Theory of emission by relativistic particles in amorphous and crystalline media

N. V. Laskin, A. S. Mazmanishvili, N. N. Nasonov, and N. F. Shul'ga

Physicotechnical Institute, Academy of Sciences of the Ukrainian SSR, Kharkov
(Submitted 24 December 1984)
Zh. Eksp. Teor. Fiz. **89**, 763–780 (September 1985)

The influence of multiple scattering and of polarization of the medium on the intensity of coherent emission of low-frequency radiation by relativistic particles in a crystal is investigated. It is shown that the method of functional integration can be used to describe the emission of radiation by high-energy particles in matter. The method is used to reproduce the results of Migdal and Ter-Mikaelyan on the effect of multiple scattering and of polarization of the medium on the emission of bremsstrahlung by fast charged particles in an amorphous medium at low frequencies, and a formula is obtained for the effect of multiple scattering by chains of atoms and of polarization of the medium on the intensity of coherent emission by ultrarelativistic particles in a crystal. It is shown that, when particles propagate in a *crystal*, multiple scattering may have a significant effect on the way they emit at much lower particle energies, and in a much greater range of emission frequencies, than in an amorphous medium. This opens up new possibilities for studying the Landau-Pomeranchuk effect on many existing accelerators.

1. INTRODUCTION

The process of emission of radiation by a fast charged particle in a medium takes place over an extended region oriented in the direction of the particle momentum. In general, if the particle collides with a large number of atoms in this region, it will not emit radiation the way it does in a low-density atomic gas. Either amplification or attenuation of the emission by relativistic particles can then occur. This conclusion was first reported by Ter-Mikaelyan,[1] who studied the emission of radiation by fast electrons in a crystal. He showed that the motion of a particle of arbitrarily large energy in a crystal may be accompanied by coherent and interference effects, and that these effects ensure that much more radiation can be emitted in a crystal than in an amorphous medium. The amplification effect occurs when the particle moves nearly parallel to one of the crystallographic axes, and if there is a large number of lattice atoms within the region in which the radiation is formed (the coherence length l).

Landau and Pomeranchuk then showed[2] that, in an amorphous body, the increase in the size of the region in which the radiation is formed with increasing energy leads to a significant reduction in bremsstrahlung. The effect they studied (the Landau-Pomeranchuk effect) occurs when the mean square multiple scattering angle within the coherence length is greater than the square of the characteristic angle of emission by a relativistic particle. Landau and Pomeranchuk[2] gave general formulas for the emission of radiation by a fast particle in a medium at low frequencies, and outlined a method for calculating the mean spectral density of the emission in an amorphous medium. They estimated the emission spectrum in the case where the suppression effect is considerable. Ter-Mikaelyan subsequently showed[2] that the bremsstrahlung from a relativistic particle in an amorphous medium is suppressed not only by multiple scattering but also by the polarization of the medium, and that the latter has an important effect on the emission in a broader range of particle energies than multiple scattering.

The quantitative theory of emission by relativistic particles in amorphous and crystalline media has, however, developed out of different approaches to the description of the particle-medium interaction, and this has made it much more difficult to exhibit general relationships for, and distinguishing features of, the emission process in these cases. The first quantitative results on the effect of multiple scattering on the emission of radiation by a fast particle in an amorphous medium were obtained by Migdal,[4] who used his own method based on the kinetic equation for the position and velocity distribution function of particles in the medium. The method was subsequently employed to investigate the effect of many other factors on the emission of radiation, including, for example, recoil in emission, photon absorption, and so on (see, for example, Ref. 5 and the references therein). The theory of emission of radiation by relativistic particles in crystals has also been developed using the first Born approximation of perturbation theory.[1,5–8] However, no quantitative results have been obtained so far on the emission spectrum of fast particles within the framework of the method outlined in Ref. 2.

Analysis of the range of validity of the Born theory of coherent emission by relativistic particles in crystals has shown[9,10] that this theory is valid if there is no particle channeling and they do not pass over potential barriers, and if the angle of scattering in the crystal within the coherence length is small in comparison with the characteristic angle of emission of radiation by the particle. New emission effects arise when either of these conditions is violated. Moreover, it has been found that by no means all processes have been investigated even within the framework of the theory of coherent emission. In particular, the effect of multiple scattering on emission has not been examined.

The emission of radiation by relativistic particles in a crystal in the presence of channeling and potential-barrier

 0038-5646/85/090438-10$04.00

crossing has attracted considerable attention in recent years (see the reviews in Refs. 10–13 and the references therein). It was shown that the curving of the trajectory as the particle moves in the continuous potential along crystallographic axes and planes produces a considerable change in the emission spectrum as compared with the Born theory.

The emission of radiation by an ultrarelativistic particle was considered in Refs. 14 and 15 in the case of a thin crystal (whose thickness T is small in comparison with the coherence length l) for different ratios between the mean square scattering angle $\overline{\vartheta^2}$ in the crystal and the square of the characteristic emission angle $\vartheta_k^2 \sim \gamma^{-2}$, where γ is the Lorentz factor of the particle. It was shown that coherence effects do not appear in emission when $\overline{\vartheta^2} \ll \vartheta_k^2$. Violation of this condition gives rise to the suppression of coherent emission, as a result of which the intensity emitted by the particle in a crystal is reduced as compared with the Born theory of coherent emission.

The present paper is devoted to low-frequency emission by ultrarelativistic particles in thick crystals ($T \gg l$), in which multiple scattering and polarization of the medium can have a considerable influence on emission.

We note that the first attempt at taking into account the effect of the polarization of the medium on the coherent emission spectrum of fast particles in a crystal was made by Ter-Mikaelyan.[16] Bazylev and Zhevago subsequently examined in detail the influence of the polarization of the medium on emission by channeled particles. They concentrated their attention on the motion of particles in a crystal under planar channeling conditions. More recently, it was shown[18,19] that the polarization of the medium had a considerable effect on emission at low frequencies, not only in the case of channeling but also when this phenomenon was absent.

There is particular interest in emission by high-energy particles in a crystal almost parallel to one of the crystallographic axes, because coherence and interference effects are then particularly well defined. It has been noted[19] that the emission of low-frequency radiation may then be significantly affected, not only by polarization of the medium, but also by multiple scattering of the particles by chains of atoms in the crystal, and multiple scattering in fact becomes the dominant effect in a broad range of particle energy and photon frequency. However, quantitative results were not obtained for this effect.

We shall show below that, when radiation is emitted by particles in a crystal, the emission effects are similar to those in an amorphous medium, but often occur at much lower particle energies and in broader frequency intervals. Moreover, it turns out that the emission of radiation by ultrarelativistic particles in a crystal gives rise to effects that are actually absent from amorphous media.

General formulas for the emission spectral density, which describe emission by ultrarelativistic particles in a thin layer of a medium, are presented in Section 2. These formulas enable us to examine how multiple scattering and polarization of the medium affect the intensity of coherent and bremsstrahlung emission by fast particles (electrons and positrons) in both crystalline and amorphous media from a unified point of view.

In a previous brief note,[20] we drew attention to the fact that the average emission spectral density due to relativistic particles in a medium could be determined by the method of functional integration. This method is used in Sections 3 and 4 below to reproduce the Migdal[4] and Ter-Mikaelyan[3,5] formulas for the effect of multiple scattering and of polarization of the medium on bremsstrahlung by high-energy particles in an amorphous medium, and to derive a formula for the effect of multiple scattering and of polarization of the medium on the intensity of low-frequency coherent emission by fast particles in a crystal.

Section 4 compares basic characteristics of the radiation emitted by ultrarelativistic particles in crystalline and in amorphous media. It is shown that an effect analogous to the Landau-Pomeranchuk effect (whereby radiation by fast particles in an amorphous medium is suppressed) is also possible in a crystal. However, in contrast to the amorphous medium, the radiation produced in a crystal is not the usual bremsstrahlung, but coherent emission by ultrarelativistic particles. The essential point is then that the conditions for the suppression of coherent emission are satisfied at much lower particle energies, and in a much wider range of emitted-photon frequencies, than in the amorphous medium.

Section 5 is devoted to the special case of emission at moderate particle energies, at which relativistic particles produce dipole radiation in a crystal. It is shown that, in contrast to amorphous media, multiple scattering in a crystal has a significant effect on the emitted radiation, not only for $\gamma^2\vartheta_l^2 \gg 1$, but also when $\gamma^2 \overline{\vartheta_l^2} \ll 1$, where $\overline{\vartheta_l^2}$ is the mean square of the particle scattering angle within the coherence length in the medium.

All the results reported below were obtained within the framework of classical electrodynamics. The use of this approximation in the description of the radiation emitted by high-energy particles in a crystal is justified if recoils on emission can be neglected, and if the particle collides with a large number of atoms in the medium within the coherence length. These conditions are satisfied in a wide range of particle energies and emitted-photon frequencies.[9,10]

2. SPECTRAL DENISTY OF LOW-FREQUENCY RADIATION EMITTED BY A RELATIVISTIC PARTICLE IN A MEDIUM

In classical electrodynamics, the spectral energy density emitted by an electron as it moves on a path $\mathbf{r}(t)$ in a medium is given by[21]

$$\frac{dE}{d\omega} = \frac{e^2}{4\pi^2} \int do \left| \left[\mathbf{k}, \int_{-\infty}^{\infty} dt \mathbf{v}(t) \exp\{i(\omega t - \mathbf{k}\mathbf{r}(t))\} \right] \right|^2, \quad (2.1)$$

where ω and $\mathbf{k}$ are, respectively, the frequency and wave vector of the radiated wave, $k^2 = \varepsilon_p(\omega)\omega^2$, $\varepsilon_p(\omega)$ is the permittivity of the medium, $\mathbf{v}(t)$ is the particle velocity, and do is the solid-angle element in the direction of emission. (Here and in what follows, we shall use the system of units in which the velocity of light is set equal to unity.)

By integrating with respect to the emission angles in (2.1), we find

$$\frac{dE}{d\omega}=\frac{e^2k}{\pi}\int_{-\infty}^{\infty}dT\int_{-\infty}^{\infty}d\tau e^{-i\omega\tau}\left[\mathbf{v}(T)\mathbf{v}(T+\tau)-\frac{1}{\varepsilon_p(\omega)}\right]$$

$$\times\frac{\sin k|\mathbf{r}(T+\tau)-\mathbf{r}(T)|}{|\mathbf{r}(T+\tau)-\mathbf{r}(T)|}. \qquad (2.2)$$

Typical values of the scattering angle of a relativistic particle in a medium are small, so that the integrand in (2.2) can be expanded in terms of the scattering angle. Using the result

$$\mathbf{v}(T+\tau)\approx\mathbf{v}(1-{}^1/_2\boldsymbol{\vartheta}^2(\tau))+\boldsymbol{\vartheta}(\tau),\quad |\boldsymbol{\vartheta}|\ll v, \qquad (2.3)$$

where $\mathbf{v}=\mathbf{v}(T)$ and $\mathbf{v}\boldsymbol{\vartheta}(\tau)=0$, we find that[1)]

$$\frac{dE}{d\omega}=-\frac{e^2k}{\pi}\int_{-\infty}^{\infty}dT\int_{-\infty}^{\infty}\frac{d\tau}{\tau}e^{-i\omega\tau}\left[\frac{1}{\varepsilon_p(\omega)}-v^2+\frac{1}{2}\boldsymbol{\vartheta}^2(\tau)\right]$$

$$\times\sin k\left\{v\tau-\frac{1}{2}\int_0^\tau dt\boldsymbol{\vartheta}^2(t)+\frac{1}{2\tau}\left(\int_0^\tau dt\boldsymbol{\vartheta}(t)\right)^2\right\}. \qquad (2.4)$$

Equation (2.4) is valid if the scattering angle $\vartheta(\Delta\tau)$ within the interval $\Delta\tau$ providing the principal contribution to the integral with respect to τ in (2.4) is small, i.e., when $\vartheta(\Delta\tau)\ll 1$. The order of magnitude of $\Delta\tau$ can be determined from the relation

$$\omega\Delta\tau-k(v\Delta\tau-\Delta\tau\boldsymbol{\vartheta}^2(\Delta\tau))\sim 1.$$

The length $l=v\Delta\tau$ traversed by the particle in the time $\Delta\tau$ is called the coherence length.[5,22] This length is of fundamental significance in the study of emission by relativistic particles in a medium because, within this length, there is significant interference between waves emitted by the particles at different points along their path.

If the scattering angle within the coherence length is small in comparison with the characteristic angle of emission by a relativistic particle $\vartheta_l\ll\gamma^{-1}$, we can expand (2.4) in terms of the small parameter $\gamma\vartheta_l$. The first order of this expansion corresponds to the dipole approximation, for which the spectral density of the emitted radiation is given by

$$\frac{dE}{d\omega}=\frac{e^2\omega}{2\pi}\int_{\delta_p}^{\infty}\frac{d\nu}{\nu^2}\left[1-2\frac{\delta_p}{\nu}\left(1-\frac{\delta_p}{\nu}\right)\right]|\mathbf{W}(\nu)|^2, \qquad (2.5)$$

where

$$\delta_p=\omega(1-v\varepsilon_p^{1/2}(\omega)),\quad \mathbf{W}(\nu)=\int_{-\infty}^{\infty}dt\dot{\boldsymbol{\vartheta}}(t)e^{i\nu t}.$$

These formulas generalize the corresponding results of Ref. 10 to the case where emission by a particle is affected by the polarization of the medium. The derivation of these formulas does not make use of any specific law of motion of the particle in the medium, so that the final formulas can be used for both amorphous and crystalline media. The difference between the emission processes in these two cases will appear only when the formulas are averaged over the scattering angles and over the positions of the atoms in the medium.

We note in this connection that the spectral density given by (2.4) is the functional $E'\{\boldsymbol{\vartheta}(\tau)\}$ of the random values of the scattering angle $\boldsymbol{\vartheta}(\tau)$ in the medium. This functional must be averaged over all the possible realizations of the random process $\boldsymbol{\vartheta}(\tau)$. The essential point then is that the functional (2.4) is a Gaussian in the random variable $\boldsymbol{\vartheta}(\tau)$. Since this random variable is also a simple Markov process, the averaging can be performed analytically by the method of functional integration.[23]

In the ensuing presentation, we shall be interested in the emission of radiation by a relativistic particle (electron or positron) in a medium at frequencies $\omega\gtrsim\gamma\omega_p$, where ω_p is the plasma frequency, i.e., we shall consider the case where both multiple scattering and polarization of the medium may have a significant effect on emission. It is well-known[5] that the permittivity of the medium can be written in the following form in this frequency range:

$$\varepsilon_p(\omega)\approx 1-\omega_p^2/\omega^2,\quad \omega_p\ll\omega,$$

and we can expand in terms of the parameter ω_p/ω.

3. FUNCTIONAL APPROACH TO THE EFFECT OF MULTIPLE SCATTERING ON EMISSION BY AN ULTRARELATIVISTIC PARTICLE IN AN AMORPHOUS MEDIUM

We shall begin by considering the emission of low-frequency radiation by an ultrarelativistic electron in an amorphous medium. We shall show that the effect of multiple scattering on emission by the particle in the medium can then be taken into account by the method outlined in Ref. 2, in which the average spectral density is calculated for a fast particle in a medium by means of functional integration.

It is well-known[5] that, in an amorphous medium, the distribution of the particles over the angles ϑ at time τ is given by

$$P(\boldsymbol{\vartheta})=\frac{1}{2\pi\sigma\tau}\exp\left(-\frac{\boldsymbol{\vartheta}^2}{2\sigma\tau}\right), \qquad (3.1)$$

where 2σ is the mean square scattering angle per unit length. The probability density that the scattering angles $\boldsymbol{\vartheta}_n=\boldsymbol{\vartheta}(n\Delta)$ at times $\vartheta_n=n\Delta$ lie in the intervals $(\boldsymbol{\vartheta}_n,\boldsymbol{\vartheta}_n+d\boldsymbol{\vartheta}_n)$ is[23]

$$d\mathscr{P}_N^{(a)}=\frac{d\boldsymbol{\vartheta}_1\ldots d\boldsymbol{\vartheta}_N}{(2\pi\sigma\Delta)^N}$$

$$\times\exp\left\{-\frac{\boldsymbol{\vartheta}_1^2}{2\sigma\Delta}-\frac{(\boldsymbol{\vartheta}_2-\boldsymbol{\vartheta}_1)^2}{2\sigma\Delta}-\ldots-\frac{(\boldsymbol{\vartheta}_N-\boldsymbol{\vartheta}_{N-1})^2}{2\sigma\Delta}\right\},$$

where $\Delta=\tau/N$.

We shall now use this relation to write down the average spectral density emitted by an electron in an amorphous medium in the form of a functional integral with respect to the Wiener measure $d_W\boldsymbol{\vartheta}$:

$$\left\langle\frac{dE}{d\omega}\right\rangle_a=\int d_W\boldsymbol{\vartheta}(\tau)E'\{\boldsymbol{\vartheta}(\tau)\}=\lim_{N\to\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}d\mathscr{P}_N^{(a)}E'\{\boldsymbol{\vartheta}(\tau)\}. \qquad (3.2)$$

Since

$$\boldsymbol{\vartheta}^2(\tau)=\frac{\partial}{\partial\mu}\exp(\mu\boldsymbol{\vartheta}^2(\tau))|_{\mu=0},$$

we find that all the terms in (2.4) containing different components of the vector $\boldsymbol{\vartheta}=(\vartheta_x,\vartheta_y)$ can be factored, so that, when the average spectral density $\langle dE/d\omega \rangle_a$ is evaluated, it is sufficient to evaluate the functional integral for only one of the components of the vector $\boldsymbol{\vartheta}$. In view of this, we have

$$\left\langle \frac{dE}{d\omega} \right\rangle_a = -\frac{2e^2\delta_p}{\pi} \int_{-\infty}^{\infty} dT \, \mathrm{Im} \int_0^{\infty} \frac{d\tau}{\tau} \left\{ \exp(-i\delta_p\tau) \times \left(1 + \frac{1}{2}\gamma_p^2 \frac{\partial}{\partial\mu}\right) Q_\omega^2 - \exp(-2i\omega\tau) \left(1 + \frac{1}{2}\gamma_p^2 \frac{\partial}{\partial\mu}\right) Q_{-\omega}^2 \right\} \Bigg|_{\mu=0}, \quad (3.3)$$

where

$$Q_\omega = \int d_w \vartheta(\tau) \exp\left\{ \mu\vartheta^2(\tau) - \frac{i\omega}{2} \int_0^\tau dt \vartheta^2(t) + \frac{i\omega}{2\tau} \left(\int_0^\tau dt \vartheta(t) \right)^2 \right\}, \quad (3.4)$$

$$\delta_p \approx \omega/2\gamma_p^2, \quad \gamma_p^2 = \gamma^2 \left(1 + \gamma^2 \frac{\omega_p^2}{\omega^2}\right)^{-1},$$

and ϑ is one of the components of the vector $\boldsymbol{\vartheta}$.

The functional integral (3.4) is a Gaussian, so that it can be evaluated analytically by a standard procedure.[23] The result is (see Appendix)

$$Q_\omega = \left\{ D(0) \left[1 - \frac{i\omega\sigma}{\tau} \int_0^\tau dt D^{-2}(t) \left(\int_t^\tau dt' D(t') \right)^2 \right] \right\}^{-1/2}, \quad (3.5)$$

where

$$D(t) = \mathrm{ch}\, r(t-\tau) + (2\sigma\mu/r)\, \mathrm{sh}\, r(t-\tau), \quad r \equiv (i\omega\sigma)^{1/2}.$$

Substituting (3.5) in (3.3), we can readily show that, to within the accuracy specified (we are discarding terms of the order of ϑ^2 and γ^{-2}), we have

$$\left\langle \frac{dE}{d\omega} \right\rangle_a = \frac{2e^2\delta_p}{\pi} T \left\{ \mathrm{Im}\left(-r \int_0^\infty d\tau\, \mathrm{cth}\, r\tau \exp(-i\delta_p\tau) \right) - \frac{\pi}{2} \right\}, \quad (3.6)$$

where T is the path length traversed by the particle in the medium.

Substituting $z = r\tau$ in (3.6), and transforming from integration with respect to the complex variable z to integration with respect to the real variable $x = \mathrm{Re}\, z$, we obtain the final expression for the mean spectral density in the form

$$\left\langle \frac{dE}{d\omega} \right\rangle_a = E_{BH}' \left(1 + \frac{\gamma^2\omega_p^2}{\omega^2}\right)^{-1} \Phi_M(s), \quad (3.7)$$

where $E'_{BH} = 4e^2\gamma^2\sigma T/3\pi$ is the spectral density corresponding to the Bethe-Heitler result for the intensity emitted by a particle in a low-density gas at low frequencies, and the Migdal function[4] is given by

$$\Phi_M(s) = 24s^2 \left\{ \int_0^\infty dx\, \mathrm{cth}\, x e^{-2sx} \sin 2sx - \frac{\pi}{4} \right\},$$

$$s = \frac{\delta_p}{2^{3/2}|r|} = \frac{1}{8}\left(1 + \frac{\gamma^2\omega_p^2}{\omega^2}\right)\left(\frac{\omega}{\omega_{LP}}\right)^{1/2}, \quad \omega_{LP} = \frac{\sigma\gamma^4}{2}. \quad (3.8)$$

The mean spectral density (3.7), obtained by the method of functional integration, is identical with the corresponding result for a fast particle in a medium (see Section 20 in Ref. 5), obtained by the Migdal method[4] in which multiple scattering in the amorphous medium is taken into account. Thus, quantitative results on the Landau-Pomeranchuk effect, in which the emission by relativistic particles in a medium is suppressed, can be obtained either by using the kinetic equation or by functional integration. We shall now show that functional integration frequently enables us to examine, from a unified point of view, how multiple scattering affects emission by particles in both amorphous and crystalline media, and we shall determine the conditions under which this approach is valid.

4. SUPPRESSION OF COHERENT EMISSION BY RELATIVISTIC PARTICLES IN CRYSTALS

The above formulas show that the spectral density emitted by fast particles in a medium depends significantly on the ratio of the square of the characteristic emission angle to the square of the scattering angle within the coherence length. When a relativistic particle moves through a crystal nearly parallel to one of the crystallographic axes or planes, there are significant correlations between its successive collisions with lattice atoms. These correlations ensure that the scattering of the particles in the crystal is stronger than that in an amorphous medium.[10,24,25] We shall show that this influences the character of the radiation emitted by particles in a crystal at low frequencies.

The difference between the scattering angle in a crystal and in an amorphous medium is greatest when the particle propagates at a small angle ψ to one of the crystallographic axes (the z axis), but well away from the densely packed crystallographic planes. Let us therefore consider in greater detail the emission process in a crystal in this case.

In this situation, correlations arise in collisions between the incident particle and atoms in an individual chain lying along the z axis in the crystal, as long as collisions with different chains of atoms may be regarded as random. The effect of correlations is that, for small ψ and ω and high energies ε, the motion and emission of a particle in a crystal are largely determined by the continuous potential of the chains of atoms in the crystal, i.e., the potential of the crystal lattice averaged over the coordinate z (Refs. 10 and 26–28).

$$U_c(\boldsymbol{\rho}) = \sum_k U(|\boldsymbol{\rho} - \boldsymbol{\rho}_k|), \quad U(\boldsymbol{\rho}) = \frac{1}{a}\int dz\, u(\mathbf{r}), \quad (4.1)$$

where $u(\mathbf{r})$ is the potential energy of the interaction between the incident particles and an individual atom in the lattice, $\boldsymbol{\rho}$ is the radius vector in the xy plane perpendicular to the crystallographic axis (z axis), $\boldsymbol{\rho}_k$ is the position of the k th chain of atoms in the xy plane, and a is the average separation between atoms in the chain.

The motion of particles in a field with this potential distribution can, in general, be either finite (channeling) or unbounded (traversing potential barriers) relative to the chains of atoms lying parallel to the z axis in the crystal. Finite motion is possible if the angle ψ between the incident beam and the chain axis is small in comparison with the critical angle for axial channeling $\psi_c = (4Ze^2/\varepsilon a)^{1/2}$, where $Z|e|$ is the charge residing on the nucleus of a crystal atom. When $\psi > \psi_c$, all the particles incident on the crystal execute infinite motion relative to the crystallographic z axis. In this range of angles ψ, the average scattering angle in the crystal assumes its maximum value,[24,25] so that we shall now confine our attention to fast particles propagating in the crystal in this particular case.

We are interested in the emission of radiation at low frequencies, for which the coherence length l is much greater than the length $2R/\psi$ (R is the atomic screening length) within which an electron is scattered as a result of collision with each chain of atoms. In this frequency range, the details of the interaction between the incident particle and each individual crystal atom are unimportant, and only the interaction between the incident particle and each chain of atoms as a whole is significant.

The scattering of a particle by the continuous field of each chain occurs at the azimuthal angle φ in the xy plane, perpendicular to the chain axis (z axis). The angle ϑ, through which the particle is scattered by a chain, is related to the azimuthal scattering angle φ by[10,27]

$$\vartheta=2\psi\sin\frac{\varphi}{2},\qquad \varphi=\varphi(b)=\pi-2b\int_{\rho_0}^{\infty}\frac{d\rho}{\rho^2}\left(1-\frac{U(\rho)}{\varepsilon_\perp}-\frac{b^2}{\rho^2}\right)^{-1/2}, \tag{4.2}$$

where b is the impact parameter for the chain, $\varepsilon_\perp = \varepsilon\psi^2/2$ is the energy associated with transverse motion of the particle, and ρ_0 is the minimum separation between the particle and the chain axis.

Multiple scattering by different chains of atoms redistribute the particles in the azimuthal angle φ. In the case that we are considering, in which the motion takes place well away from densely packed crystallographic planes, collisions between the particle and the different chains of atoms can be regarded as random. In the simplest case, the distribution of the particles in the angle φ at depth z satisfies the kinetic equation

$$\frac{df(\varphi,z)}{dz}=na\psi\int_{-\infty}^{\infty}db[f(\varphi+\varphi(b);z)-f(\varphi;z)], \tag{4.3}$$

where n is the density of atoms in the crystal.

Equation (4.3) was first used in Refs. 29 and 30 to describe scattering of channeled ions in a crystal. It was shown in Refs. 10, 24, and 31 that (4.3) is valid for ultrarelativistic particles not only in the case of channeling, but also when this phenomenon is absent.[2)]

The general solution of (4.3) is a complicated function of the azimuthal angle φ and depth of penetration of the particle in the crystal.[10] Substantial simplification arises when $\psi \gg \psi_c$. In this range of ψ, we can expand (4.2) in powers of the parameter $U/\varepsilon_\perp \sim \psi_c^2/\psi^2$. In the first approximation, we have

$$\vartheta\approx\psi\varphi(b),\qquad \varphi(b)\approx\frac{1}{\varepsilon_\perp}\frac{\partial}{\partial b}\int_{-\infty}^{\infty}dxU((x^2+b^2)^{1/2}). \tag{4.4}$$

For $\psi \gg \psi_c$, the characteristic azimuthal scattering angle in a crystal is small in comparison with unity, so that the function f in (4.3) can be expanded in terms of the angle $\varphi(b)$, which leads to the equation

$$\frac{df(\varphi,z)}{dz}=\frac{1}{2}\overline{\varphi^2}\frac{d^2}{d\varphi^2}f(\varphi,z),\qquad \overline{\varphi^2}=na\psi\int_{-\infty}^{\infty}db\varphi^2(b), \tag{4.5}$$

where $\varphi(b)$ is given by (4.4).

Equations (4.4) and (4.5) show that, if the initial angular distribution of the particles has the form of a delta-function $f(\varphi,0)=\delta(\varphi)$, the distribution at time $\tau = z/v$ is the Gaussian

$$f(\varphi,\tau)=(2\pi\sigma_c\tau)^{-1/2}\exp(-\vartheta^2/2\sigma_c\tau); \tag{4.6}$$

where $\vartheta = \psi\varphi$ and $\sigma_c = \psi^2\overline{\varphi^2}$ is the mean square scattering angle in the crystal per unit length. The probability density that the particle scattering angles in the crystal $\vartheta(n\Delta) = \vartheta_n = \psi\varphi(n\Delta)$ at time $t_n = n\Delta$ lie within the intervals $(\vartheta_n, \vartheta_n + d\vartheta_n)$ is given by

$$d\mathscr{P}_N=\frac{d\vartheta_1\ldots d\vartheta_N}{(2\pi\sigma_c\Delta)^{N/2}}\exp\left\{-\frac{\vartheta_1^2}{2\sigma_c\Delta}-\ldots-\frac{(\vartheta_N-\vartheta_{N-1})^2}{2\sigma_c\Delta}\right\}. \tag{4.7}$$

We shall now use this expression to write the mean spectral density emitted by a relativistic electron in a crystal at low frequencies in the form of a functional integral over the Wiener measure:

$$\left\langle\frac{dE}{d\omega}\right\rangle_c=\int d_W\vartheta(\tau)E'\{\vartheta(\tau)\}=\lim_{N\to\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}d\mathscr{P}_NE'\{\vartheta(\tau)\}, \tag{4.8}$$

where $E'\{\vartheta(\tau)\}$ is the functional defined by (2.4).

Formula (4.8) differs from the corresponding formula (3.2) for an amorphous medium only by the fact that $\boldsymbol{\vartheta}(\tau)$ is a two-component process in the latter case whereas, in a crystal, $\vartheta(\tau)$ is a one-component process (scattering is possible only at the azimuthal angle φ) and by the fact that the mean square scattering angle in the crystal is different from that in the amorphous medium. Moreover, the spectral density (4.8) differs from the corresponding expression (3.3) for an amorphous medium only by the fact that, in (3.3), we must replace $Q^2_{\pm\omega}$ with $Q_{\pm\omega}$ and, in the expression for Q_ω, given by (3.4), we must replace σ with σ_c.

As a result, we find that

$$\left\langle\frac{dE}{d\omega}\right\rangle_c=NE_{BH}'\left(1+\frac{\gamma^2\omega_p^2}{\omega^2}\right)^{-1}\Phi(s_c), \tag{4.9}$$

where

$$N=\frac{\sigma_c}{2\sigma},\quad s_c=\frac{1}{8}\left(1+\frac{\gamma^2\omega_p^{\,2}}{\omega^2}\right)\left(\frac{\omega}{\omega_c}\right)^{1/2},\quad \omega_c=2N\omega_{\rm LP},$$

$$\Phi(s_c)=48s^2\left\{-\frac{\pi}{4}+\int_0^\infty dx\frac{\exp(-2s_cx)}{(x\,\mathrm{sh}\,x)^{1/2}}\right.$$

$$\times\left[\sin 2s_cx+\frac{1}{8s_c}\frac{\mathrm{ch}\,x-1}{\mathrm{sh}\,x}\right.$$

$$\left.\left.\times(\sin 2s_cx+\cos 2s_cx)\right]\right\}. \qquad (4.10)$$

Formula (4.9) describes the effect of both multiple scattering and polarization of the medium on the coherent emission intensity due to fast particles in a crystal at low frequencies.

Let us consider some limiting cases of (4.9). For small and large values of the argument, the function $\Phi(s_c)$ assumes the form

$$\Phi(s_c)\approx\begin{cases}1, & s_c\gg 1\\ 6\eta s, & s_c\ll 1\end{cases}, \qquad (4.11)$$

where

$$\eta=\int_0^\infty dy\,y^{-1/2}(\mathrm{sh}\,y)^{-3/2}(\mathrm{ch}\,y-1)\approx 1{,}333.$$

Thus, when $s_c\gg 1$, we have

$$\left\langle\frac{dE}{d\omega}\right\rangle_c\approx NE_{BH}'\left(1+\frac{\gamma^2\omega_p^{\,2}}{\omega^2}\right)^{-1}. \qquad (4.12)$$

If in addition to $s_c\gg 1$ we have $\omega\gg\gamma\omega_p$, the formula given by (4.12) becomes identical with the corresponding result of the theory of coherent emission by relativistic particles interacting with chains of atoms in a crystal.[5,10] When $\omega\lesssim\gamma\omega_p$, Eq. (4.12) shows that the polarization of the medium reduces the emitted intensity relative to the Born theory of coherent emission.

As $s_c\to 0$, Eq. (4.9) provides a more accurate value for the coefficient in the corresponding result reported in Ref. 19, where it was derived on the basis of qualitative estimates of the spectrum emitted by a relativistic particle in the crystal for $s_c\to 0$. According to (4.9) and (4.12), the result in this case is

$$\left\langle\frac{dE}{d\omega}\right\rangle_c\approx\left(\frac{\omega}{\omega_c}\right)^{1/2}NE_{BH}'\ll NE_{BH}'. \qquad (4.13)$$

Thus, when $s_c\ll 1$ $(\omega\ll 2N\omega\ \mathrm{LP})$, coherent emission by particles in a crystal is substantially suppressed.

For arbitrary values of s_c, the function $\Phi(s_c)$ can be found numerically. Figure 1 shows the calculated values of this function and of the Migdal function $\Phi_{\rm M}(s)$ in (3.8). These curves show that the functions $\Phi(s_c)$ and $\Phi_{\rm M}(s)$ are very similar to one another. The values of s_c and s for given ε and ω may, however, be very different, so that the conditions under which there is a significant change in the nature of the emitted radiation in crystals may be significantly different from those in amorphous media.

Let us now compare the basic characteristics of the radiation emitted by relativistic particles in a crystal and in an amorphous medium. If the potential due to an individual atom of the medium is the screened Coulomb potential

$$u(r)=(Ze^2/r)\exp(-r/R),$$

then, according to (4.6), we have

$$\sigma_c=4\pi^2Z^2e^4n\varepsilon^{-2}(R/\psi a). \qquad (4.14)$$

In the case of motion in an amorphous medium, it is known[5] that[3)]

$$\sigma=4\pi Z^2e^4n\varepsilon^{-2}\ln 183Z^{-1/3}, \qquad (4.15)$$

so that the quantity N in (4.9) is given by

$$N=\frac{\pi}{2\ln 183Z^{-1/3}}\frac{R}{\psi a}. \qquad (4.16)$$

Formulas (3.8) and (4.9) show that, when $\omega\lesssim\gamma\omega_p$, the polarization of the medium has a considerable effect on the emission of radiation, and the effect occurs in the same frequency range for both crystals and amorphous media. Multiple scattering affects the emission of radiation in amorphous media and in the crystals for frequencies $\omega\lesssim\omega_{\rm LP}$ and $\omega\lesssim 2N\omega_{\rm LP}$, respectively. For small values of the angle ψ $(R/a\gg\psi\gg\psi_c)$, Eq. (4.16) shows that $N\gg 1$, so that the character of the radiation emitted by a particle in a crystal becomes substantially modified at lower particle energies and in a wider frequency range than in the amorphous medium.

Figure 2 shows the calculated emission spectra due to electrons with $\varepsilon=1$ (a) and 10 (b) GeV in an amorphous medium (broken curves) and in a crystal (solid lines). The electron beam is incident on a tungsten crystal at the angle $\psi=2$ mrad (the condition $\psi>\psi_c$ must be satisfied) to the crystallographic $\langle 100\rangle$ axis.

These results show that, when the particle radiates in a crystal, the frequency range in which multiple scattering has a significant effect on emission is much greater than the corresponding frequency range in an amorphous medium. A further important point is that, at very high particle energies, the effect of multiple scattering on emission in an amorphous medium cannot be treated independently of the polarization of the medium, but this *is* possible in a crystal. The condition for this is that $\omega_{\rm LP}\ll\gamma\omega_p\ll 2N\omega_{\rm LP}$. These inequalities are satisfied, for example, when an electron with $\varepsilon=1$ GeV propagates in a tungsten crystal at an angle $\psi=2$ mrad with respect to the $\langle 100\rangle$ axis. When $\varepsilon=10$ GeV, it is readily verified that $\gamma\omega_p\lesssim\omega_{\rm LP}\ll 2N\omega_{\rm LP}$, so that, at this energy, there is a frequency interval in which the effect of multiple scattering can be treated independently of the polarization of

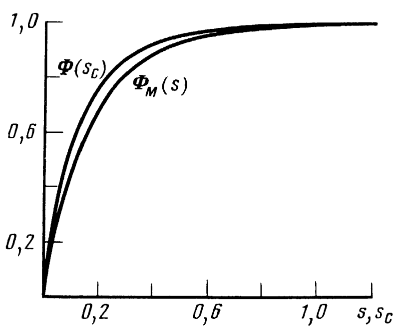

FIG. 1.

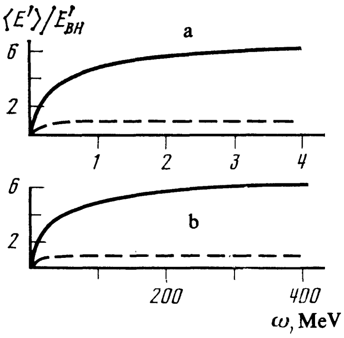

FIG. 2.

the medium both in the crystal and the amorphous medium.

Thus, when relativistic particles propagate in a crystal, the suppression of coherent emission (analog of the Landau-Pomeranchuk effect in which bremsstrahlung is suppressed in an amorphous medium) can occur at much lower particle energies than in the amorphous medium. This provides us with new possibilities for studying the Landau-Pomeranchuk effect on existing accelerators.

5. EMISSION OF LOW-ENERGY PHOTONS BY RELATIVISTIC PARTICLES IN A CRYSTAL IN THE DIPOLE APPROXIMATION OF CLASSICAL ELECTRODYNAMICS

The results obtained in the last Section refer to particles with a Gaussian angular distribution of the form given by (4.6). The condition for this to be so is that, firstly, the particle move in the crystal at an angle ψ that is much greater than the critical angle ψ_c for axial channeling and, secondly, the mean square of the azimuthal scattering angle for a particle interacting with the chains of atoms within the coherence length be small in comparison with unity. A separate examination of the emission of radiation by a particle in a crystal is necessary when these conditions are violated. In general, this type of analysis can be performed in the dipole approximation for a relativistic particle radiating in a medium $\gamma^2\overline{\vartheta_l^2}\ll 1$.

It is well-known[10,32] that, in the dipole approximation, the spectral density radiated in an amorphous medium is the same as the corresponding Bethe-Heitler expression for the intensity emitted by a fast particle at low frequencies. We shall show that, in contrast to an amorphous medium, multiple scattering of a particle by chains of atoms in a crystal can have a significant effect on the intensity of coherent emission at low frequencies when the condition $\gamma^2\overline{\vartheta_l^2}\ll 1$ is satisfied.

In the dipole approximation, the spectral density due to a relativistic particle at low frequencies is given by (2.5). In the emitted-photon frequency range in which we are interested, the length $l=2\gamma^2/\omega$ in which the radiation evolves is much greater than the length $2R/\psi$, in which the particle acceleration $\dot{\boldsymbol{\vartheta}}(t)$ differs from zero during a collision with a chain of crystal atoms. In this frequency range, the quantity $|\mathbf{W}(\nu)|^2$ in (2.5) can be written in the form

$$|\mathbf{W}(\nu)|^2\approx\sum_{n,k}\boldsymbol{\vartheta}_n\boldsymbol{\vartheta}_k\exp[i\nu(t_n-t_k)], \qquad (5.1)$$

where ϑ_n is the scattering angle in the nth collision with a chain of atoms and t_n is the time at which the collision takes place. The expression given by (5.1) must be averaged over the random values of the scattering angle in the crystal.

As before (Section **4**), we are interested in the radiation emitted when a particle propagates in a crystal at a small angle ψ to the crystallographic z axis, but well away from the densely packed crystallographic planes. The particle is then scattered by each chain of atoms at the azimuthal angle φ in the plane perpendicular to the z axis. Multiple scattering by different chains of atoms redistribute the particles in angle φ. The resultant change in the azimuthal scattering angle after collisions with n chains of atoms is given by

$$\Phi_n=\sum_{i=1}^{n}\varphi_i.$$

The components of the vector $\boldsymbol{\vartheta}_n$ in the xy plane perpendicular to the crystal z axis are, respectively, given by

$$(\vartheta_n)_x=\psi(\cos\Phi_n-\cos\Phi_{n-1}),\quad (\vartheta_n)_y=\psi(\sin\Phi_n-\sin\Phi_{n-1}). \qquad (5.2)$$

In our case, particle collisions with different chains of atoms in the crystal are random and independent, so that, in (5.1), we can explicitly average over the azimuthal angles φ_n. If we define the average of $\cos\varphi$ during scattering by an individual chain of atoms by the expression

$$\langle\cos\varphi\rangle=\sigma_l^{-1}\int d\sigma_l(\varphi)\cos\varphi,$$

where σ_l and $d\sigma_l(\varphi)$ are, respectively, the total and differential scattering cross sections of the chain, and if we recall that

$$\langle\cos(\Phi_n-\Phi_k)\rangle=\langle\cos\varphi\rangle^{|n-k|},\quad \langle\sin\varphi\rangle=0,$$

we find that

$$\langle|W(\nu)|^2\rangle=2\psi^2\sum_{n,k}\cos\nu(t_n-t_k)\,(\langle\cos\varphi\rangle^{|n-k|}-\langle\cos\varphi\rangle^{|n-k+1|}). \qquad (5.3)$$

This expression is still to be averaged over the quantities τ_j defining the interval of time between jth and $j-l$ th collisions of the particle with the atomic chains. Since the collisions are random, the distribution over τ_j is (see, for example, Ref. 33)

$$f(\tau_j)=\frac{1}{\bar{\tau}}\exp\left(-\frac{\tau_j}{\bar{\tau}}\right), \qquad (5.4)$$

where $\bar{\tau}=(\psi\sigma_l na)^{-1}$ is the average mean free time between successive collisions with the chains of atoms. Since

$$t_n=\sum_{j=1}^{n}\tau_j,$$

we obtain the following expression for the mean value of the function $\cos\nu(t_n-t_k)$ over the random quantities τ_j:

$$\overline{\cos\nu(t_n-t_k)}=\int\ldots\int\prod_j d\tau_j f(\tau_j)\cos\nu(t_n-t_k)$$
$$=\operatorname{Re}(1+i\nu\bar{\tau})^{-|n-k|}. \qquad (5.5)$$

We are interested in radiation emitted in a crystal whose thickness is large in comparison with the coherence length l, so that the contribution of path segments of scale l at entry and exit from the crystal to the emitted radiation can be neglected. We shall now average (5.3) over τ_j, using (5.5) and then, having substituted $n = k + m$, we extend the summation over m to the interval $(-\infty, \infty)$. After summing the geometric progression, we find that

$$\overline{\langle|W(\nu)|^2\rangle}=4\mathcal{N}\psi^2\left\langle \sin^2\frac{\varphi}{2}\right\rangle \nu^2\left[\nu^2+\frac{4}{\bar{\tau}^2}\left\langle \sin^2\frac{\varphi}{2}\right\rangle\right]^{-1}, \tag{5.6}$$

where $\mathcal{N} = T/v\bar{\tau}$ is the number of collisions between the particle and the chains of atoms in the crystal.

We now substitute (5.6) in (2.5) and obtain the following expression for the spectral density emitted by a relativistic particle in the crystal:

$$\left\langle\frac{d\bar{E}}{d\omega}\right\rangle=T2e^2\psi^2\omega_d\left[3\pi\left(1+\frac{\gamma^2\omega_p{}^2}{\omega^2}\right)\right]^{-1}F(x), \tag{5.7}$$

where

$$\omega_d=\frac{4\gamma^2}{\bar{\tau}}\left\langle \sin^2\frac{\varphi}{2}\right\rangle, \quad x=\frac{\omega}{\omega_d}\left(1+\frac{\gamma^2\omega_p{}^2}{\omega^2}\right),$$

$$F(x)={}^3x({}^3/_2-x^2)\operatorname{arcctg} x+3x^2(1-{}^1/_2\ln(1+x^{-2})). \tag{5.8}$$

No mention has been made of the specific interaction between the particle and the field of an individual chain of atoms in the derivation of (4.7), so that this formula can be used to consider the emission of radiation by particles in a crystal, both for $\psi\gg\psi_c$, i.e., when the particle angular distribution is Gaussian and the average azimuthal scattering angle for a particle colliding with each chain of atoms is small, and for $\psi\lesssim\psi_c$, i.e., when these conditions are not satisfied. All that is necessary is that dipole radiation be produced by the relativistic particle in the crystal, i.e., $\gamma^2\,\overline{\vartheta_l^2}\ll1$.

Let us now consider some limiting cases of (5.7). To do this, we note that the asymptotic behavior of $F(x)$ is

$$F(x)\approx\begin{cases}1-\dfrac{7}{20x^2}+\ldots, & x\gg1\\ \dfrac{3}{4}x[\pi+2x\ln(ex^2)+\ldots], & x\ll1\end{cases} \tag{5.9}$$

Formulas (5.7) and (5.9) show that, when $\gamma^2\,\overline{\vartheta_l^2}\ll1$, the spectral density depends significantly on the relationship between the frequencies ω, ω_d and $\gamma\omega_p$.

When $\omega_d\ll\gamma\omega_p$, we have from (5.7) and (5.9)

$$\left\langle\frac{d\bar{E}}{d\omega}\right\rangle=T2e^2\psi^2\omega_d\left[3\pi\left(1+\frac{\gamma^2\omega_p{}^2}{\omega^2}\right)\right]^{-1} \tag{5.10}$$

If, in addition to $\omega_d\ll\gamma\omega_p$, we have $\psi\gg\psi_c$, we can expand in terms of ψ_c/ψ in (5.9). In the first approximation to this expansion, (5.9) becomes identical with the corresponding result of the theory of coherent emission with allowance for the effect of polarization on emission, as given by (4.12).

The quantity ω_d in (5.7) depends on the sign of the particle charge and on the relationship between the angles ψ and ψ_c. The maximum value of this quantity is attained for $\psi\lesssim\psi_c$. In this range of values of ψ, we have $\omega_d\sim4\gamma^2naR\psi_c$. [When $\psi\lesssim\psi_c$, the motion and emission of relativistic positrons in a crystal can be described assuming a continuous potential of the chain of atoms of the form[11,12,34] $U_1(\rho)=U_0R/\rho$, where $U_0=2Ze^2/a$. In a field with this potential distribution, $\omega_d=2\pi\gamma^2naR\psi_c^2/\psi$).] Comparing this value of ω_d with $\gamma\omega_p$, we find that the following inequality is satisfied in a wide range of energies:

$$(\omega_d/\gamma\omega_p)^2\sim4\gamma naR^2\gg1$$

and, consequently, there are three regions of ω in which the character of the radiation emitted by particles in a crystal is significantly different:

$$\omega\lesssim\gamma\omega_p,\quad \gamma\omega_p\lesssim\omega\lesssim\omega_d,\quad \omega_d\lesssim\omega.$$

When $\omega\lesssim\gamma\omega_p$, the polarization of the medium has a considerable effect on the emitted radiation. When $\omega\gg\omega_d$ (this corresponds to a coherent length l, which is small compared with the mean free path $v\bar{\tau}$ of the particle between successive collisions with chains of atoms) and the radiation is determined only by the properties of the interaction between the particles and the individual chains of atoms. In this frequency range and for $\omega\lesssim\gamma\omega_p$, formula (5.7) becomes identical with (5.10). On the other hand, for frequencies in the range $\gamma\omega_p\ll\omega\ll\omega d$, (5.7) and (5.9) show that

$$\langle d\bar{E}/d\omega\rangle={}^1/_2Te^2\psi^2\omega. \tag{5.11}$$

Formula (5.11) shows that, for frequencies $\gamma\omega_p\ll\omega\ll\omega_d$, multiple scattering of the particles by chains of atoms in the crystal leads to a rapid reduction in the spectral density with decreasing radiated-photon frequency.

The formulas obtained in this Section are valid if the dipole approximation can be used for the radiation emitted by particles in a crystal, i.e., if $\gamma^2\,\overline{\vartheta_l^2}\ll1$. When $\psi\sim\psi_c$, we have the order-of-magnitude result $\overline{\vartheta_l^2}\sim\psi_c^2$, so that the inequality $\gamma^2\,\overline{\vartheta_l^2}\sim\gamma^2\psi_c^2\ll1$ gives rise to a restriction on the particle energy ε. We note in this connection that there is a range of energies ε in which both $\gamma\psi_c\ll1$ and $\gamma\omega_p\ll\omega_d$ are satisfied. In particular, these inequalities are satisfied when $1/4naR^2\ll\gamma\ll ma/4Ze^2$.

The authors are greatly indebted to A. I. Akhiezer, M. L. Ter-Mikaelyan, and S. V. Peletminskiĭ for supporting this research and for numerous discussions of the questions touched upon in this review. They are also indebted to E. L. Feĭnberg for a number of interesting discussions on the history of the problem.

APPENDIX

Let us find the mathematical expectation (3.4) by the well-known procedure based on the evaluation of functional integrals[23] of Gaussian form. We shall do this by taking

$$Q_\omega=\lim_{N\to\infty}\int_{-\infty}^{\infty}\frac{dq}{\pi^{1/2}}e^{-q^2}Q_\omega(N,q), \tag{A.1}$$

where

$$Q_\omega(N,q)=\int_{-\infty}^{\infty}\ldots\int\frac{d\vartheta_1\ldots d\vartheta_N}{(2\pi\sigma\Delta)^{N/2}}\exp\left\{-\sum_{n=0}^{N-1}\frac{(\vartheta_{n+1}-\vartheta_n)^2}{2\sigma\Delta}\right.$$

$$\left.+\mu\vartheta_N^2-\frac{i}{2}\omega\Delta\sum_{n=1}^{N}\vartheta_n^2+q\Delta\left(\frac{2i\omega}{\tau}\right)^{1/2}\sum_{n=1}^{N}\vartheta_n\right\},\qquad \vartheta_0=0.$$

If we now change the variables in accordance with the expression $y_n=(2\sigma\Delta)^{-1/2}\vartheta_n$, we obtain

$$Q_\omega(N,q)=\int_{-\infty}^{\infty}\ldots\int\frac{dy_1\ldots dy_N}{\pi^{N/2}}\exp\left\{b\sum_{n=1}^{N}y_n-\sum_{n,m=1}^{N}A_{nm}y_ny_m\right\},\tag{A.2}$$

where $b=2q\Delta(i\omega\sigma/\Delta)^{1/2}$, and the nonzero elements of the matrix A are

$$A_{nn}=2+i\omega\sigma\Delta^2,\quad n=1,\ldots,(N-1),$$
$$A_{nn}=1+i\omega\sigma\Delta^2-2\mu\sigma\Delta,\quad n=N,$$
$$A_{n,n+1}=A_{n+1,n}=-1,\quad n=1,\ldots,(N-1).$$

Since the matrix A is positive-definite, it can be reduced to a diagonal form with the aid of a unitary matrix U, i.e.,

$$(U^{-1}AU)_{nk}=a_n\delta_{nk},$$

where $a_n>0$, $n=1,\ldots,N$. If we now transform to new variables z_k in accordance with the formula

$$y_n=\sum_{k=1}^{N}U_{nk}z_k,$$

we find that (A.2) can be expressed in terms of the eigenvalues a_n:

$$Q_\omega(N,q)=(\det A)^{-1/2}\int_{-\infty}^{\infty}\ldots\int dz_1\ldots dz_N\exp\left[b\sum_{k,n=1}^{N}U_{nk}z_k\right]$$
$$\times\prod_{j=1}^{N}(2\pi\sigma_j)^{-1/2}\exp\left(-\frac{z_j^2}{2\sigma_j^2}\right),\tag{A.3}$$

where $\sigma_j^2=(2a_j)^{-1}$. Formula (A.3) can also be written in the form

$$Q_\omega(N,q)=(\det A)^{-1/2}\exp\left(\frac{1}{2}\sigma_N^2\right),$$

$$\sigma_N^2=\frac{1}{2}b^2\sum_{n,m=1}^{N}(A^{-1})_{nm},$$

where A^{-1} is the inverse of A.

We now introduce a set of quantities D_n that are the minors of order $N-n+1$ of the determinant of the matrix A, lying in its bottom right-hand corner. According to Refs. 35 and 36, we then have

$$\sigma_N^2=\frac{1}{2}b^2\sum_{n=1}^{N}(D_nD_{n+1})^{-1}\left(\sum_{k=n}^{N}D_{k+1}\right)^2,$$

where $D_{n+1}=1$. Hence,

$$Q_\omega(N,q)=\frac{1}{D_1^{1/2}}\exp\left\{\frac{b^2}{4}\sum_{n=1}^{N}(D_nD_{n+1})^{-1}\left(\sum_{k=n}^{N}D_k\right)^2\right\}.\tag{A.4}$$

Since we must take the limit as $N\to\infty$ in (A.1), we shall investigate the behavior of the quantities D_n in this case. We first note that

$$D_n=(2+i\omega\sigma\Delta^2)D_{n+1}-D_{n+2},\quad 1\leqslant n\leqslant N-2,\tag{A.5}$$

and

$$D_N=1+i\omega\sigma\Delta^2-2\mu\sigma\Delta,\quad D_N-D_{N-1}=2\sigma\mu\Delta+o(\Delta^2).\tag{A.6}$$

If, instead of D_n, we write $D(n\Delta)$, then, in the limit as $N\to\infty$, the quantity $D(n\Delta)$ will tend to the value at $t=n\Delta$ of the continuous function $D(t)$ (Fredholm determinant[23]), which is a solution of the differential equation

$$\frac{d^2}{dt^2}D(t)=i\omega\sigma D(t)$$

subject to the initial conditions $D(N\Delta)=D(\tau)=1$ and $D'(\tau)=2\sigma\mu$, which follows from the recurrence relation (A.5) and conditions (A.6). The solution is

$$D(t)=\operatorname{ch}r(t-\tau)+(2\sigma\mu/r)\operatorname{sh}r(t-\tau),\quad r=(i\omega\sigma)^{1/2}.\tag{A.7}$$

We now rewrite (A.4) in the form

$$Q_\omega(N,q)$$
$$=\frac{1}{D^{1/2}(\Delta)}\exp\left\{-q^2\frac{i\omega\sigma}{\tau}\Delta^3\sum_{n=1}^{N}[D(n\Delta)D((n+1)\Delta)]^{-1}\right.$$
$$\left.\times\left[\sum_{k=n}^{N}D(k\Delta)\right]^2\right\}.$$

Integrating with respect to q, passing to the limit as $N\to\infty$, and replacing the quantities $D(n\Delta)$ with the function $D(t)$, which is continuous at $t=n\Delta$, we finally obtain (3.5). According to (A.7) and (3.5), when $\mu=0$,

$$Q_\omega=\left(\frac{r\tau}{\operatorname{sh}r\tau}\right)^{1/2},\quad\frac{\partial}{\partial\mu}Q_\omega=\frac{2\sigma}{r}\operatorname{th}\frac{r\tau}{2}Q_\omega.\tag{A.8}$$

[1)]In the derivation of (2.4), we have used the transformations given in Ref. 2. However, (2.4) is somewhat different from the corresponding result in Ref. 2. The difference is due to the fact that, when the formula for the spectral density due to a relativistic particle in a medium was derived in Ref. 2, the terms discarded were of the same order as those retained (see Ref. 10 in this connection).

[2)]Equation (4.3) is valid if the scattering of particles in the crystal is largely determined by the continuous potential due to the chains of atoms. This requirement is satisfied if the average scattering angle in the crystal is much greater than the average scattering angle of particles due to scattering by thermal vibrations of lattice atoms. At high particle energies, this requirement is satisfied in a much greater range of angles ψ as compared with the angles $\psi\lessdot\psi_c$ for which the phenomenon of particle channeling is known to occur.[10,31]

[3)]Expression (4.15) was obtained with logarithmic precision (see the discussion of how σ can be obtained, given at the end of Section 19 in Ref. 5).

[1]M. L. Ter-Mikaelyan, Zh. Eksp. Teor. Fiz. **25**, 296 (1953).
[2]L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **92**, 735 (1953).
[3]M. L. Ter-Mikaelyan, Dokl. Akad. Nauk SSSR **94**, 1033 (1954).
[4]A. B. Migdal, Dokl. Akad. Nauk SSSR **96**, 49 (1954).
[5]M. L. Ter-Mikaelyan, Vliyanie sredy na elektromagnitnye protsessy pri vysokikh energiyakh (Effect of the Medium on Electromagnetic Processes at High Energies), Academy of Sciences of the Armenian SSR, Erevan, 1969.

[6]H. Überall, Phys. Rev. 103, 1055 (1956).
[7]G. Diambrini, Rev. Mod. Phys. **40**, 611 (1968).
[8]V. Timm. Fortschr. Phys. **17**, 765 (1971).
[9]A. I. Akhiezer, V. F. Voldyshev, and N. F. Shul'ga, Fiz. Elem. Chastits. At. Yadra **10**, 51 (1979) [Sov. J. Part. Nucl. **10**, 19 (1979)].
[10]A. I. Akhiezer and N. F. Shul'ga, Usp. Fiz. Nauk **134**, 561 (1982) [Sov. Phys. Usp. **25**, 541 (1982)].
[11]V. A. Bazylev and N. K. Zhevago, Usp. Fiz. Nauk **134**, 604 (1982) [Sov. Phys. Usp. **25**, 565 (1982)].
[12]V. V. Beloshitsky and F. F. Komarov, Phys. Rep. **93**, 117 (1982).
[13]V. G. Baryshevskiĭ, Kanalirovanie, izluchenie i reaktsii v kristallakh pri vysokikh energiyakh (Channeling, Emission and Reactions in Crystals at High Energies), Belorussian State University, Minsk, 1982.
[14]N. F. Shul'ga and S. P. Fomin, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 126 (1978) [JETP Lett. **27**, 117 (1978)].
[15]N. F. Shul'ga, V kn.: Problemy sovremennoĭ teoreticheskoĭ fiziki. Sb. nauchnykh trudov (in: Problems of Modern Theoretical Physics. Collection of Scientific Papers), Naukova Dumka, Kiev, 1982, p. 256.
[16]M. L. Ter-Mikaelyan, Author's Thesis, FIAN, Moscow, 1961.
[17]V. A. Bazylev and N. K. Zhevago, Zh. Eksp. Teor. Fiz. **73**, 1697 (1977) [Sov. Phys. JETP **46**, 891 (1977)].
[18]A. L. Avakyan, N. K. Zhevago, and Yan Shu, Zh. Eksp. Teor. Fiz. **82**, 584 (1982) [Sov. Phys. JETP **55**, 341 (1982)].
[19]N. N. Nasonov and N. F. Shul'ga, Ukr. Fiz. Zh. **27**, 789 (1982).
[20]N. V. Laskin, A. S. Mazmanishvili, and N. F. Shul'ga, Dokl. Akad. Nauk SSSR **277**, 850 (1984) [Sov. Phys. Dokl. **29**, 638 (1984)].
[21]L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Nauka, Moscow, 1974.
[22]E. L. Feĭnberg, Usp. Fiz. Nauk **132**, 255 (1980) [Sov. Phys. Usp. **23**, 629 (1980)].
[23]I. M. Gel'fand and A. M. Yaglom, Usp. Mat. Nauk **11**, 77 (1956).
[24]S. P. Fomin and N. F. Shul'ga, Preprint KhFTI AN Ukr.SSR 79-42, Kharkov, 1979.
[25]N. F. Shul'ga, V. I. Truten', and S. P. Fomin, Zh. Tekh. Fiz. **51**, 2279 (1982) [Sov. Phys. Tech. Phys. **27**, 1399 (1982)].
[26]J. Lindhard, Usp. Fiz. Nauk **99**, 249 (1969) [translated from Mat.-Fyz. Medd. Dan Vid. Selsk. **34** (No. 14) (1965)].
[27]D. S. Gemmell, Rev. Mod. Phys. **46**, 129 (1974).
[28]M. A. Kumakhov and G. Shirmer, Atomnye stolknoveniya v kristallakh (Atomic Collisions in Crystals), Atomizdat, Moscow, 1980.
[29]V. V. Beloshitskiĭ and M. A. Kumakhov, Fiz. Tverd. Tela (Leningrad) **15**, 1588 (1973) [Sov. Phys. Solid State **15**, 1061 (1973)].
[30]I. A. Golovchenko, Phys. Rev. B **13**, 4672 (1976).
[31]V. I. Truten', S. P. Fomin, and N. F. Shul'ga, Preprint KhFTI AN Ukr.SSR 82-11, Kharkov, 1982.
[32]M. I. Ryazanov, Elektrodinamika kondensirovannogo veshchestva (Electrodynamics of Condensed Matter), Nauka, Moscow, 1984.
[33]S. M. Rytov, Vvedenie v statisticheskuyu radiofiziku (Introduction to Statistical Radiophysics), Nauka, Moscow, 1976, Sec. 1.
[34]N. F. Shul'ga, V. I. Truten', and S. P.Fomin, Zh. Eksp. Teor. Fiz. **87**, 250 (1984) [Sov. Phys. JETP **60**, 145 (1984)].
[35]E. W. Montroll, Commun. Pure Appl. Mat. **5**, 415 (1952).
[36]N. V. Laskin and A. S. Mazmanishvili, Preprint KhFTI AN Ukr.SSR 83-1, Kharkov, 1983.

Translated by S. Chomet