Diffusion of charged particles in a random magnetic field

K. P. Zybin and Ya. N. Istomin

P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow

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The motion of charged particles in a magnetic field **B** having a regular component $\mathbf{B}_0$ and large-scale fluctuations $\delta\mathbf{B}$ is analyzed. The effect of small-scale scattering on the motion of the particles in a large-scale magnetic field is taken into account. The diffusion coefficient ($D_\perp$) of the particles across the regular magnetic field $\mathbf{B}_0$ is calculated. For small-scale fluctuations of a general type there is always a diffusion across the field $\mathbf{B}_0$ with a coefficient $D_\perp = D_\parallel \overline{\delta B^2}/B_0^2$, where $D_\parallel$ is the coefficient for diffusion along the magnetic field. This diffusion arises from the scattering of particles by small-scale fluctuations.

1. In several problems in plasma physics it is necessary to deal with the motion of charged particles in random electromagnetic fields which have very different correlation lengths and correlation times. One such problem is the effect of collisions on transport in large-scale fluctuational fields[1–3]; another is the diffusion of cosmic rays in the galactic magnetic field.[4]

Let us consider the motion of particles in a magnetic field $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}(|\delta\mathbf{B}| \ll |\mathbf{B}_0|)$ strong enough that the Larmor radius r_H and the cyclotron resolution period $2\pi/\omega_H$ are much smaller than all the scale lengths and scale times of the problem. A particle can then be assumed "tied" to a magnetic line of force and constrained to move exclusively along it. In the absence of a random field component $\delta\mathbf{B}$, the particle moves across the magnetic $\mathbf{B}_0$ by diffusion:

$$r_\perp = (D_m S)^{1/2}, \qquad (1)$$

where S is the distance traversed by the particle along the magnetic field, and

$$D_m = \int_0^{+\infty} \frac{\overline{\delta B(0)\,\delta B(z)}}{B_0^2}\,dz = \frac{\overline{\delta B^2}}{B_0^2} L_c$$

is the so-called magnetic diffusion coefficient.[5] The z axis is directed along the field $\mathbf{B}_0$, and L_c is the correlation length along z of the random process $\delta\mathbf{B}$. We further assume that the particle moves along $\mathbf{B}_0$ not freely but diffusively, with a mean free path l_f ($r_H \ll l_f \ll L_c$) and with a diffusion coefficient $D_\parallel$ due to any scattering process, involving either collisions or a scattering by fluctuations. At first glance it appears that we would have

$$S = (D_\parallel t)^{1/2}. \qquad (2)$$

Substituting (2) into (1), we find[6,7]

$$r_\perp = D_m^{1/2} D_\parallel^{1/4} t^{1/4}. \qquad (3)$$

Expression (3) means that the motion of a particle across the magnetic field is not ordinary diffusion as it is usually understood:

$$D_\perp = \lim_{t\to\infty} \frac{d}{dt}\overline{r_\perp^2} = 0.$$

The motion described by (3) might be called "second-order diffusion." However, it is not completely correct to use expression (2): Expression (1) presupposes that as a line moves a distance $S \gg L_c$ away from the initial point $S_0 = 0$ along a line of force it is displaced an average distance $r_\perp = (D_m S)^{1/2}$ from the unperturbed magnetic surface. Until we have traversed a distance $S \sim L_c$, the line of force undergoes essentially no excursion from the unperturbed magnetic surface. Consequently, a particle on such a line of force undergoes an equally small displacement in the transverse direction. In expression (2), on the other hand, a particle which returns repeatedly is displaced a distance $s = (D_\parallel t)^{1/2}$, but this displacement is quite different from the length S of the line of force, which is assumed in (1).

Let us consider a simple example in which the path length S can easily be calculated exactly. Figure 1 shows a pattern of lines of the magnetic field **B**. At distance L_c there is an equiprobable transition to either the line of force which is the continuation of the given line of force or to the two adjacent lines of force, separated from the given line of force by a distance $\delta = L_c \overline{(\delta B^2)}^{1/2}/B_0$. After a time $\Delta t = L_c^2/D_\parallel$, the particle will thus probably deviate a distance δ from the line of force. As it then moves by diffusion along the line of force, the particle may return to its original line of force or a neighboring one, but there is a large probability (2/3) that it will be on a line of force which is separated from the original line by a distance δ until the distance L_c is traversed. The path traversed by the particle is thus the sum of individual segments with a length on the order of L_c.

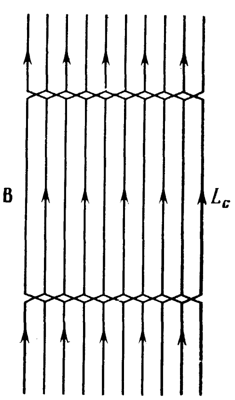

FIG. 1.

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We can thus find $\overline{V}$, the average velocity of the particle along the line of force:

$$\overline{V}=L_c/\Delta t=D_{\|}/L_c.$$

At times $t>\Delta t$, the average distance traversed by a particle along $\mathbf{B}$ is thus

$$S=\overline{V}t=D_{\|}t/L_c. \tag{4}$$

Substituting (4) into (1), we find

$$r_{\perp}=(D_m D_{\|}t/L_c)^{1/2},$$

i.e.,

$$D_{\perp}=D_m D_{\|}/L_c=(\overline{\delta B^2}/B_0^2)D_{\|}.$$

This example shows that the superposition of two independent random processes, as in our problem, leads to ordinary diffusion, not second-order diffusion.

2. We can prove this assertion in a general form. We introduce $f(\mathbf{r},\mathbf{v},t)$, the particle distribution function under the conditions specified above. The kinetic equation for this distribution function is

$$\partial f/\partial t+v_{\|}\mathbf{h}\nabla f=L(\mathbf{v})f. \tag{5}$$

The operator $L(\mathbf{v})$ describes the scattering of particles by small-scale functions, $\mathbf{h}=\mathbf{B}/|B|$ is a unit vector along the direction of the magnetic field $\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B}$, and $v_{\|}$ is the velocity of the particle along the magnetic field. We introduce the average values $\bar{f}$ and $\mathbf{B}_0$ and the fluctuating values of δf and $\delta\mathbf{B}$ in (5), under the assumption that the fluctuations are small:

$$f=\bar{f}+\delta f,\qquad |\delta f|\ll|f|,$$
$$\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B},\qquad |\delta\mathbf{B}|\ll|\mathbf{B}_0|.$$

We take an average of Eq. (5) over the ensemble of realization of the random quantity. Following Ref. 3, we find

$$\partial\bar{f}/\partial t+v_{\|}\mathbf{h}_0\nabla\bar{f}=L(\mathbf{v})\bar{f}+I_{fl},$$
$$I_{fl}=\frac{\partial}{\partial r_i}\int_{-\infty}^{+\infty}\int_{-\infty}^{t}\frac{\overline{\delta B_{i\perp}(\mathbf{r},t)\delta B_{l\perp}(\mathbf{r}',t')}}{B_0^2}v_{\|}G(\mathbf{r},\mathbf{v},t|\mathbf{r}',\mathbf{v}',t')$$
$$\times v_{\|}'\frac{\partial}{\partial r_l}\bar{f}(\mathbf{r},\mathbf{v}',t)\,d\mathbf{r}'\,d\mathbf{v}'\,dt', \tag{6}$$
$$\partial G/\partial t+v_{\|}\mathbf{h}_0\nabla G=L(\mathbf{v})G+\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{v}-\mathbf{v}').$$

In order to calculate the diffusion coefficient in Eq. (6), we need to know the properties of the operator $L(\mathbf{v})$. We restrict the discussion to Hermitian operators L having a set of discrete negative eigenvalues $\lambda_0,\lambda_1,\dots,\lambda_n,\dots$

If the fluctuations scatter the particles in such a way that the distribution function tends toward isotropy, the operator L must have these properties. Let us explain this assertion. The operator L describes the relaxation of the distribution function in a homogeneous medium. Its eigenvalues are the reciprocal relaxation times of the corresponding kinetic moments, so that they must be negative real numbers. The operator L is thus Hermitian. We also note that the distribution function $\bar{f}$ satisfies a continuity equation; it follows that the operator L has at least one eigenvalues $\lambda_0=0$. For simplicity we also assume that only one eigenvalue is zero. If there are several such values, the derivation becomes more complicated, but the final result—Eq. (17)—is the same.

For the Hermitian operator L we introduce a set of orthonormal bar and ket vectors in the standard way:

$$L|n,\mathbf{v}\rangle=-\lambda_n|n,\mathbf{v}\rangle,$$
$$\langle n,\mathbf{v}|L^{+}=-\lambda_n\langle n,\mathbf{v}|,\qquad\langle m,\mathbf{v}|n,\mathbf{v}\rangle=\delta_{mn}. \tag{7}$$

The operator L^{+} is the Hermitian adjoint of L.

We turn now to the solution of the second equation of system (6). We write G in the form

$$G=\int g(\mathbf{k})\exp\{-\lambda\tau+i\mathbf{k}(\mathbf{r}-\mathbf{r}')\}d\mathbf{k},\quad\tau=t-t'.$$

Substituting into (6), we find

$$-\lambda g(\mathbf{k})+ik_{\|}v_{\|}g(\mathbf{k})=L(\mathbf{v})g(\mathbf{k})+(2\pi)^3\delta(\mathbf{v}-\mathbf{v}'). \tag{8}$$

The quantity $k_{\|}v_{\|}$ in (8) serves as a perturbation since we have $k_{\|}v_{\|}\sim\lambda l_f/L_c$, $l_f<L_c$ where l_f is the mean free path of the particles with respect to scattering by the small-scale fluctuations of $L(\mathbf{v})$. We expand Eq. (8) in the small parameter l_f/L_c. In the zeroth approximation we find the following expression for the Green's function G from (8):

$$G(\mathbf{r},\mathbf{v},t|\mathbf{r}',\mathbf{v}',t')=\sum_{n=0}^{\infty}|n,\mathbf{v}\rangle\exp(-\lambda_n\tau)\delta(\mathbf{r}-\mathbf{r}')\langle n,\mathbf{v}'|. \tag{9}$$
$$\tau>0$$

In first-order perturbation theory we have

$$\lambda_l^{(1)}=ik_{\|}\langle l,\mathbf{v}|v_{\|}|l,\mathbf{v}\rangle; \tag{10}$$

the physical meaning of $\lambda_0^{(1)}$ is that it is a quantity proportional to the average velocity of the particles. We choose a coordinate system in which we have $\lambda_0^{(1)}=0$.

Taking into account small terms of second order, we find the diffusion corrections:

$$\lambda_l^{(2)}=k_{\|}^2\sum_{n=1}^{\infty}\langle l,\mathbf{v}|v_{\|}|n,\mathbf{v}\rangle\frac{1}{\lambda_n}\langle n,\mathbf{v}|v_{\|}|l,\mathbf{v}\rangle. \tag{11}$$

Here $\lambda_0^{(2)}=D_{\|}k_{\|}^2$, where $D_{\|}$ is the coefficient of the diffusion along the magnetic field $\mathbf{B}_0$ due to scattering by small-scale fluctuations.

Correction (11) is important only for the eigenvalue λ_0, since $\lambda_0^{(0)}=\lambda_0^{(1)}=0$. For the other eigenvalues, (11) is small. Using (11), we can write (9) as

$$G(\mathbf{r},\mathbf{v},t|\mathbf{r}',\mathbf{v}',t')=|0,\mathbf{v}\rangle(2\pi D_{\|}\tau)^{-1/2}\delta(\mathbf{r}_{\perp}-\mathbf{r}_{\perp}')$$
$$\times\exp\{-(z-z')^2/4D_{\|}\tau\}\langle 0,\mathbf{v}'| \tag{12}$$
$$+\sum_{n=1}^{\infty}|n,\mathbf{v}\rangle\delta(\mathbf{r}-\mathbf{r}')\exp(-\lambda_n\tau)\langle n,\mathbf{v}'|.$$

Here the vector $\mathbf{B}_0$ is directed along the z axis. Substituting (12) into (6), we find

$$I_{fl}=\sum_{n=0}^{\infty}\frac{\partial}{\partial r_i}\int_{-\infty}^{+\infty}\int_{-\infty}^{t}v_{\|}|n,\mathbf{v}\rangle\frac{\overline{\delta B_{i\perp}\delta B_{l\perp}}}{B_0^2}\Theta_n(\mathbf{r}-\mathbf{r}',\tau)\frac{\partial}{\partial r_l} \tag{13}$$
$$\times\langle n,\mathbf{v}'|v_{\|}'|\bar{f}(\mathbf{v}',\mathbf{r},t)\rangle d\mathbf{v}'d\mathbf{r}'dt',$$

where

$$\Theta_n(\mathbf{r}-\mathbf{r}',\tau)=\begin{cases}(2\pi D_\parallel\tau)^{-1/2}\delta(\mathbf{r}_\perp-\mathbf{r}_\perp')\exp\{-(z-z')^2/4D_\parallel\tau\},\\ n=0\\ \exp(-\lambda_n\tau)\delta(\mathbf{r}-\mathbf{r}'),\quad n\geqslant 1.\end{cases}$$

Since we are interested in the solution of Eq. (6) over times $t\gg\lambda^{-1}$, we have $L(\mathbf{v})\bar{f}^{(0)}=0$ in the zeroth approximation; i.e., the function $\bar{f}^{(0)}$ is

$$\bar{f}^{(0)}=N(\mathbf{r},t)\,|0,\mathbf{v}\rangle. \tag{14}$$

From (13) and (14) we find

$$\langle 0,\mathbf{v}|I_{fl}\rangle=\frac{\partial}{\partial r_i}\sum_{n=1}^{\infty}\langle 0,\mathbf{v}|v_\parallel|n,\mathbf{v}\rangle\frac{1}{\lambda_n}\langle n,\mathbf{v}|v_\parallel|0,\mathbf{v}\rangle$$
$$\times\frac{\overline{\delta B_{i\perp}\delta B_{l\perp}}}{B_0^2}\frac{\partial}{\partial r_l}N. \tag{15}$$

The summation in (15) begins with $n=1$, since we have $\lambda_0^{(1)}=0$ according to (10). Using (11) and (7), we can write expression (15) in the form

$$\langle 0,\mathbf{v}|I_{fl}\rangle=\frac{\partial}{\partial r_i}\frac{\overline{\delta B_{i\perp}\delta B_{l\perp}}}{B_0^2}D_\parallel\frac{\partial N}{\partial r_l}. \tag{16}$$

Assuming the fluctuations $\delta\mathbf{B}$ to be isotropic, we finally find our diffusion equation from (6) and (14):

$$\frac{\partial N}{\partial t}=\frac{\partial}{\partial r_i}\{D_\parallel h_{0i}h_{0l}+D_\perp(\delta_{il}-h_{0i}h_{0l})\}\frac{\partial N}{\partial r_l}, \tag{17}$$

where the transverse diffusion coefficient is $D_\perp=(\overline{\delta B^2}/B_0^2)D_\parallel$.

Over times $t\gg L_c^2/D_\parallel$ and scale lengths $L\gg L_c$, the motion across the magnetic lines of force is thus purely diffusive. In the literature, however, we find the assertion[8] that for any ordinary diffusion process describable by a parabolic equation, i.e., a heat-conduction equation, the diffusion coefficient is zero in a random magnetic field. We thus consider the simplest approximation of a diffusion process which leads to a heat-conduction equation, Brownian motion of a particle in a random velocity field.

3. We assume that the motion of a charged particle is described by the equation

$$d\mathbf{r}/dt=V(t)\mathbf{h}, \tag{18}$$

where $\mathbf{h}$ is a unit vector along the direction of the magnetic field, given by

$$\mathbf{h}=(\mathbf{B}_0+\delta\mathbf{B})/|\mathbf{B}_0+\delta\mathbf{B}|,$$

and $V(t)$ is a Gaussian random function with a correlation scale time τ_c so small in comparison with the typical time for a change $\delta\mathbf{B}$ that it can be approximated by the expression

$$\langle V(t)V(t')\rangle=2D_\parallel\delta(t-t').$$

The random process $V(t)$, which is statistically independent of $\delta\mathbf{B}$, describes diffusive motion of a particle along a magnetic field with the diffusion coefficient $D_\parallel$.

We introduce the probability density for finding a particle at the time t at the point $\mathbf{r}$:

$$F(\mathbf{r},t)=\overline{f(\mathbf{r},t)}=\overline{\langle\varphi(\mathbf{r},t)\rangle}=\overline{\langle\delta(\mathbf{r}-\mathbf{r}(t))\rangle}.$$

Averaging over the random process $V(t)$ (the angle brackets indicate the averaging) by the method of functional integration,[9] we find from (18) a standard diffusion equation:

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial r_i}D_\parallel h_ih_k\frac{\partial}{\partial r_k}f. \tag{19}$$

The further averaging over the random field $\delta\mathbf{B}$ is usually carried out by a standard method involving a splitting up of f into an average function F and a small random increment δf. Substituting this expression into (19), we find the following equations for F:

$$\frac{1}{D_\parallel}\frac{\partial F}{\partial t}=\frac{\partial}{\partial r_i}h_{i0}h_{k0}\frac{\partial}{\partial r_k}F+\frac{\partial}{\partial r_i}\overline{b_ib_k}\frac{\partial}{\partial r_k}F$$
$$+\frac{\partial}{\partial r_i}\overline{(b_ih_{0k}+b_kh_{0i})\frac{\partial}{\partial r_k}\delta f}+\frac{\partial}{\partial r_i}\overline{b_ib_k\frac{\partial}{\partial r_k}\delta f}, \tag{20a}$$

$$\frac{1}{D_\parallel}\frac{\partial}{\partial t}\delta f=\frac{\partial}{\partial r_i}h_{i0}h_{k0}\frac{\partial}{\partial r_k}\delta f+\frac{\partial}{\partial r_i}\overline{b_ib_k}\frac{\partial}{\partial r_k}\delta f$$
$$+\frac{\partial}{\partial r_i}(b_ih_{0k}+b_kh_{0i})\frac{\partial F}{\partial r_k}-\frac{\partial}{\partial r_i}\overline{b_ib_k\frac{\partial}{\partial r_k}\delta f}+\frac{\partial}{\partial r_i}(b_ib_k$$
$$-\overline{b_ib_k})\frac{\partial F}{\partial r_k}+\frac{\partial}{\partial r_i}(b_ib_k-\overline{b_ib_k})\frac{\partial}{\partial r_k}\delta f+\frac{\partial}{\partial r_i}(b_ih_{0k}+b_kh_{0i})\frac{\partial}{\partial r_k}\delta f$$
$$-\frac{\partial}{\partial r_i}\overline{(b_ih_{0k}+b_kh_{0i})\frac{\partial}{\partial r_k}\delta f}, \tag{20b}$$

where $b_i=\delta B_i/B_0; |b_i|\ll 1$. Assuming $\delta f\sim b$ and retaining in (20) those terms which are quadratic in $\mathbf{b}$, we find that the second term on the right side of Eq. (20a) cancels out completely. This cancellation is interpreted as meaning that there is no transverse diffusion.[8] However, it can be seen from (20b) that the terms of the type $b_k(\partial/\partial r_k)\delta f$ are not quadratic in $\mathbf{b}$ but terms of first order, since the width of a perturbation δf in the transverse direction is $\delta\sim L_c(\overline{b}^2)^{1/2}$. For this reason, we cannot carry out a successive expansion in powers of $\mathbf{b}(\mathbf{r})$ in (20b), since all the quantities containing $b_k(\partial/\partial r_k)\delta f$ are terms of first order in $\mathbf{b}(\mathbf{r})$, not of second order, as was assumed in Ref. 8. To find the answer we take the following approach: We carry out an averaging first over the random process $\mathbf{b}(\mathbf{r})$ and then over $V(t)$. The result will of course be independent of the order of the averaging steps, but in the second case the series in $\mathbf{b}(\mathbf{r})$ is found in a natural way:

$$\frac{\partial\varphi}{\partial t}=-V(t)\frac{\partial\varphi}{\partial z}+V(t)\frac{\partial}{\partial r_k}\int_{-\infty}^{t}V(\tau)$$
$$\times\overline{b_k(\mathbf{r})b_i(\mathbf{r}(\tau))}\,d\tau\frac{\partial}{\partial r_i}\varphi(\mathbf{r},t). \tag{21}$$

From (21) we find

$$D_{\perp ik}=\int_{-\infty}^{t}\langle V(t)V(\tau)\overline{b_{k_\perp}(\mathbf{r})b_{i_\perp}(\mathbf{r}(\tau))}\rangle d\tau,$$

which gives us the following expression in first order in b^2:

$$D_{\perp ik}=D_\parallel\overline{b_ib_k}.$$

This expression agress with the results above.

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