the terms appearing in $\Sigma_{(2)}^{+-}$ and $\Sigma_{(3)}^{++}$ are comparable in magnitude.

Let us go directly to the calculation of the diagrams shown in Fig. 1. The main source of difficulty here is the presence of the total vertices (the hatched squares). They differ from the bare vertices by the presence of the exchange

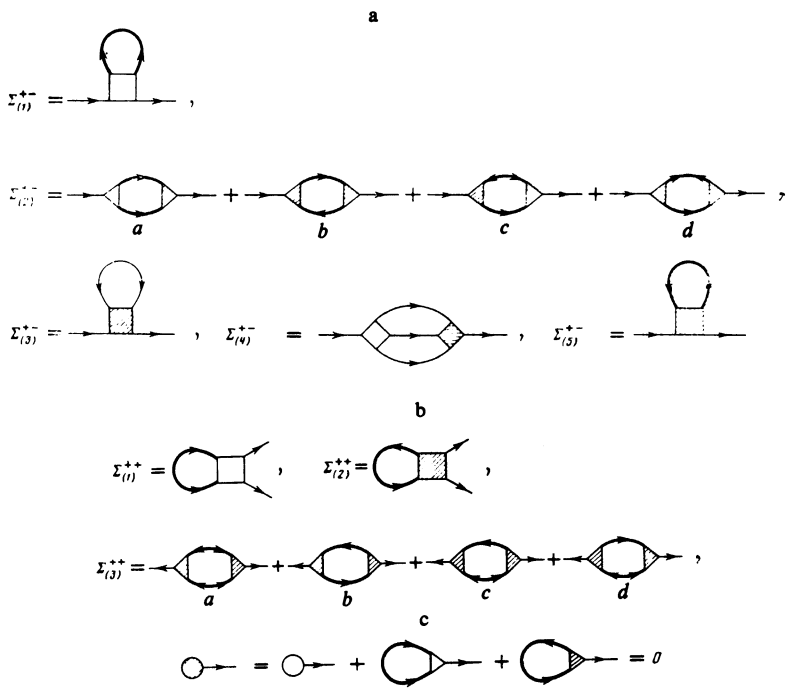

FIG. 1. Digrams which determine the self-energy parts Σ^{+-} (a) and Σ^{++} (b) and the position of the quantization axis (c).

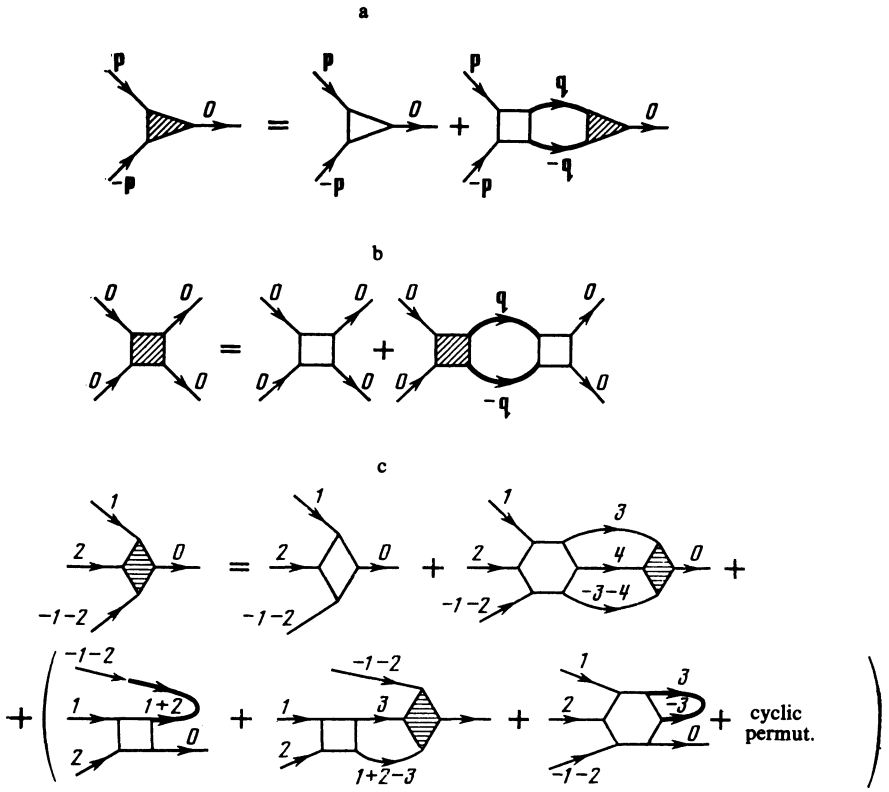

FIG. 2. Graphs for the total vertices appearing in the diagrams in Fig. 1.

renormalization. Allowance for the renormalization is necessary since the exchange interaction between virtual magnons is strong, and the inclusion of "exchange ladders" for arbitrary S does not give rise to an additional small parameter.

The graphical equations for the total vertices of interest to us are given in Figs. 2a–2c. Calculation of the diagrams a and b is quite simple, since the kernel of the corresponding integral equations factors: the substitution $\Phi = a + b(1-\nu_{\mathbf{k}})$ (Φ is the total vertex) reduces each of these equations to a system of two algebraic equations. On solving the corresponding system of equations we get (the total vertices are denoted by the letter Φ with indices):

$$\Phi^{(3)}_{\mathbf{p},-\mathbf{p};0}=-i(2S)^{1/2}\sin\theta\cos\theta\tilde{\alpha}\left\{1+(1-\nu_{\mathbf{p}})\times\left[\frac{1}{2S(1-1/2S)^{1/2}}-\frac{qW}{4S}\right]\right\}, \tag{26}$$

$$\Phi^{(4)}_{0,0;0,0}=-\frac{\tilde{\alpha}}{4}(3\sin^2\theta-2). \tag{27}$$

The calculation of $\Sigma^{++}_{(1)}$ is done in an analogous way. Since it is linear in α we cannot neglect the temperature terms in calculating this quantity. The result is

$$\Sigma^{++}_{(1)}=-\tilde{\alpha}S\sin^2\theta\left\{1-(1-1/2S)^{-1/2}+(1-\nu_{\mathbf{k}})\left[\frac{1}{2S(1-1/2S)^{1/2}}-\frac{qW}{4S}\right.\right.$$
$$\left.\left.-\frac{q}{2NS}\sum_{\mathbf{p}}\frac{JZS}{\varepsilon^{(0)}_{\mathbf{p}}}n_{\mathbf{p}}\right]-\frac{q}{2NS}\sum_{\mathbf{p}}\frac{A_{\mathbf{p}}}{\varepsilon^{(0)}_{\mathbf{p}}}n_{\mathbf{p}}+\left(\frac{\tilde{\alpha}}{JZS}\right)\times\left[\left(1-\frac{q}{4}\right)\sin^2\theta-\cos^2\theta\right]\left(\lambda+\frac{2}{N}\sum_{\mathbf{p}}\frac{JZS}{\varepsilon^{(0)}_{\mathbf{p}}}n_{\mathbf{p}}\right)\right\}, \tag{28}$$

where

$$\lambda=W+\frac{1}{2S(1-1/2S)},\qquad W=\frac{1}{N}\sum_{\mathbf{k}}\frac{1}{1-\nu_{\mathbf{k}}}. \tag{29}$$

Here W is the Watson integral.[27] After using formulas (26)–(28), we obtain an analytical expression for the diagrams $\Sigma^{+-}_{(1),(5)}$

$$\Sigma^{+-}_{(1)}=\frac{3\tilde{\alpha}^2S}{4JZ}\left[1-\left(1-\frac{1}{S}\right)^{1/2}\right]\sin^4\theta\left(\lambda+\frac{2}{N}\sum_{\mathbf{p}}\frac{JZS}{\varepsilon^{(0)}_{\mathbf{p}}}n_{\mathbf{p}}\right), \tag{30}$$

$$\Sigma^{+-}_{(5)}=-2\tilde{\alpha}(3\sin^2\theta-2)\frac{1}{N}\sum_{\mathbf{p}}\frac{A_{\mathbf{p}}}{\varepsilon^{(0)}_{\mathbf{p}}}n_{\mathbf{p}} \tag{31}$$

and for the angular deviation of the quantization axis from the Z axis we get

$$\cos\theta=\cos\theta_0\left[1-\frac{\tilde{\alpha}\lambda}{2JZS}\sin^2\theta_0+\frac{2}{S}\varphi_\theta(T)\right], \tag{32}$$

where

$$\cos\theta_0=h/h_c,\qquad h_c=\tilde{\alpha}S.$$

The explicit form of the temperature dependence of the angle between the magnetic moment and the Z axis is

$$\varphi_\theta(T)=\frac{1}{N}\sum_{\mathbf{p}}\frac{A_{\mathbf{p}}+\frac{1}{2}\tilde{B}_0}{\varepsilon_{\mathbf{p}}^{(0)}}n_{\mathbf{p}}$$

$$=\begin{cases}\left(\dfrac{T}{4\pi JS}\right)^2\left(\dfrac{2J}{\tilde{\alpha}\sin^2\theta_0}\right)^{1/2}\zeta(2), & JZS\gg\tilde{\alpha}S\sin^2\theta_0\gg T,\\ \left(\dfrac{T}{4\pi JS}\right)^{1/2}\zeta(3/2), & JZS\gg T\gg\tilde{\alpha}S\sin^2\theta_0.\end{cases}\qquad(33)$$

An expression for the temperature renormalization of the angle θ to first order in $1/S$ was obtained in Ref. 28. Knowledge of the total vertex $\Phi^{(3)}_{p,-p;0}$ also yields analytical expressions for the diagrams $\Sigma^{+-}_{(2)}$ and $\Sigma^{++}_{(3)}$. Since the integrals corresponding to the diagrams $\Sigma^{+-}_{(2),b,c,d}$ and $\Sigma^{++}_{(3)}$ are determined by small characteristic wave vectors, for $T=0$ we have to the required accuracy

$$\Sigma^{+-}_{(2)}=\Sigma^{+-}_{(2),a}=-\frac{2\tilde{\alpha}^2\lambda}{JZ}\sin^2\theta\cos^2\theta,\qquad \Sigma^{++}_{(3)}=0.\qquad(34)$$

Allowance for $\Sigma^{++}_{(3)}$ and $\Sigma^{+-}_{(2),b,c,d}$ is necessary for a correct calculation of the temperature dependence of the spin wave energy in the region $T\ll\tilde{\alpha}S\sin^2\theta_0$ [see formula (21)]. The expressions for the temperature parts of $\Sigma^{+-}_{(2)}$ and $\Sigma^{++}_{(3)}$ are rather awkward and we shall not give them here. However, the total temperature correction to the spectrum from ternary terms for $T\ll\tilde{\alpha}S\sin^2\theta_0$ can be written rather compactly:

$$(\varepsilon_k^{(0)}+\Delta\varepsilon_k^{(3)})^2=(\varepsilon_k^{(0)})^2\Big[1-\frac{36}{S}\operatorname{ctg}^2\theta_0\frac{1}{N}\sum_{\mathbf{p}}\frac{A_{\mathbf{p}}+\tilde{B}_0}{\varepsilon_{\mathbf{p}}^{(0)}}n_{\mathbf{p}}$$

$$\times(\ln T/\tilde{\alpha}S\sin^2\theta_0)\Big]+\frac{8A_0^2}{S}\operatorname{ctg}^2\theta_0\frac{1}{N}\sum_{\mathbf{p}}\frac{B_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}^{(0)}}n_{\mathbf{p}}.\qquad(35)$$

The remaining diagrams are expressed explicitly in terms of the total scattering amplitude $\tilde{\Phi}^{(4)}_{1,2,-1-2;0}\equiv\Phi_{1,2,-1-2;0}$:

$$\Sigma^{+-}_{(3)}=\frac{3}{2}\frac{\tilde{\alpha}S\sin^2\theta}{(1-1/2S)^{1/2}}\frac{1}{N}\sum_{\mathbf{p}}\Phi_{\mathbf{p},-\mathbf{p},0;0}\frac{(1+2n_{\mathbf{p}})}{\varepsilon_{\mathbf{p}}^{(0)}},\qquad(36)$$

$$\Sigma^{+-}_{(4)}=-\frac{3\sin^2\theta}{(1-1/2S)^{1/2}}\frac{\tilde{\alpha}}{JZ}[1-(1-1/S)^{1/2}]$$

$$\times\frac{1}{N^2}\sum_{1,2}\frac{\Phi_{1,2,-1-2;0}}{3-\nu_1-\nu_2-\nu_{1+2}},\qquad(37)$$

$$\Sigma^{++}_{(2)}=6\Phi_{0,0,0;0}\frac{1}{N}\sum_{\mathbf{p}}\frac{A_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}^{(0)}}n_{\mathbf{p}}.\qquad(38)$$

The dressed vertex $\Phi_{1,2,-1-2;0}$ is determined by the set of graphs describing three-particle processes. This set includes diagrams which form a single ladder sequence, i.e., a series in the number of three-particle exchanges due to the sixth-order anharmonicity (12), (14), and also diagrams describing virtual exchanges of two or three interacting particles (double ladder).[1] Allowance for these latter diagrams complicates the calculation of $\Phi_{1,2,-1-2;0}$ considerably, since in this case the kernel of the corresponding integral equation does not factor. Nevertheless, in this problem the vertices of interest, $\Sigma^{++}_{(2)}$ and $\Sigma^{+-}_{(3)}+\Sigma^{+-}_{(4)}$ [see (36)–(38)], can be calculated explicitly. The intermediate steps, however, are rather awkward and have been relegated to the Appendix. The final result is

$$\Sigma^{+-}_{(3)}+\Sigma^{+-}_{(4)}=\frac{\tilde{\alpha}^2\sin^4\theta}{4JZ}\Big[4-q-3S\Big(1-\Big(1-\frac{1}{S}\Big)^{1/2}\Big)\Big]$$

$$\times\Big(\lambda+\frac{2}{N}\sum_{\mathbf{p}}\frac{JZS}{\varepsilon_{\mathbf{p}}^{(0)}}n_{\mathbf{p}}\Big),\qquad(39)$$

$$\Sigma^{++}_{(2)}=\frac{\tilde{\alpha}}{2}(4-q)\sin^2\theta\frac{1}{N}\sum_{\mathbf{p}}\frac{A_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}^{(0)}}n_{\mathbf{p}}.\qquad(40)$$

By summing the analytical expressions for the constituent parts of the diagrams Σ^{+-} and Σ^{++} [formulas (30)–(34) and (39), (40)] and substituting them into (21), we finally get, with allowance for (10) and (32) ($\varepsilon_{\mathbf{k}}$ is the total energy):

$$\varepsilon_{\mathbf{k}}^2=JZS(1-\nu_{\mathbf{k}})(C^2+JZS(1-\nu_{\mathbf{k}})),\qquad(41)$$

where

$$C=(2\tilde{\alpha}S)^{1/2}\Big[\sin^2\theta_0+\frac{\tilde{\alpha}}{JZS}\Big(W+\frac{1-S}{2S(1-1/2S)}\Big)\sin^4\theta_0$$

$$-\frac{2}{S}(1+\cos^2\theta_0)\varphi_\varepsilon(T)\Big]^{1/2},\qquad(42)$$

and the explicit form of the function $\varphi_\varepsilon(T)$ is known in the limiting cases of extremely low or relatively high temperatures:

$$\varphi_\varepsilon(T)$$

$$=\begin{cases}24\pi^2\left(\dfrac{T}{4\pi JS}\right)^4\left(\dfrac{2J}{\tilde{\alpha}\sin^2\theta_0}\right)^{5/2}\zeta(4)\\ \times\Big[1+\dfrac{18\cos^2\theta_0}{1+\cos^2\theta_0}\Big(\ln\dfrac{T}{\tilde{\alpha}S\sin^2\theta_0}+O(1)\Big)\Big];\\ \qquad\qquad\tilde{\alpha}S\sin^2\theta_0\gg T,\\ =\left(\dfrac{T}{4\pi JS}\right)^{3/2}\zeta(3/2),\quad T\gg\tilde{\alpha}S\sin^2\theta_0.\end{cases}\qquad(43)$$

Of course, formula (42) takes into account only the leading temperature and quantum corrections. Terms $\sim(\tilde{\alpha}/J)^{3/2}$ and $(\tilde{\alpha}/J)^{1/2}\varphi_\varepsilon(T)$ have been dropped.

At $T=0$ and in the absence of external field formula (42) goes over to the corresponding expression that was obtained in Ref. 17 with the aid of additional assumptions (see the Introduction).

It is seen from (42) that to leading order in $\tilde{\alpha}/J$ the quantum effects reduce to a renormalization of the anisotropy constant $\alpha\to\tilde{\alpha}$, but in the general case the quasiparticle energy depends on the spin S in a complicated manner.

The spin wave velocity goes to zero at the critical point of the phase transition

$$h_c^T\approx h_c\Big(1-\frac{2}{S}\varphi_\varepsilon(T)\,|_{\theta_0=0}\Big),$$

which differs from h_c on account of the temperature renormalization caused by the two-body interaction of magnons. (There is no quantum renormalization of h_c, since the spin structure is collinear above the critical point.) One can de-

termine h_c^T exactly for an arbitrary relationship between the constants α and J by calculating the temperature correction to the homogeneous precession frequency in the collinear phase and using the fact that the gap in the spectrum goes to zero at the transition point. This procedure leads to the following result:[3)]

$$h_c^T = h_c\left[1-\frac{2}{S}\left(1+\frac{\alpha}{\alpha_c}\right)^{-1}\left(\frac{T}{4\pi JS}\right)^{1/2}\zeta(3/2)\right], \quad (44)$$

where

$$\alpha_c = JZS\left[W-\frac{1}{2S}(W-1)\right]^{-1}. \quad (45)$$

The magnetization vector has the following nonzero components:

$$M_x = \frac{2\mu}{v_0}\langle S_x\rangle, \quad M_z = \frac{2\mu}{v_0}\langle S_z\rangle$$

(μ is the Bohr magneton, and v_0 is the volume of the unit cell). To the adopted accuracy we have

$$M_x = \frac{2\mu S}{v_0}\left\{\sin^2\theta_0\left[1+\frac{\tilde{\alpha}\cos^2\theta_0}{JZS}\left(W+\frac{1}{2S(1-1/2S)}\right)\right] - \frac{2}{S}\left[2\varphi_0(T)-\sin^2\theta_0\varphi_M(T)\right]\right\}^{1/2}, \quad (46)$$

$$M_z = \frac{2\mu S}{v_0}\cos\theta_0\left\{1-\frac{\tilde{\alpha}\sin^2\theta_0}{2JZS}\left[W+\frac{1}{2S(1-1/2S)}\right] + \frac{1}{S}\varphi_M(T)\right\}, \quad (47)$$

where

$$\varphi_M(T) = \frac{1}{N}\sum_{\mathbf{p}}\frac{A_{\mathbf{p}}+B_0}{\varepsilon_{\mathbf{p}}^{(0)}}n_{\mathbf{p}}. \quad (48)$$

The asymptotic behavior of the function $\varphi_M(T)$ is the same as that of $\varphi_\varepsilon(T)$ [see formula (43)] but without the logarithmic factors. At the transition point

$$M_x = 0, \quad M_z = \frac{2\mu S}{v_0}\left\{1-\frac{1}{S}\left(\frac{T}{4\pi JS}\right)^{1/2}\zeta(3/2)\right\}. \quad (49)$$

This result is completely natural, since the spin structure is collinear above the transition point and the deviation of the magnetization from its nominal value is due solely to thermal fluctuations.

In actuality, owing to the presence of dipole-dipole interactions the magnetization does not reach saturation at any finite value of the field (see Refs. 2 and 4 and, in application to our particular model, Ref. 17). We note in this regard that earlier in this paper, in the calculation of the spectrum and magnetization in the noncollinear phase, we neglected the complications due to dipole interactions. Such a procedure is justified if the dimensionless anisotropy constant is much greater than 4π.

In concluding this section let us consider the problem of calculating the magnon lifetime. It is well known[2] that because of the presence of the δ function the leading contribution to the damping $\gamma_{\mathbf{k}}$ at low temperature comes from processes involving the smallest number of quasi-particles: in this case three-particle decay and coalescence. The magnon lifetime $\tau_{\mathbf{k}}^{(3)} = 1/\gamma_{\mathbf{k}}^{(3)}$ with respect to these processes was determined to leading order in α/J by Bar'yakhtar *et al.*[28] As we concluded earlier, in this approximation allowance for quantum effects reduces to just a renormalization of the anisotropy constant $\alpha\to\tilde{\alpha}$. Therefore, for determining $\gamma_{\mathbf{k}}$ for arbitrary S one need only make the replacement $\alpha\to\tilde{\alpha}$ in the corresponding formulas of Ref. 28. The correctness of this assertion can be checked directly by calculating the sum of the ladder diagrams; this I have done, calculating the damping at zero temperature (the intermediate steps are extremely awkward).

We note that because of the noncollinearity of the spin structure, the magnon lifetime is finite even at $T=0$. In the region of comparatively large k, when $1\gg ak\gg ak_c = (\tilde{\alpha}/J)^{1/2}\sin\theta_0$, the damping increases linearly with temperature, while for small $k(k\ll k_c)$ the exponent of the temperature varies depending on the relationship between $JS(ak_c)^2$ and T. It should be emphasized that the damping of the quasiparticles turns out to be rather weak, since the smearing of the spectrum is always small in comparison to the value found above for the energy renormalization.

For $h=0$ the damping due to the three-particle anharmonicity (16) goes to zero. Therefore, in the region of small fields the leading contribution to γ is from other processes. In this regard it is of interest to compare the formulas for $\gamma_{\mathbf{k}}^{(3)}$ with the results of Bar'yakhtar *et al.*,[30] who calculated the quasiparticle damping (for $h=0$) due to the fourth-order anharmonicity ($\gamma_{\mathbf{k}}^{(4)}$). Such a comparison shows that at small wave vectors $k\ll k_c$, even in the case of "intermediate" temperature $k/k_c \ll T/JS(ak_c)^2 \ll 1$,

$$\frac{\gamma^{(3)}}{\gamma^{(4)}} \sim \left(\frac{h}{h_c}\right)^2\left(\frac{JS(ak_c)^2}{T}\right)^4\left(\frac{T}{JS(ak)^2}\right)\left(\frac{J}{\tilde{\alpha}}\right)^{1/2}, \quad (50)$$

i.e., the damping due to fourth-order processes can be manifested only in extremely weak external fields. Most likely this damping will actually not be manifested at all, since the Hamiltonian always contains third-order anharmonicities due to the dipole-dipole interaction. Very large values of the anisotropy constant would be needed in order for the four-particle process to compete with the three-particle dipole process.

3. LOW-DIMENSIONAL SYSTEMS

It is known that the role of effects due to quasiparticles is larger in two-dimensional ($2D$) and $1D$ magnets than in $3D$ magnets, and allowance for fluctuations (both quantum and classical) can qualitatively alter the spin structure from that given by the classical description. In this section we look at the results of a calculation of the fluctuational corrections to the spin wave spectrum and magnetization of $2D$ and $1D$ easy-plane ferromagnets [the corresponding spin Hamiltonians are the low-dimensional analogs of Hamiltonian (2)].

Let us begin with the $2D$ case. At $T=0$ the calculation of the quantum corrections is completely analogous to the calculation we have done for $3D$ systems, since, as before, the leading contribution to the renormalization comes from integrals for which the characteristic wave vectors obey

$ak_{\text{char}} \gg (\tilde{\alpha}/J)^{1/2}$. The only change for the $3D$ case is the value of the small parameter used in selecting the diagrams: in $2D$ magnets this parameter is the logarithm of the ratio of the exchange to the anisotropy. The answers in the $2D$ case differ from the formulas of Sec. 2 [(42), (46), (47)] only in that the Watson integral W (which is formally divergent for $D = 2$) must be replaced by the more exact expression

$$W \to W' = \frac{JZS}{N} \sum_{\mathbf{k}} \frac{1}{A_{\mathbf{k}}} \approx \frac{Z}{4\pi} \ln\left(\frac{J}{\tilde{\alpha} \sin^2 \theta_0} \right). \qquad (51)$$

It is seen that in $2D$ magnets the quantum effects are stronger than in the analogous $3D$ systems, but they are not enough to completely smear out the order in the XY plane.[4)]

For $T \neq 0$ the Mermin-Wagner theorem[33] states that there is no long-range order. This is confirmed by the divergence of the temperature corrections to the magnetization M_X in $2D$ space [see (33) and (46)]: $N^{-1} \Sigma_{\mathbf{k}} (A_{\mathbf{k}}/\varepsilon_{\mathbf{k}}^{(0)}) n_{\mathbf{k}}$ diverges logarithmically. However, the temperature corrections to the spin wave velocity and to the magnetization along the Z axis remain finite. With allowance for these corrections, we have, outside the fluctuation region,

$$M_{z,(2)}^{T} = M_{z,(2)}^{(0)} \left[1 + \frac{1}{S} \varphi_M^{(2)}(T) \right],$$
$$C_{(2)}^{T} = C_{(2)}^{(0)} \left[1 - \frac{2}{S} (1 + 2 \operatorname{ctg}^2 \theta_0) \varphi_\varepsilon^{(2)}(T) \right]^{1/2}, \qquad (52)$$

where

$$\varphi_\varepsilon^{(2)}(T) = \begin{cases} \left(\frac{T}{JS}\right)\left(\frac{T}{\tilde{\alpha} S \sin^2 \theta_0}\right)^2 ((1+18 \operatorname{ctg}^2 \theta_0)/(1+2 \operatorname{ctg}^2 \theta_0)) \\ \quad \times \frac{\zeta(3)}{4\pi}, \quad T \ll \tilde{\alpha} S \sin^2 \theta_0, \\ = \left(\frac{T}{JS}\right)\left(\ln \frac{T}{\tilde{\alpha} S \sin^2 \theta_0}\right) \frac{1}{4\pi}, \quad T \gg \tilde{\alpha} S \sin^2 \theta_0, \end{cases} \qquad (52')$$

and the asymptotic behavior of $\varphi_M^{(2)}(T)$ for $T \ll \tilde{\alpha} S \sin^2 \theta_0$ differs from (52') only by the absence of the factor containing $\cot^2 \theta_0$. Here $M_{z,(2)}^{(0)}$ and $C_{(2)}^{(0)}$ are the values of the quantities at $T = 0$.

The finding of a finite spin wave velocity for $0 < T \ll JS$ agrees with general ideas about the structure of the low-temperature phase of $2D$ magnets having a two-component order parameter:[34,35] there is no long-range order, but the correlation length is infinite (the correlators decay by a power law), and so the low-lying excitation are spin waves.

Let us turn to a description of $1D$ magnets. Here the long-range order is destropyed even at $T = 0$ because of the logarithmic divergences of the quantum corrections $\sim N^{-1} \Sigma_{\mathbf{k}} (\varepsilon_{\mathbf{k}}^{(0)})^{-1}$. However, as in $2D$ space for $T \neq 0$, the spin wave velocity remains finite. To calculate the spin wave velocity we must take into account not only the loop graphs consisting of the anomalous Green functions but also the loop graphs consisting of the normal Green functions, since in $1D$ space all integrals are determined by the same characterisitic wave vector $ak_{\text{char}} \sim (\tilde{\alpha}/J)^{1/2}$. For the same reasons all the third-order diagrams appearing in $\Sigma_{(2)}^{+-}$ and $\Sigma_{(3)}^{++}$ are important. Without dwelling on the details of the calculations, let us give the final results:

$$C_{(1)}^{(0)} \approx (2\tilde{\alpha} S)^{1/2} \sin \theta_0 \left[1 + \frac{1}{\pi S} \left(\frac{2\tilde{\alpha}}{J} \right)^{1/2} \frac{1}{\sin \theta_0} \right]^{1/2}, \qquad (53)$$

$$M_{z,(1)}^{(0)} = \frac{2\mu S}{v_0} \cos \theta_0 \left[1 - \frac{1}{2^{1/2} \pi S} \left(\frac{\tilde{\alpha}}{J} \right)^{1/2} \sin \theta_0 \right]. \qquad (53')$$

The presence of a finite spin wave velocity means that the correlators of the transverse X and Y components of the spin fall off a large distances in accordance with a power law (see Refs. 22, 36, and 37 for details). Strictly speaking, formulas (53) and (53') are valid for $(h_c - h)/h_c \gg \alpha/J$, i.e., outside the immediate vicinity of the phase transition point. However formula (53') correctly describes the field dependence of the longitudinal magnetization in the opposite limiting case as well, i.e., for $(h_c - h)/h_c \ll \tilde{\alpha}/J$:

$$M_{z,(1)}^{(0)} \propto \frac{h}{h_c} \left[1 - O\left(1 - \frac{h}{h_c} \right)^{1/2} \right]. \qquad (54)$$

The reason for this is that the value of the longitudinal-susceptibility exponent for this orientational transition at $T = 0$ is given exactly in the framework of the Gaussian theory [$\alpha = 1/2$, Eq. (54); see Sec. 4 for details].

For $T \neq 0$ the long-range order in $1D$ space is always destroyed, and the correlation of the spins in the XY plane falls off exponentially. The fact that the correlation length is finite means that a $1D$ magnet is in the paramagnetic phase at any nonzero temperature. The destruction of the long-range order occurs through a breakup of the spin structure into domains. At low temperatures the dimensions of the domains are exponentially large and so for wave vectors that are not too small the magnons can be treated as elementary excitations above a homogeneous ferromagnetic ground state. The velocity of such magnons is finite:

$$C_1^{T} = C_1^{(0)} \left[1 - \frac{2}{S} (1 + 2 \operatorname{ctg}^2 \theta_0) \varphi_\varepsilon^{(1)}(T) \right]^{1/2} \qquad (55)$$

[the notation is the same as in (52)]; the asymptotic forms of this function are:

$$\varphi_\varepsilon^{(1)}(T) = \begin{cases} \frac{1}{2^{5/2}\pi} \left(\frac{T}{JS}\right)^{1/2} \left(\frac{T}{\tilde{\alpha} S \sin^2 \theta_0}\right)^{3/2} \\ \times ((1+12 \operatorname{ctg}^2 \theta_0)/(1+2 \operatorname{ctg}^2 \theta_0)) \zeta(2), \quad T \ll \tilde{\alpha} S \sin^2 \theta_0 \\ R(\theta) \left(\frac{T}{JS}\right)^{1/2} \left(\frac{T}{\tilde{\alpha} S \sin^2 \theta_0}\right)^{1/2}, \quad T \gg \tilde{\alpha} S \sin^2 \theta_0, \quad R(\theta) \sim 1 \end{cases} \qquad (55')$$

4. PHASE TRANSITIONS IN EASY-PLANE FERROMAGNETS

In this section we discuss the critical properties of easy-plane ferromagnets near the low-temperature phase transitions. These transitions can occur both upon a change in the magnetic field and upon a change in the single-ion anisotropy constant. In the latter case the transition is brought on purely by quantum effects: At large $\tilde{\alpha}/J$ the quantum fluctu-

ations are so strong that under certain conditions they destroy the order in the XY plane. The properties of the low-temperature phase transitions are conveniently studied by first contructing an effective classical functional H_{eff} with the same value of the partition function as for the original quantum functional.[38] For an easy-plane ferromagnet one can construct H_{eff} by making use of the equivalence of the low-temperature limit of the Vaks-Larkin-Pikin[39] diagram technique to the diagram technique of the classical theory of phase transitions.[40] This analogy has been used in Refs. 11 and 41.

Let us first take $S = 1$. Then we easily obtain the following formula for the effective classical functional:

$$H_{eff} = \frac{1}{2}\sum_{\mathbf{k}}\sum_{\omega}[K^{(0)}(\mathbf{k},\omega)]^{-1}|\sigma_{\mathbf{k},\omega}|^2 + \frac{JZ}{8N}\sum_{\mathbf{k}}\sum_{\omega}|\sigma|^4+\ldots, \tag{56}$$

where $\sigma_{\mathbf{k},\omega}$ is the two-component order parameter, and $\omega = 2\pi nT$ is the Matsubara frequency. The appearance of this frequency in the classical functional (56) reflects the quantum nature of the original spin Hamiltonian.

The bare propagator in the strong-field and weak-field regions is:[5)]

$$K^{(0)}(\mathbf{k},\omega) = \begin{cases} [\Delta + i\lambda_1\omega + \lambda_2\omega^2 + J(ka)^2]^{-1}, & h < h_1 = {}^1/_2\alpha(1-2JZ/\alpha)^{1/2}, \quad (57) \\ [\delta + i\omega + J(ka)^2]^{-1}, & h > h_c = \alpha/2. \quad (58) \end{cases}$$

Here we have used the notation

$$\Delta = \frac{\alpha}{2}\left(1-\frac{2JZ}{\alpha}\right)\left[1-\left(\frac{h}{h_1}\right)^2\right], \quad \delta = 2(h-h_c), \quad \lambda_1 = \frac{2h}{\alpha}, \quad \lambda_2 = \frac{1}{2\alpha}. \tag{59}$$

The condition $\Delta = 0$ determines the line of phase transitions $\alpha_c = \alpha_c(h)$ in the mean field approximation, while the condition $\delta = 0$ determines the critical field $h = h_c$ for the orientational transition.

For $T \neq 0$ the Matsubara frequency is discrete, and so its appearance in functional (56) does not affect the critical behavior. The values of the exponents in this case are the same as for the transition at the Curie point.[38] The situation is different at $T = 0$: the Matsubara frequency becomes a continuous variable, and the wave vector, as it were, acquires an additional dimension. For orientational transitions, as was shown in Refs. 42–46, the linear frequency dependence of $K^{(0)}$ [see (58)] leads to an extremely peculiar critical behavior: Below the critical dimensionality $D_{\text{cr}} = 2$ the values of the three exponents ν, η, and γ remain the same as in Landau theory, while the rest of the exponents are given by the scaling relations: $\beta = D/4$, $\alpha = (2-D)/2$, $\delta = (D+4)/D$.

The transition upon a change in the anisotropy constant has been considered[11] in $3D$ space only for $h = 0$. According to (57), in the absence of field the frequency appears in the effective functional to the same power as does the wave vector, and so the critical behavior of a D-dimensional quantum system at $T = 0$ turns out to be the same as the behavior in the vicinity of T_c for its $(D+1)$-dimensional classical analog. This has been confirmed for $1D$ systems by the results of numerical experiments.[47]

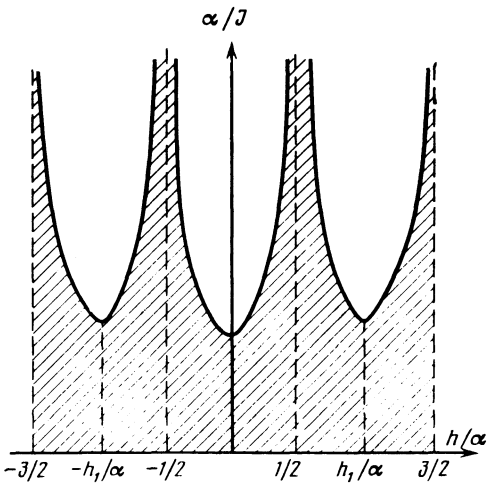

FIG. 3. Phase diagram in the variables α/J and h/α for the cases $S = 2$. The hatched region corresponds to the ordered phase. The equations of the critical lines $\alpha_c(h)$ in the mean field approximation are found in Ref. 49; $h_1 \approx 0.95\alpha$.

It is seen from (57) that the imposition of a magnetic field sharply alters the characteristic of the critical behavior: the propagator acquires a linear frequency term which is much larger than the quadratic term in the region of the characteristic frequencies $\omega_{\text{char}} \sim \Delta/\lambda_1$. For this reason the values of the exponents in a nonzero field are the same as for the orientational transition at the point h_c. We note that the difference in the structure of propagators (57) and (58) in the high-frequency region [the presence of a quadratic frequency term in (57)] does not affect the values of the exponents but leads only to a shift in the transition point $\alpha_c(h)$ from the value obtained in the mean field approximation. (Recall that there is no such shift for the orientational transition).[9]

For arbitrary S the phase diagram of an easy-plane ferromagnet in the $(\alpha/J, h/\alpha)$ plane consists of repeating blocks of length $\Delta h = \alpha$ (see Fig. 3).[48] In each of the blocks there is a single value of the field h_m at which the transition following a change in the anisotropy constant is characterized by the same exponents as is the transition in the equivalent classical system in a space of one-higher dimension. In the mean field approximation

$$h_m = \frac{\alpha}{2m}[3m^2 - S^2 - S + (S(S+1) - m(m+1))^{1/2}(S(S+1) - m(m-1))^{1/2}]. \tag{60}$$

At all the remaining values of h the critical exponents are the same as for the orientational transition at the point $h = h_c$.

For finite, but extremely low temperatures the quantum critical behavior will be manifested everywhere except in the

immediate vicinity of the transition point. The transition from the quantum critical behavior to the classical critical behavior will occur when the characteristic frequencies ω_{char} acquire a substantially discrete nature, i.e., when $\omega_{char} \sim 2\pi T$ or (for $S=1$)

$$\frac{\alpha-\alpha_c^T}{\alpha_c^T} \sim \begin{cases} (h/\alpha_c^T)(T/\alpha_c^T), & T/h \ll 1, \\ (T/\alpha_c^T)^2, & T/h \gg 1. \end{cases} \quad (61)$$

Here it must be taken into account that at room temperature the transition point itself is shifted to lower fields:

$$\frac{\alpha_c^{(0)}-\alpha_c^T}{\alpha_c^{(0)}} \sim \begin{cases} (h/\alpha_c^{(0)})^{1/2}(T/\alpha_c^{(0)})^{1/2}, & T/h \ll 1, \\ (T/\alpha_c^{(0)})^2, & T/h \gg 1. \end{cases} \quad (62)$$

Here $\alpha_c^{(0)}$ is the value of α_c at $T=0$.

Unfortunately, the experimental observation of such a transition in $3D$ space is hardly feasible, since the quantum critical behavior of $3D$ systems is described by Landau theory and the classical fluctuation region begins at values

$$\left|\frac{\alpha-\alpha_c^T}{\alpha_c^T}\right| \leqslant \left(\frac{T}{\alpha_c^T}\right)^2, \quad (63)$$

which for $T \ll h$ lie well inside the region of classical critical behavior and for $T \gg h$ lie close to the boundary of this region.

Quantum critical behavior near phase transitions both on a charge in the anisotropy constant and on a change in the magnetic field may well be observable in experiments on quasi-one-dimensional easy-plane ferromagnets such as $CsNiF_3$.

I would like to take this opportunity to thank M. I. Kagan for many discussions of this study at various stages and for many comments. I am also grateful to A. F. Andreev, D. A. Yablonskiĭ, and V. N. Krivoruchko for a discussion of the results.

APPENDIX

The integral equation for the total amplitude $\Phi_{1,2,-1-2;0}$ is written (see Fig. 2c):

$$\Phi_{1,2,-1-2;0} = \tilde{\psi}_4^{rel} - \frac{6}{N^2}\sum_{3,4}\left(\frac{\Phi_{3,4,-3-4;0}}{3-\nu_3-\nu_4-\nu_{3+4}} + \frac{1}{4}\frac{JZS}{\varepsilon_3^{(0)}} \times B_3(1+2n_3)\right)(\psi_6^{ex}) - \frac{1}{2NS}\sum_3\left[\left(\frac{\Phi_{3,1+2-3,-1-2;0}}{3-\nu_3-\nu_{1+2-3}-\nu_{1+2}} + \frac{JZS}{6}\frac{B_{1+2}}{\varepsilon_{1+2}^{(0)}}(1+2n_{1+2})\right)(\psi_4^{ex}) + \dots\right], \quad (A.1)$$

where the ellipses denote a cyclic permutation, and the potentials ψ_4^{ex} and ψ_6^{ex} and the bare amplitude $\tilde{\psi}_4^{rel}$ are given by (13), (14), and (17), respectively. Let us first consider the case $T=0$. We introduce the notation:

$$X=\frac{1}{N^2}\sum_{1,2}\frac{\Phi_{1,2,-1-2;0}}{I_{1,2}}, \quad Y=\frac{1}{N^2}\sum_{1,2}\Phi_{1,2,-1-2;0},$$
$$Z=\frac{1}{N}\sum_1\frac{\Phi_{1,-1,0;0}}{1-\nu_1}, \quad F=\frac{1}{N}\sum_1\Phi_{1,-1,0;0}, \quad (A.2)$$
$$E=\frac{1}{N^2}\sum_{1,2}\frac{\Phi_{1,2-1-2;0}}{1-\nu_1}, \quad R=\frac{1}{N^2}\sum\frac{\Phi_{1,2-1-2;0}}{I_{1,2}(1-\nu_1)},$$

$$I_{1,2}=3-\nu_1-\nu_2-\nu_{1+2}.$$

Substituting the combination Y, Z, E, F into the right-hand side of (A.1) and summing over the wave vectors, we obtain a system of four equations in six unknowns:

$$Y=\delta_1+3X[1-(1-1/S)^{-1/2}], \quad (A.3)$$

$$\left(1-\frac{q}{4S}\right)F=\delta_2+\delta_3X+\delta_4Y-\frac{q}{4S}Z, \quad (A.4)$$

$$\left(1-\frac{q}{4S}\right)E=\delta_5-\frac{q}{2S}R+\delta_6X+\delta_7Y, \quad (A.5)$$

$$\delta_8Z=\delta_9+\delta_{10}X+\delta_{11}Y+\delta_{12}F-\frac{q}{2S}\left[\left(1-\frac{q}{4S}\right)E+\frac{q}{2S}R\right]. \quad (A.6)$$

The expressions for the coefficients $\delta_1-\delta_{12}$ are awkward, and we shall not give them here.

Using (A.5), we can eliminate the dependence on E and R on the right-hand side of (A.6). We can then write a closed system of linear equations for F, Y, and Z in which X plays a role of a parameter. Solving this system we find

$$Z=K+2[1-(1-1/S)^{1/2}]X, \quad (A.7)$$

where

$$K=\frac{(1-1/2S)\lambda}{1-(1-1/S)^{1/2}}\left[\frac{4-q}{3S}-(1-(1-1/S)^{1/2})\right]\tilde{\psi}_4^{rel}, \quad (A.8)$$
$$\lambda=W+\frac{1}{2S(1-1/2S)}.$$

After summing the terms in (36) and (37) we are satisfied that, with allowance for condition (A.7), the desired quantity

$$\Sigma_{(3)}^{+-}+\Sigma_{(4)}^{+-}=\frac{3}{2}\frac{\tilde{\alpha}\sin^2\theta_0}{JZ(1-1/2S)^{1/2}}K \quad (A.9)$$

does not depend on X. In an analogous way we can calculate the total scattering amplitude in the limit of zero wave vectors [see (40)]:

$$\Phi_{0,0,0;0}=\lim_{q\to 0}\Phi_{q,-q,0;0}.$$

Generalization of these results to the case $T \neq 0$ does not present any difficulties, since to the required accuracy the temperature dependence of the vertex $\Phi_{1,2,-1-2;0}$ is determined solely by the temperature parts of its constituent loop diagrams which consist of anomalous Green functions (see Fig. 2c). The result is that for $T \neq 0$ the quantity λ should be replaced by $\tilde{\lambda}$, where

$$\tilde{\lambda}=\lambda+2\frac{JZS}{N}\sum_p\frac{n_p}{\varepsilon_p^{(0)}}. \quad (A.10)$$

[1)] An alternative version is to work with finite sums but with two types of quasiparticles—bosons and fermions.[6]

[2)] We are talking, of course, only about slightly anisotropic magnets, for which $\alpha/J \ll 1$. When $\alpha/J \gg 1$, the long-range order in the XY plane an be completely smeared out on account of strong quantum fluctuations.[11,18,19]

[3)]A correct calculation of the temperature correction to the homogeneous precession frequency of a uniaxial collinear ferromagnet in the Dyson-Maleev formalism is given in Ref. 29.

[4)]This assertion is apparently also valid for $2D$ noncollinear systems which lack a small parameter—antiferromagnets and the XY model.[31,32]

[5)]For $h > h_c$ formula (56) can also be obtained by the coherent states technique.[42,43]

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