

Kinematic resonance and memory effect in free-mass gravitational antennas

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Detailed studies are made of two effects in the motion of free masses subject to the influence of gravitational waves—kinematic resonance and the memory effect. In the first of these, besides the oscillatory motion there is a systematic change in the distance between the bodies if they become free in an appropriate phase of the gravitational wave. The second effect takes the form that the distance between a pair of bodies will, in general, be different from the original distance after they have been influenced by a pulse of gravitational radiation. Possible practical applications of these effects in three different experimental programs are discussed. Allowance for these effects should lessen the requirements on the detection systems and ultimately raise the sensitivity of gravitational antennas.

1. INTRODUCTION

The sensitive element of a gravitational-wave antenna is an oscillator or a set of free test bodies. For example, in a laser interferometer the role of the test bodies is played by mirrors, the distance between which can be deduced from the interference pattern. Under the influence of a gravitational wave, the distance between the mirrors changes slightly, and this change is to be detected in the experiment. Besides the change in the distance between the bodies, there is a change in their relative velocity. Measurement of variations of the velocity is the basis of the proposals that use Doppler tracking of space probes.

The motion of test bodies (particles) in the field of a gravitational wave has to a large degree been studied.¹ It has much in common with the motion of charges in the field of an electromagnetic wave. To describe the motion of closely spaced particles separated by a distance l much less than the length λ_g of the gravitational wave, it is convenient to use a locally inertial frame of reference. It is the closest one can get to the global inertial frame of reference generally used to describe the motion of charges. If the gravitational wave propagates in the direction of the x axis, the particles basically execute motion in the yz plane, but they also have a component of the motion along the x axis.² Under the influence of a weak gravitational wave, as under a weak electromagnetic wave, the amplitude of the particle oscillations along the axis is small compared with the amplitude of the oscillations in the yz plane.

If the distance between the particles is $l \gtrsim \lambda_g$, then the use of a locally inertial frame of reference (attached to one of the particles) does not result in any gain and even complicates the analysis. In this case, it is necessary to work directly with the metric describing the given wave field. However, the general features of the relative motion remain unchanged.

One is usually interested in the oscillatory motion of a test particle, it being assumed that on average it is at rest. However, in the case of a signal of not too long duration or in the case of a restricted time of observation of a monochromatic signal there is also practical interest in comparing the positions and velocities of the particles before and after the

action of the signal. The kinematic quantities that characterize the final state are determined by the shape of the signal and by the initial conditions. The clarification of the details of this motion together with some practical recommendations are the subject of this paper.

2. THEORETICAL ANALYSIS OF KINEMATIC RESONANCE AND MEMORY EFFECT

The gravitational field of a weak plane wave propagating along the x^1 direction is determined by the metric

$$ds^2 = c^2 dt^2 - dx^{1^2} - (1-a) dx^{2^2} - (1+a) dx^{3^2} + 2b dx^2 dx^3, \quad (1)$$

$$a = a(u), \quad b = b(u), \quad u = x^0 - x^1.$$

For a monochromatic wave, a and b are harmonic functions proportional to e^{iqu} . Since the curvature tensor can be expressed in terms of $d^2 a/du^2$ and $d^2 b/du^2$, the space-time region occupied by a pulse of finite duration, $u_1 \leq u \leq u_2$, is restricted to values of u for which $d^2 a/du^2 \neq 0$ or $d^2 b/du^2 \neq 0$. Outside this region, space-time is flat.

To analyze the relative motion of particles separated by a distance $l \gtrsim \lambda_g$, it is necessary to use the metric (1) directly. But if $l \ll \lambda_g$, a locally inertial frame of reference can be used, and this makes it possible to give a Newtonian interpretation of the forces which act. We begin with closely spaced particles considered in such a frame. We take the origin $x = y = z = 0$ of the frame at one of the test particles. We shall treat the motion in only the leading nonvanishing approximation and, in particular, ignore the motion along the x axis. Then the equations of motion of a nearby test particle reduce to the so-called geodesic deviation equations^{1,3} and take the form

$$\ddot{x} = 0, \quad \ddot{y} = -\frac{1}{2}(\ddot{a}y + \ddot{b}z), \quad \ddot{z} = -\frac{1}{2}(\ddot{b}y - \ddot{a}z), \quad (2)$$

where the dot denotes differentiation with respect to the time. The quantities \ddot{a} and \ddot{b} are taken along the world line of the origin $x = y = z = 0$. In other words, we obtain the equation of motion

$$m\mathbf{x} = \mathbf{F},$$

where \mathbf{F} is determined by the values of the curvature tensor at the origin and by the position of the particle under study.

The force \mathbf{F} can also be expressed in terms of an ordinary Newtonian potential φ (see, for example, Ref. 4):

$$\mathbf{F} = -m \text{grad } \varphi, \quad \varphi = -\frac{1}{4} \ddot{a}(z^2 - y^2) + \frac{1}{2} \dot{b}yz.$$

If the initial coordinates of the particle are $(x, y, z) = (l^1, l^2, l^3)$ and its initial velocity so small that over the whole time of action of the wave the displacement satisfies $\Delta l \ll l = (l^{12} + l^{22} + l^{32})^{1/2}$, then in the expression for \mathbf{F} we can replace x, y, z by l^1, l^2, l^3 . In other words, if this condition is satisfied, a force that depends only on the time acts on each particle. We assume that the force is nonzero in the interval $t_1 \leq t \leq t_2$.

Qualitatively, the problem reduces to one of ordinary mechanics and the very simple equation $m\ddot{\xi} = F(t)$. It follows from this equation that for the given particle the final velocity v_2 and the initial velocity v_1 are related by

$$v_2 - v_1 = \frac{1}{m} \int_{t_1}^{t_2} F(t) dt \quad (3)$$

and, accordingly, for the coordinates

$$\xi_2 - \xi_1 = \frac{1}{m} \int_{t_1}^{t_2} \left(\int_{t_1}^t F(\tau) d\tau \right) dt + v_1(t_2 - t_1). \quad (4)$$

Thus, a particle initially at rest ($v_1 = 0$) is displaced in the general case by the force during the time $t_2 - t_1$ to a new position and acquires a certain velocity.

To be specific, we take

$$\frac{1}{m} F(t) = \frac{1}{2} h \omega^2 l \sin(\omega t + \psi), \quad 0 \leq t \leq \hat{\tau}.$$

Then

$$\begin{aligned} v_2 - v_1 &= h \omega l \sin\left(\frac{\omega \hat{\tau}}{2} + \psi\right) \sin \frac{\omega \hat{\tau}}{2}, \quad \xi_2 - \xi_1 \\ &= h l \left[\frac{\omega \hat{\tau}}{2} \cos \psi - \cos\left(\frac{\omega \hat{\tau}}{2} + \psi\right) \sin \frac{\omega \hat{\tau}}{2} \right] + v_1 \hat{\tau}. \end{aligned} \quad (5)$$

Since we are interested in the displacement caused by the wave, and not the initial velocity, it is necessary to assume $v_1/c \ll h l / \lambda_g$ or, simply, $v_1 = 0$.

As can be seen from the expression (5), the final velocity of the particle does not differ from the initial velocity if the force acts for an integral number of periods ($\omega \hat{\tau} = 2n\pi$) for arbitrary phase ψ or has a definite initial phase $\psi = n\pi - \omega \hat{\tau} / 2$ for fixed duration $\hat{\tau}$. There is certainly no systematic change in the velocity proportional to $\hat{\tau}$. However, such a change is possible for the coordinate of the particle through the first term in the expression (5) for $\xi_2 - \xi_1$. This term leads to the relative displacement

$$\frac{\xi_2 - \xi_1}{l} = \frac{\Delta l}{l} = \frac{1}{2} h \omega \hat{\tau} \cos \psi. \quad (6)$$

The expression (6) has the same form as the expression for the amplitude of an oscillator on which a monochromatic resonant force acts. In a steady regime, it is well known that $\Delta l / l \approx h Q$, where Q is the quality of the oscillator. In our case accumulation of a response is limited, not by the Q of the oscillator, but by the duration of the signal that acts on the

free particle. We call this effect kinematic resonance.

Such effects are also encountered in a different formulation of the problem. For example, the systematic displacement along the x axis of the position of a particle released in the plane yz of the wave front (so-called drift of particles in the field of a gravitational wave, see Ref. 5) is also determined by the duration of the applied force and its phase. The resultant displacement of the particle can in this case be measured near the point of its release, and not far from it, if the particle is forced to reflect between a pair of mirrors.

In order to distinguish under experimental conditions displacement due to the kinematic resonance from the neglected influence of the initial velocity, it is helpful to have, say, two particles released into free motion at times differing by a half-period. Then in accordance with (6) the systematic displacement of these particles must have opposite signs. Particularly favorable are the phases $\psi = 0$ and $\psi = \pi$.

It is important that if a short signal with a definite time shape is applied the resulting displacement between the particles can be "remembered." This means that a particle at rest before the arrival of the signal is still in a state of rest relative to the locally inertial frame of reference after the action of the force has ceased but, in general, in a different position and at a different distance from the origin. For this, it is necessary that the integral on the right-hand side of (4), extended to the complete time of action of the force, be nonzero. This effect, which we call the memory effect, was pointed out for the first time in Ref. 6 in connection with the analysis of the gravitational radiation that arises from close encounters of gravitating bodies. From the practical point of view, this effect is important because it permits a measurement to be made, not during a short burst of gravitational radiation, but over a much longer time, during which the particles can still be assumed to be free. Specifically, in a program looking for bursts of duration 10^{-3} sec by means of a laser interferometer the time of measurement of the resulting displacement can be 10^{-1} sec, which promises practical advantages.

These results can be readily generalized in accordance with Eqs. (2) to the case of arbitrary polarization of the wave and an arbitrary initial position of the particle being studied. Suppose that at the initial time $t = 0$ the particle is at rest in the locally inertial frame and has coordinates l^1, l^2, l^3 . Then subsequently, as follows from (2), its instantaneous coordinates are

$$\begin{aligned} x(t) &= l^1, \quad y(t) = l^2 - \frac{1}{2} [a(t) - a(0)] l^2 - \frac{1}{2} [b(t) - b(0)] l^3 \\ &\quad + \frac{1}{2} t [\dot{a}(0) l^2 + \dot{b}(0) l^3], \\ z(t) &= l^3 - \frac{1}{2} [b(t) - b(0)] l^2 + \frac{1}{2} [a(t) - a(0)] l^3 \\ &\quad + \frac{1}{2} t [\dot{b}(0) l^2 - \dot{a}(0) l^3]. \end{aligned}$$

If moreover $a(t) \equiv 0, b(t) \equiv 0$ before the arrival of the wave, then

$$y(t) = l^2 - \frac{1}{2} a(t) l^2 - \frac{1}{2} b(t) l^3, \quad z(t) = l^3 - \frac{1}{2} b(t) l^2 + \frac{1}{2} a(t) l^3.$$

The position and velocity of the particle when the force ceases to act are determined by the values of the functions $a(t)$ and $b(t)$ and their derivatives at this time.

The actual dependence of the force on the time is determined by the physical nature of the radiation source. In the quadrupole approximation, the wave correction to the metric far from the source has the form

$$h_{ik} \propto \ddot{D}_{ik}(t-r)/r,$$

where D_{ik} is the reduced quadrupole moment of the source (see, for example, Refs. 1, 3, 7). The force $F(t)$ can be expressed in terms of \ddot{h}_{ik} . Thus, the integrals in (3) and (4) are determined by the difference between the initial and final values of \ddot{D}_{ik} and \dot{D}_{ik} , respectively. For a system nonstationary in time that executes a finite motion, these quantities are equal to zero. Radiation pulses from such sources leave test particles in the same position and with the same velocity as before the arrival of the pulse. However, the pulses that arise, for example, from a close encounter of gravitating bodies (gravitational Bremsstrahlung) lead to $\xi_2 - \xi_1 \neq 0$ although $v_2 - v_1 = 0$ (Ref. 6).¹¹ To see this, let us consider in somewhat more detail the collision of two attracting bodies moving in the yz plane.³ For the wave radiated in the x direction the important components of D_{ik} are $D_{22} - D_{33}$ and D_{23} . Their asymptotic values are

$$\begin{aligned} D_{22} - D_{33} &\propto (2 - e^2)t^2, & D_{23} &\propto (e^2 - 1)^{1/2}t^2, & t \rightarrow -\infty, \\ D_{22} - D_{33} &\propto (2 - e^2)t^2, & D_{23} &\propto -(e^2 - 1)^{1/2}t^2, & t \rightarrow +\infty, \end{aligned} \quad (7)$$

where e depends on the impact parameter; $e = 1$ in the case of a head-on collision. It follows from the expressions (7) that the force due to the metric component h_{23} (for $e \neq 1$) gives rise to a nonzero resultant displacement $\xi_2 - \xi_1$. (The time dependence of pulses of gravitational Bremsstrahlung was considered in Refs. 6, 9, and 10.)

Thus far we have considered closely spaced particles, and we have described the motion of one of them relative to a locally inertial frame of reference attached to the other particle.

If the distance between the particles satisfies $l \gtrsim \lambda_g$, then, as already noted, it is more convenient to work directly with the metric (1). It is well known that the world lines of free particles are space-time geodesics. Numbered among these are the lines $x^1 = \text{const}$, $x^2 = \text{const}$, $x^3 = \text{const}$, i.e., the world lines of the particles that realize the synchronous coordinate system in which the metric (1) is specified. The fate of these particles before the arrival of the pulse ($u < u_1$) and after it has passed ($u > u_2$) is determined by the functions $a(u)$ and $b(u)$ in these regions. Since space-time is flat outside the pulse, $a(u)$ and $b(u)$ can have only the form of linear functions of u there:

$$a(u) = a_1 u + a_2, \quad b(u) = b_1 u + b_2, \quad (8)$$

where a_1, a_2, b_1, b_2 are arbitrary constants. On the boundary hypersurfaces $u = u_1$, $u = u_2$ the functions a , b themselves and their first derivatives must be continuous. The second derivatives i.e., the components of the curvature tensor, can have discontinuities. If we were to assume that $a(u) = 0$ and $b(u) = 0$ for $u < u_1$ and $u > u_2$, we should overdetermine the physical properties of the gravitational-wave pulse. In this case, particles at rest before the arrival of the wave (x^i

$= \text{const}$) remain in a state of rest and at the same separation as they had initially after the wave has passed. However, this is not always so.

Suppose these particles with world lines $x^i = \text{const}$ are at rest before the arrival of the wave, i.e., $a(u)$ and $b(u)$ are equal to zero for $u < u_1$. The behavior of these particles after the wave has passed depends on the values of the functions $a(u)$ and $b(u)$ and their first derivatives at $u = u_2$. These values determine the constants in the expressions (8) and thus the further behavior of the particles carrying the coordinates $x^i = \text{const}$. As we have already noted, for a real pulse of gravitational radiation the constants a_2 and b_2 may be nonzero. The resulting displacement, being proportional to the distance l between the particles (we recall that it is perfectly permissible to have $l \gg \lambda_g$), can reach appreciable values (see the definite estimates below). For example, the distance between particles situated on the x^2 axis after the passage of the pulse, i.e., for $u \geq u_2 = ct_2$, is $l = l_0(1 - \frac{1}{2}a_2)$, where l_0 is the distance before the arrival of the pulse.

Distant particles for which l is of the order of λ_g or much greater than λ_g can also be subject to kinematic resonance. For them, it is first of all necessary to make more precise the concept of zero velocity at the initial time. For closely spaced particles considered in a locally inertial frame of reference, the condition $v_1 = 0$ meant a zero initial velocity with respect to the "rigid frame" attached to one of the particles that realized the locally inertial frame. (We shall not dwell on the well-known conditions of applicability of the concept of a rigid frame.) Such a construction is impossible for a distant particle. However, for both close and distant particles it is possible to introduce a universal concept of a vanishing initial velocity based on the requirement that the frequencies of a light signal emitted from one particle and received by the other be equal. For closely spaced particles, such a definition is equivalent to the condition $v_1 = 0$ with respect to a locally inertial frame. Since the frequency of a light signal measured with respect to the synchronous coordinate system (1) varies periodically along the path of a ray propagating in the field of a monochromatic gravitational wave,⁵ free particles whose world lines differ somewhat from $x^i = \text{const}$ will have vanishing initial velocity. The tangent 4-vector u^α to the geodesics in which we are interested has small (of order h) nonvanishing spatial components. As a result, the distance to a distant free particle released with zero initial velocity will have a resonance (proportional to t) component besides the ordinary periodic term. For closely spaced particles, the resonance contribution (6) can be rewritten in the form

$$\Delta l = \pi h (l/\lambda_g) c \hat{\tau} \cos \psi.$$

For distant particles, the factor l/λ_g is replaced by a quantity of order unity, so that the resonance term takes the form $\Delta l \approx \pi h c \hat{\tau} \cos \psi$.

The simple arguments put forward in Sec. 2 lead to some practical recommendations.

3. PROPOSALS FOR EXPERIMENTS

1. In the designs of the two large laser gravitational antennas currently created^{11,12} with $l \approx 5 \times 10^5$ cm it is assumed

that the response detection time τ_{det} must be less than the duration $\tau_g \approx 10^{-3}$ sec of the gravitational burst. However, as was shown above, antennas "remember" the effect of a gravitational burst from a source of gravitational Bremsstrahlung. Since in these antennas the mirrors are pendulums (oscillators) with a period of mechanical vibrations of about $\tau_0 \approx 1$ sec, they can with good accuracy be regarded as free particles for $\tau \approx 0.1 \tau_0$, i.e., during this time, they keep a "memory" of the applied pulse. It is clear that the possibility of increasing the averaging time from $\tau_{\text{det}} \approx 10^{-3}$ sec to $\tau_{\text{det}} \approx 0.1$ sec greatly lessens the requirements on the optical detection system.

2. In the program to look for gravitational waves from binary stars by means of satellites¹³ revolving around the Earth with a period near 3×10^7 sec and separated from each other by a distance of order $l \approx 2 \times 10^{11}$ cm, relative oscillations of the satellites with amplitude of order $\Delta l \approx 10^{-9}$ cm are expected. If two satellites are in the plane of a gravitational wave front for the time $\tau \approx 2 \times 10^5$ sec and the conditions of kinematic resonance are satisfied for them, the systematic approach (or separation) of the satellites due to the kinematic resonance will be $\Delta l \approx \pi h (l/\lambda_g) c \hat{\tau} \approx 10^{-7}$ cm, i.e., two orders of magnitude greater. Of course, as in the other cases of the use of kinematic resonance, it is necessary to foresee a way of separating the effect due to the kinematic resonance from the monotonic change in the distance associated with a small neglected relative velocity.

3. The program mentioned above¹³ can also be used to look for short bursts of gravitational radiation. According to the estimates of Ref. 14, there may be bursts with $h \approx 10^{-13}$, of duration $\tau_g \approx 10^4$ sec, with one event occurring about every 10^6 sec. Such a pulse produces a $\Delta l \approx \frac{1}{2} h l \approx 10^{-2}$ cm. The memory effect makes it possible to use a long detection time, which may be of order $\tau_{\text{det}} \approx 2 \times 10^5$ sec.

4. In the Precision Optical Interferometry in Space program (POINTS¹⁵) it is proposed to achieve a resolution in the measurement of angular displacements of stars at the level $\alpha \approx 2 \times 10^{-12}$ rad (in order to measure general relativistic effects having the order of magnitude φ^2/c^4). This program can also be used to look for low-frequency gravitational waves. Suppose there are two stars in approximately the same direction on the sky and such that the distance l_1 between the interferometer and the first star is of the order of a few light years, while the distance l_2 to the second star is of the order of tens or hundreds of light years. If a pulse of gravitational radiation passes the interferometer and the first star, the memory effect

must remember the change in their relative distance and, as a result, displace the apparent position of one star relative to the other by an angle $\alpha_g \approx h$. This change can be detected if the angular position of the stars is measured at an interval of about one year. The expected bursts¹⁴ with $h \approx 10^{-13}$ lead to $\alpha_g \approx 10^{-13}$. During one year, the interferometer may detect a random (due to about 30 bursts) angular displacement of the first star relative to the second by $\Delta \alpha \approx 3 \times 10^{-13}$ rad. Thus, a raising of the resolution by an order of magnitude in this program may lead to the possibility of detecting low-frequency bursts of gravitational radiation. It should be emphasized that at approximately the same level ($h \approx 10^{-13}$ for $\nu \approx 10^{-7}$ Hz) effects associated with the presence of a fossil background gravitational-wave noise can be expected.²

¹In electrodynamics it is possible to have so-called strange electromagnetic waves, which lead to a memory effect for the velocity, i.e., to $v_2 - v_1 \neq 0$.⁸ Analogous gravitational waves probably exist.

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