Effective c-number field of a superfluid without a condensate

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It has been shown previously [Sov. Phys. JETP **58**, 722 (1983)] that the "exotic" properties of a Bose condensate [of the quasiaverage $\psi = \langle\hat{\psi}\rangle$], which stem from the anomalous fluctuations in ψ caused by phase degeneracy, rule out the identification of a "microscopic" basis of a phenomenological c-number field of a superfluid with this condensate. A microscopic definition of the c-number field as a quasiaverage is written by replacing the original field operator $\hat{\psi}$ by an effective operator $\hat{\tilde{\psi}}$. The need for this replacement is even more obvious when the fluctuations resulting from the degeneracy destroy the condensate entirely: $\psi = 0$ [the superfluidity, either two-dimensional ($T>0$) or one-dimensional ($T=0$), of the Bose system is retained]. When the condensate vanishes, $\psi = 0$, the effective field $\tilde{\psi} = \langle\hat{\tilde{\psi}}\rangle$ is conserved. Furthermore, regardless of the presence of a condensate, this field can be represented as the wave function of some effective condensate of quasiparticles which determines the ground state of the superfluid. At $T>0$, this field, along with the thermal excitations, determines the density matrix of the superfluid. The effective condensate not only reveals the quantum-mechanical nature of the c-number field of a superfluid, in terms of a macroscopically filled one-particle level (by analogy with an ideal Bose gas and ordinary "classical" fields) but also constitutes a necessary and sufficient condition for superfluidity. It associates with the superfluidity an effective long-range order which is conserved even if the initial long-range order is disrupted by fluctuations. The manifestations of two types of an infrared anomaly of the anharmonicity — one stemming from a phase degeneracy and one stemming from an increase in the fluctuation amplitude toward a transition point — are analyzed for a Bose system without a condensate.

I. INTRODUCTION

1. A unique property of a superfluid is the presence of a subsystem which is described phenomenologically by a c-number field of macroscopic amplitude, so that the quantum-mechanical nature of the system has some direct macroscopic manifestations: a dissipationless mass transfer[1] and a quantization of the circulation of the velocity[2] at all $T\geqslant 0$, as well as interference field effects near T_c (Ref. 3). The phenomenological c-number field which results from a spontaneous breaking of the gauge symmetry describes, in addition to the "superfluid" macroscopic motion, the phase transition itself.[3,4] The microscopic nature of the c-number field of a superfluid Bose liquid was analyzed in Ref. 5. The field theory of superfluidity[6–8] identifies as the "natural microscopic basis" of the c-number field the quasiaverage of the field operator, $\psi = \langle\hat{\psi}\rangle$, which is a condensate of the original bosons. In this theory, which makes use of the analogy with an ideal Bose gas,[9] the Bose system as a whole is represented as a classical nonlinear field ψ with fluctuating normal modes $\hat{\psi}'$, for which the anharmonicity of the zero-point and thermal vibrations are taken into account in succession by diagrams of a field perturbation theory:

$$\hat{\psi}=\psi+\hat{\psi}'; \quad \hat{H}=H(\hat{\psi},\hat{\psi}^+)=H(\psi,\psi^*)+H_2(\hat{\psi}',\hat{\psi}'^+)+\hat{H}_{int}. \quad (1)$$

The description of the macroscopic motion $H(\psi,\psi^*)$, of the broken symmetry,

$$H^{(\mu)}(\psi_0\neq 0)=\min H^{(\mu)}(\psi), \qquad H^{(\mu)}(\psi)=H(\psi,\psi^*)-\mu\int\psi^*\psi\,d\mathbf{r},$$

of the spectrum $\hat{H}_2$, and of the interaction of the excitations $\hat{H}_{int}$ directly from (1) requires a single refinement, viz., allowance for anharmonicity, i.e., replacement of the initial Hamiltonian $H(\hat{\psi},\hat{\psi}^+)$ in (1) by an "effective" Hamiltonian $\tilde{H}(\hat{\psi},\hat{\psi}^+)$, which by definition contains exact vertices instead of zeroth-approximation vertices.

As is shown in Ref. 5, however, this refinement disrupts the analogy with a classical field "without pathology": The exact vertices are radically different from the zeroth-approximation vertices; in particular, the two-prong vertex $\Sigma_{12}(0)$ (the primary characteristic of the long-wave properties of the field ψ and of the spontaneous transition to $\psi_0\neq 0$, which tends toward zero only toward the transition point.

$$\Sigma_{12}{}^{B}(0)=n_0V_0, \qquad n_0=|\psi_0|^2\to 0,$$

according to description (1), vanishes identically upon the replacement $\hat{H}=\tilde{H}$ (Refs. 10 and 11). In the exact approach, most of the characteristics which are related to $\Sigma_{12}(0)$ in the first approximation [the hybridization of the "particles and holes"— the creation and annihilation operators, which transform the quadratic spectrum into a linear spectrum; the velocity of sound $c_B = (\Sigma_{12}^B(0)/m)^{1/2}$; the equilibrium value

$$\langle\hat{\psi}\rangle=n_0{}^{1/2}, \qquad n_0=\Sigma_{12}{}^{B}(0)/V_0,$$

etc.], lose their direct connection with $\Sigma_{12}(0)=0$. On the other hand, the connection between $\Sigma_{12}(0)$ and the longitudinal susceptibility to perturbations of the condensate,

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$|\langle\hat{\psi}\rangle|$, is conserved,

$$\chi_{\parallel}(p\to 0)=-{}^1/_4\Sigma_{12}(0),$$

in contrast with the standard picture of a spontaneous breaking of the gauge symmetry of a classical field,[5] $\chi_{\parallel}(p\to 0)\to\infty$.

The vanishing of $\Sigma_{12}(0)$ occurs even in the "quasiharmonic" model

$$\alpha=mp_0^{d-2}V_0\ll 1,\quad \beta=(n/p_0^d)^{1/2}\sim 1/\alpha^{1/2}, \tag{2}$$

where each integration with respect to momentum in the diagrams characterizing the anharmonicity is accompanied by a small factor α. Here p_0 is a characteristic momentum of the interaction potential, and d is the number of spatial dimensions. The ratio of the potential energy to the kinetic energy is small ($\sim\alpha$) for a pair of particles, but over a unit volume it is ~ 1, and it is this circumstance which is largely responsible for the interesting features of this model. The source of this "infrared anomaly of the anharmonicity" is the large interaction of long-wave field modes ($V_{p\to 0}\neq 0$), whose energy $\varepsilon=cp$ is small according to the Goldstone theorem. There are no anomalies in the hydrodynamic variables n and $\mathbf{v}$, where the interaction of modes vanishes in the limit $p\to 0$, and the effective Hamiltonian in the long-wave limit (the Landau Hamiltonian[1]),

$$\hat{H}(\hat{n},\hat{\mathbf{v}})=\int[m\hat{\mathbf{v}}\hat{n}\hat{\mathbf{v}}/2+\varepsilon(\hat{n})]d\mathbf{r},$$

becomes the same as the original Hamiltonian, $H(\hat{n},\hat{\mathbf{v}})=H(\hat{\psi},\hat{\psi}^+)$, if we set $\varepsilon(\hat{n})=V_0\hat{n}^2/2$. The reason for the difference in variables is that $V=(\hbar/m)\nabla\varphi$ is linear, and $\psi=|\psi|e^{i\varphi}$ nonlinear, in the phase φ: the "broken-symmetry variable," whose fluctuations are large because of the degeneracy,

$$g_{\varphi\varphi}\infty(\varepsilon^2+c^2p^2)^{-1},\quad \langle|\varphi_p|^2\rangle\infty 1/p^2(T>0);\ 1/p(T=0).$$

If the choice of variables is to reflect both the hydrodynamic nature of the weakly interacting (approximately independent) long-wave degrees of freedom of the superfluid (the linearity of these degrees of freedom in $\hat{n}$ and $\hat{\varphi}$) and the specific field nature of the state (the quantization of the circulation and the spontaneous breaking of the gauge symmetry), we must introduce a variable of the field type—the effective field operator $\tilde{\psi}=\tilde{\psi}_L+\tilde{\psi}_{sh}$—while retaining the linearity in $\hat{n}$ and $\hat{\varphi}$ in the long-wave region. The long-wave component of the field $\tilde{\psi}$,

$$\hat{\tilde{\psi}}_L=(\langle\hat{n}_L\rangle)^{1/2}e^{i\langle\hat{\varphi}\rangle}\left[1+\frac{\hat{n}_L-\langle\hat{n}\rangle}{2\langle\hat{n}\rangle}+i(\hat{\varphi}_L-\langle\hat{\varphi}\rangle)\right], \tag{3}$$

is found by linearizing the expression

$$(\langle\hat{n}\rangle+\hat{n}')^{1/2}\exp\{i(\langle\hat{\varphi}\rangle+\hat{\varphi}')\},$$

which characterizes (somewhat arbitrarily) the original field $\hat{\psi}$. Because of the "long-wave" nature, we need no refinement in the definition of $\hat{\mathbf{v}}$ or $\hat{\varphi}$ (Refs. 12 and 13). In the short-wave region, where $\hat{n}$ and $\hat{\varphi}$ lead to an ultraviolet di vergence, the field $\tilde{\psi}=\tilde{\psi}_{sh}$ must be approximately $\hat{\psi}_{sh}=\hat{\psi}-\hat{\psi}_L$ (in the case $n_0\neq 0$, the identification $\tilde{\psi}_{sh}=\hat{\psi}_{sh}$ is permissible; more on this below). A representation of the type in (1) in terms of $\tilde{\psi}$ in the long-wave region,

$$\begin{aligned}\hat{H}&=H(\hat{n},\hat{\mathbf{v}})=H(n+\hat{n}',\mathbf{v}+\hat{\mathbf{v}}')=H(n,\mathbf{v})+\hat{H}_2+\hat{H}_{int}\\&=H'(\hat{\tilde{\psi}},\hat{\tilde{\psi}}^+)=H(\tilde{\psi},\tilde{\psi}^*)+H_2(\hat{\tilde{\psi}}',\hat{\tilde{\psi}}'^+)+\hat{H}'_{int},\\&\qquad\hat{\tilde{\psi}}=\tilde{\psi}+\hat{\tilde{\psi}}',\ \tilde{\psi}=\langle\hat{\tilde{\psi}}\rangle,\end{aligned} \tag{4}$$

corresponds to a converging perturbation theory. We note that $H(\tilde{\psi},\tilde{\psi}^*)$, thought of as a functional, is not the same as $H'(\tilde{\psi},\tilde{\psi}^+)$ The quadratic term is identical in the two Hamiltonians, but $H'_{\rm int}$ differs from $H_{\rm int}$ in that it corresponds to a weak interaction, which vanishes in the long-wave limit, of the modes of the field $\tilde{\psi}$ [the field $\tilde{\psi}$ appears in $H'(\tilde{\psi},\tilde{\psi}^+)$ not only "as part of" $\tilde{\psi}=\tilde{\psi}+\tilde{\psi}'$ but also separately; consequently, despite the similarity between $H(\tilde{\psi},\tilde{\psi}^*)$ and $H(\psi,\psi^*)$ the infrared anomaly of the anharmonicity does not "regenerate" in the variables $\tilde{\psi}$].

The exact vertesx $\tilde{\Sigma}_{12}(0)$ in terms of the "adequate" variables $\tilde{\psi}$ is nonvanishing and plays the same important role as in the first approximation; i.e.,

$$c=(\tilde{\Sigma}_{12}(0)/m)^{1/2},\quad |\tilde{\psi}|^2\infty\tilde{\Sigma}_{12}(0).$$

The effective Hamiltonian in the variables $\tilde{\psi}$, $\tilde{H}=\tilde{H}'(\tilde{\psi},\tilde{\psi}^+)$, is analogous to $H'(\tilde{\psi},\tilde{\psi}^+)$ and in fact describes the macroscopic motion of the superfluid at $T=0$,

$$\tilde{H}(\tilde{\psi},\tilde{\psi}^*)=\int[|\nabla\tilde{\psi}|^2/2m+\varepsilon(|\psi|^2)]d\mathbf{r},$$

and the broken symmetry,

$$\tilde{H}^{(\mu)}(\tilde{\psi}_0\neq 0)=\min\tilde{H}^{(\mu)}(\tilde{\psi})$$

in terms of a classical nonlinear field "without pathologies." It also describes the spectrum of $\tilde{H}_2$ and the interaction of the perturbations, $\tilde{H}'_{\rm int}$. The generalization of $\tilde{\psi}$ to $T>0$ indicates a microscopic basis for the decomposition of the superfluid into two subsystems (which interact in a peculiar way; more on this below): a classical field at $T=0$ and thermal excitations. This generalization demonstrates that the choice of the quantity $(\rho_s/m)^{1/2}$ instead of $|\psi|=n_0^{1/2}$ as the modulus of the superfluid-ordering parameter in the phenomenological theory[4] is not by chance. In terms of $\tilde{\psi}$ the anomaly resulting from the phase degeneracy disappears, but the anomaly stemming from the "mode damping" in the limit $T\to T_c$ remains: $\tilde{\chi}_{\parallel}(0)\infty\tilde{\Sigma}_{12}(0)^{-1}\to\infty$. Near T_c (in the "fluctuation region") the small parameter of the model, (2) is lost; this situation is analogous to that of a real field.

2. In place of $\hat{n}_L$ and $\hat{\varphi}_L$ in the definition of $\tilde{\psi}$ we could also use, to advantage the "polar coordinates" $\tilde{n}_L$ and $\tilde{\varphi}_L$ of the variable $\hat{\psi}_L$, the long-wave part of ψ (with momenta $|p|\leqslant q_0$), provided that $q_0\gg p_c$ (p_c is the "momentum of the infrared anomaly of the anharmonicity," at which the infrared increase in the field diagrams becomes important[5]). However, we can eliminate the ambiguity from the definition of $\tilde{\psi}$ by requiring that the normal modes of the field $\tilde{\psi}$ agree with the exact quasiparticle operators. This definition not only eliminates the infrared anomaly of the anharmoni-

city, which obstructs the analogy with a nonlinear classical field, but also indicates the similarity with an ideal Bose gas—a similarity which is more profound than that in the original form of London's concept[9] of field theory (Section 2). Here $\tilde{\psi} = \langle \tilde{\psi} \rangle_{T=0}$ means the wave function of a one-particle level of certain "condensate quasiparticles," and the set of these quasiparticles represents an effective condensate. This set gives an exact description of the ground state. The quantum-mechanical nature of the c-number superfluid field is thus revealed in terms of a macroscopically filled one-particle level or coherent state, by analogy with ordinary "classical" fields such as radio waves, a laser beam, or a sound wave in a crystal in the limit $T \to 0$ and also an ideal gas of massive bosons. At $T > 0$, some of the quasiparticles undergo a transition from the effective condensate to a gas of "above-condensate quasiparticles." In the description of the macroscopic motion and long-wave vibrations of the effective condensate one can see the special interaction of the condensate and the gas: an "entrainment" and a "hybridization" (even if the interaction of quasiparticles is not taken into account explicitly).

3. The microscopic nature of the c-number field and the analogy between a superfluid and a classical nonlinear field or an ideal Bose gas are particularly interesting in the case in which the original boson condensate vanishes: $n_0 = 0$. This question is the subject of the present paper. The disappearance of the original condensate (a small number of dimensions[14] and the nature of the boundaries[15]) use a manifestation of the same infrared anomaly of an anharmonicity which has led in cases considered previously[5] to only a pathology of the properties of the original condensate (an infinite longitudinal susceptibility). There are several considerations which suggest that the basic properties of the state, primarily the superfluidity, may be retained in this case.[15,16] In the present paper we study the elimination of the infrared anomaly of the anharmonicity in the case $n_0 = 0$, the determination of $\tilde{\psi}$, and the existence of an effective condensate with $\langle \tilde{\psi} \rangle \neq 0$ in the absence of the original condensate, $[\hat{\psi}\rangle = 0$ (Section 3). We show that the effective condensate forms a microscopic basis for the superfluid component, regardless of whether the original condensate is retained, and it emerges as a necessary and sufficient condition for superfluidity. We refine the nature of the "disappearance" of the original condensate. The conservation of the macroscopic filling number N_0 (the unbounded increase in N_0 with the volume V, although slower than that of V) is also a condition for superfluidity (Section 4). We note that in the case $n_0 = 0$ we have $\Sigma_{12}(p) \equiv 0$; i.e., the "hybridization" of the particles and "holes" does in fact disappear (although the spectrum remains sonic). The characteristics of the system in terms of $\hat{\psi}$ and $\tilde{\psi}$ are quite different not only in the long-wave region but also in the short-wave region. We cannot replace $\tilde{\psi}_{sh}$ by $\hat{\psi}_{sh}$; physically, the "binary" condensate disappears along with the "one-particle" condensate. Although the effective condensate is, in a sense, made up of the original condensate—the one-particle condensate and the "higher-order" condensates (pairs, trios, etc.)—the disappearance of all the original condensates does not contradict the retention of an effective condensate; the explanation is simply that the approximation of the effective condensate by a finite set of original condensates, growing more slowly than V, is not an adequate approximation (Section 5). With $n_0 = 0$ ($d = 2$, $T > 0$; $d = 1$, $T = 0$), the long-wave field properties differ even more radically from the "harmonic" (Bogolyubov approximation), but in terms of $\tilde{\psi}$ all the anomalies vanish, and we see the reappearance of the picture of a classical field with a broken gauge symmetry and quantized fields:

$$\tilde{\Sigma}_{12}(p) \neq 0, \quad \tilde{\Sigma}_{12}(0) \infty |\tilde{\psi}|^2.$$

If $n_0 \neq 0$, the Bogolyubov approximation in terms of $\tilde{\psi}$ for model (2) (the thermal "emptying" of the effective condensate is taken into account) breaks down only in the immediate vicinity of T_c. In the case $n_0 = 0$ (i.e., $d = 2$), this approximation breaks down even at $T \sim T_c$. This result can be understood by noting that here the "singular" excitations, which must be taken into account in a special way, become important: the Kosterlitz-Thouless transition[17] (Section 6).

The long-range order in terms of the parameter of the Ginzburg-Pitaevskiĭ λ transition,[4] $\tilde{\psi}$, for which the effective condensate is the microscopic basis (Section 2), is thus not disrupted in the cases $d = 2, T > 0$ and $d = 2, T = 0$ (Section 4). It can be shown that this eventuality is ruled out by Rice's calculations[14]: A path integration with the Ginzburg-Landau Hamiltonian $F = \tilde{H}(\tilde{\psi},\tilde{\psi}^*)$ over $|\tilde{\psi}|$, $\tilde{\varphi}$ (where $\tilde{\psi} = |\tilde{\psi}| e^{\tilde{\varphi}}$) indicates a disruption of the long-range order in the correlation function $\langle \tilde{\psi}(\mathrm{r}) \tilde{\psi}^*(0) \rangle$. [The difference between $\tilde{H}(\tilde{\psi},\tilde{\psi}^*)$ and $\tilde{H}'(\tilde{\psi},\tilde{\psi}^+)$ does not rule out the use of $\tilde{H}(\tilde{\psi},\tilde{\psi}^*)$, since it is actually the quadratic part $\tilde{H}_2$, which is identical for $\tilde{H}$ and $\tilde{H}'$, which figures in the calculations of Ref. 14.] The error here is that the nonlinear transformation of variables is not legitimate in the effective Hamiltonian [e.g., $\tilde{H}(\psi,\psi^*) \neq \tilde{H}(n,\varphi)$, although $H(\psi,\psi^*) = H(n,\varphi)$; see Ref. 5]. The result of Ref. 14 is valid only in application to the original order parameter $\psi(\langle \psi \rangle = n_1^{0/2})$, for $\langle \psi(\mathbf{r}) \psi^*(0) \rangle$. Furthermore, this error is cancelled here by another error: The form used for $\tilde{H}(\psi,\psi^*)$ in Ref. 14 ignores the infrared anomaly of the anharmonicity [when this anomaly is taken into account, $\tilde{H}_2$ (ψ,ψ^*) gives us $\langle \psi(\mathbf{r}) \psi^*(0) \rangle$ (without a long-range order) through an immediate integration over ψ,ψ^*]. In the calculation of Ref. 14 for $\langle \psi(\mathbf{r}) \psi^*(0) \rangle$, it would have been correct to work directly from $\tilde{H}(n,\varphi)$. A path integral with $\tilde{H}$ (or, more precisely, $\tilde{H}_2$) can be used, but only in terms of "its own" variables; if the variables are appropriate (a weak anharmonicity), the calculations lead to not only the exact result for the correlation function of these variables (the Lagrangian $\hat{L}_2 = G^{-1}$) but also an approximate result for other correlation functions. The additional refinements of Ref. 14 are required in the case of superconductivity [an elongation of the type $\nabla \psi \to (\nabla - ie/\hbar c)\psi$ of the derivatives in $\tilde{H}(\psi,\psi^*)$ leads to a gap, while electrical neutrality leads to a new mode without a gap; see Section 4].

The infrared anomaly of the anharmonicity is also ignored in $\tilde{H}(\psi,\psi^*)$ in Josephson's approach (see Ref. 31 in Ref. 5). Although the nature of $\chi_{\parallel}(\varepsilon = 0, p \to 0)$, which reflects the infrared anomaly, is not used in the derivation of the relation between the density of the original condensate and ρ_s $(T \to T_c)$, it is still more accurate to speak in terms of an effective condensate instead of the original condensate here [the original condensate may be absent even at $T=0$ $(d=3)$, if there are intense short-wave zero-point vibrations].

2. EFFECTIVE CONDENSATE AND THE SUPERFLUID COMPONENT

1. The quasiparticle concept associates in an approximate way the weakly perturbed states of interacting and noninteracting systems. To what extent can we identify the c-number field of a superfluid with the c-number field of an ideal Bose gas? The latter is of course identified with the wave function of a condensate level (normalized to the number of particles, N), as in (5), or with an eigenvalue of a condensate operator, as in (6) [and, simultaneously, of the field as a whole, (7)]. A coherent state of a "condensate-mode oscillator" $|\Phi'_{0,1}\rangle$, (6), (7), is approximately the same as the "N-fold excited" state $|\Phi_{0,1}\rangle$, (5), because of the macroscopic nature of $N \gg 1$. Here

$$|\Phi_0\rangle=\frac{1}{(N!)^{1/2}}(\hat{a}_0^+)^N|0\rangle, \quad |\Phi_1\rangle=\frac{1}{(N!)^{1/2}}(\hat{\psi}_0^+)^N|0\rangle; \qquad (5)$$

$$\hat{a}_0|\Phi_0'\rangle=N^{1/2}|\Phi_0'\rangle, \quad \hat{a}_{p\neq 0}|\Phi_0'\rangle=0, \qquad (6)$$

$$|\Phi_0'\rangle=\exp[N^{1/2}(\hat{a}_0^+-\hat{a}_0)]|0\rangle, \quad \hat{\psi}_0|\Phi_1'\rangle=\psi_0(\mathbf{r})|\Phi_1'\rangle;$$

$$\hat{\psi}(\mathbf{r})|\Phi_0'\rangle=(N/V)^{1/2}|\Phi_0'\rangle,$$

$$|\Phi_0'\rangle=\exp\left[(N/V)^{1/2}\int d\mathbf{r}(\hat{\psi}^+(\mathbf{r})-\hat{\psi}(\mathbf{r}))\right]|0\rangle; \qquad (7)$$

$$\begin{gathered}\hat{\psi}(\mathbf{r})|\Phi_1'\rangle=\psi_0(\mathbf{r})|\Phi_1'\rangle;\\ |\Phi_0^*\rangle=\Pi\hat{b}_p^+|\Phi_0\rangle, \quad \hat{b}_p^+=\hat{a}_p^+\hat{\chi}_0,\\ \hat{\chi}_0=(\hat{a}_0^+\hat{a}_0+1)^{-1/2}\hat{a}_0, \quad \hat{\chi}_0\hat{\chi}_0^+=1.\end{gathered} \qquad (8)$$

Here $|\Phi_1\rangle$ is a state with an inhomogeneous condensate, and $|\Phi_0^*\rangle$ is a normalized excited state. A trivial answer to the question of the c-number field comes from a simple model for a superfluidity in which the interaction for the above-condensate particles is taken into account only in the form of the "external field" of the condensate. The wave functions here are identical to the "ideal-gas" wave functions (5)–(8) (although the gas is not a superfluid). The superfluidity is provided by a gap in the spectrum:

$$\begin{gathered}\hat{H}^{(\mu)}\to\hat{H}_0^{(\mu)}=H(\hat{a}_0)+\sum_{p\neq 0}\varepsilon_p a_p^+a_p, \quad \varepsilon_p=\varepsilon_p^0+n_0(V_0+V_p)-\mu,\\ \varepsilon_p^0=p^2/2m, \quad n_0=N_0/V=N/V=n,\\ H(\hat{a}_0)=-\mu\hat{a}_0^+\hat{a}_0+(V_0/2V)(\hat{a}_0^+\hat{a}_0)^2.\end{gathered} \qquad (9)$$

The function $\psi_0(\mathbf{r})$ in $|\Phi_1'\rangle$ satisfies a nonlinear equation; the excitations are orthogonal to $\psi_0(\mathbf{r})$. If the "anomalous terms" $(a_\mathbf{p}^+a_{-\mathbf{p}}^+ + a_\mathbf{p}a_{-\mathbf{p}})$ are also taken into account, the above-condensate states transform into quasiparticles. The ideal-gas description corresponds to a linearization of $\hat{H}$ with respect to ψ' in (1), simulating in an excellent way the numerous properties of a superfluid:[5]

$$\begin{gathered}|\Phi_\mathbf{B}'\rangle=\hat{U}_\mathbf{B}'|\Phi_0'\rangle,\\ \hat{U}_\mathbf{B}=e^{\hat{R}_\mathbf{B}}, \quad \hat{a}_p\to\hat{\alpha}_p=\hat{U}_\mathbf{B}\hat{a}_p\hat{U}_\mathbf{B}^{-1}=\hat{a}_p\operatorname{ch}2\chi(p)\\ -\hat{a}_{-p}^+\operatorname{sh}2\chi(p), \quad \hat{R}_\mathbf{B}=\sum_{p\neq 0}\chi(p)(a_p^+a_{-p}^+-a_pa_{-p}),\end{gathered} \qquad (10)$$

$$\left.\begin{matrix}\operatorname{ch}2\chi(p)\\ \operatorname{sh}2\chi(p)\end{matrix}\right\}=\pm[(\varepsilon_p^0+nV_p\pm\varepsilon_p)/2\varepsilon_p^\mathbf{B}]^{1/2},$$

$$\varepsilon_p^\mathbf{B}=[\varepsilon_p^0(\varepsilon_p^0+2nV_p)]^{1/2},$$

$$\hat{a}_0|\Phi_\mathbf{B}'\rangle=N_0^{1/2}|\Phi_\mathbf{B}'\rangle, \quad \alpha_{p\neq 0}|\Phi_\mathbf{B}'\rangle=0, \quad |\Phi_\mathbf{B}^*\rangle=\prod_{p\neq 0}\hat{\alpha}_p^+|\Phi_\mathbf{B}'\rangle.$$

The linearization of $\hat{H}$ is not appropriate, however, because of the infrared anomaly of the anharmonicity which is generated by the interaction of particles with $\mathbf{p}\neq 0$. The problem of the nature of the c-number field can be solved without resorting to approximations at all. We denote by $\hat{U}$ a unitary transformation of the ideal-gas state $|\Phi_0\rangle$ into the exact ground state of a Bose liquid: $|\Phi\rangle = \hat{U}|\Phi_0\rangle$. Introducing the operators

$$\begin{gathered}\hat{\tilde{a}}_0=\hat{U}\hat{a}_0\hat{U}^{-1}, \quad \hat{\tilde{\alpha}}_p=\hat{U}\hat{a}_p\hat{U}^{-1},\\ \hat{\Psi}=\hat{U}\hat{\psi}\hat{U}^{-1}=\frac{\hat{\tilde{a}}_0}{V^{1/2}}+\frac{1}{V^{1/2}}\sum_p\hat{\tilde{\alpha}}_pe^{ipr},\end{gathered} \qquad (11)$$

we write the ground state as a condensate of quasiparticles of an effective condensate, (12), or a coherent state of an effective-condensate mode $\hat{\tilde{a}}_0$ (of the effective field $\hat{\Psi}$), (13), (14):

$$\begin{gathered}|\Phi\rangle=\hat{U}|\Phi_0\rangle=\hat{U}\left[\frac{(\hat{a}_0^+)^N}{(N!)^{1/2}}\right]|0\rangle=\frac{(\hat{\tilde{a}}_0^+)^N}{(N!)^{1/2}}\left|0\right\rangle\\ \times(|\tilde{0}\rangle\equiv\hat{U}|0\rangle=|0\rangle),\end{gathered} \qquad (12)$$

$$\begin{gathered}\hat{\tilde{a}}_0|\Phi'\rangle=N^{1/2}|\Phi'\rangle, \quad \hat{\tilde{a}}_p|\Phi'\rangle=0,\\ |\Phi'\rangle=\hat{U}|\Phi_0'\rangle=\exp[N^{1/2}(\hat{\tilde{a}}_0^+-\hat{\tilde{a}}_0)]|0\rangle,\end{gathered} \qquad (13)$$

$$\Psi(\mathbf{r})|\Phi'\rangle=(N/V)^{1/2}|\Phi'\rangle,$$

$$|\Phi'\rangle=\exp\left\{(N/V)^{1/2}\int d\mathbf{r}(\hat{\Psi}^+(\mathbf{r})-\hat{\Psi}(\mathbf{r}))\right\}|0\rangle \qquad (14)$$

[see (5)–(8)]. To describe the excited states $|\Phi^*\rangle$ we write $|\Phi\rangle$ as the "vacuum," $\tilde{b}_\mathbf{p}|\Phi\rangle=0$ (the approximation of independent one-particle excitations):

$$\begin{gathered}\hat{U}^{-1}(\hat{\tilde{b}}_\mathbf{p}\hat{U}|\Phi_0\rangle)=0=b_\mathbf{p}|\Phi_0\rangle\to\hat{\tilde{b}}_\mathbf{p}=\hat{U}\hat{b}_\mathbf{p}\hat{U}^{-1}, \quad \hat{\tilde{b}}_\mathbf{p}^+=\hat{\tilde{\alpha}}_\mathbf{p}^+\hat{\tilde{\chi}}_0,\\ \hat{\tilde{\chi}}_0=(\hat{\tilde{a}}_0^+\hat{\tilde{a}}_0+1)^{-1/2}\hat{\tilde{a}}_0=\hat{U}\hat{\chi}_0\hat{U}^{-1}, \quad \hat{\chi}_0\hat{\tilde{\chi}}_0^+=1,\end{gathered} \qquad (15)$$

$$|\Phi^*\rangle=\prod_{p\neq 0}\hat{\tilde{b}}_\mathbf{p}^+|\Phi\rangle=\hat{U}|\Phi_0^*\rangle=\hat{U}\prod_{\mathbf{p}\neq 0}\hat{b}_\mathbf{p}^+|\Phi_0\rangle \qquad (16)$$

[see (8)]. Since $\hat{H}_{\rm int}$ commutes with $\hat{N}$ and $\hat{\mathbf{P}}=\sum_\mathbf{p}\mathbf{p}\hat{a}_\mathbf{p}^+\hat{a}_p$, $\hat{U}$ has the same properties in the homogeneous case. We can thus write $\hat{U}$ as

$$\hat{U}=e^{\hat{R}},\quad \hat{R}^{+}=-\hat{R},\quad \hat{R}=\Big\{\sum_{\mathbf{p}} f^{(2)}(p)\,\hat{a}_{\mathbf{p}}{}^{+}\hat{a}_{-\mathbf{p}}^{+}\hat{\chi}_0{}^2 + \frac{1}{N^{1/2}}\sum_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0}[f_1^{(3)}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\,\hat{a}_{\mathbf{p}_1}{}^{+}\hat{a}_{\mathbf{p}_2}{}^{+}\hat{a}_{\mathbf{p}_3}{}^{+}\hat{\chi}_0{}^3 + f_2^{(3)}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) \times \hat{a}_{\mathbf{p}_1}{}^{+}\hat{a}_{\mathbf{p}_2}{}^{+}\hat{a}_{-\mathbf{p}_3}\hat{\chi}_0] + \frac{1}{N}\sum_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4=0}[\ldots]+\ldots\Big\}-\text{H.a.} \tag{17}$$

In contrast with (10), $\hat{U}$ transforms not only the excitation operators but also the operator representing a macroscopically filled mode. Correspondingly, $\hat{a}\to\hat{\tilde{a}}_0\neq\hat{a}_0$. The coefficients $1/N^{1/2}$, $1/N$, ... , at the front of the sums in (17) are chosen such that the N-independent expectation values

$\langle\hat{\psi}(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_2)\rangle$, $\langle\hat{\psi}(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_2)\hat{\psi}(\mathbf{r}_3)\rangle,\ldots$ between $|\Phi'\rangle$

correspond to N-independent functions $f_i^{(n)}$. For model (2), we can find the functions $f_i^{(n)}$ by, for example, minimizing the ground-state energy $\langle\Phi|\hat{H}|\Phi\rangle$ (at $T=0$) or the free energy, taking the gas of excitations into account (at $T>0$, outside the "fluctuation region"). In general, $\hat{U}$ is determined by the circumstance that $\tilde{b}_{\mathbf{p}\to 0}$ diagonalizes $\tilde{H}_2(\hat{n},\hat{\mathbf{v}})$. The nonlinear corrections in $\hat{n}_{\mathbf{p}}$ and $\hat{v}_{\mathbf{p}}$ are given by perturbation theory[12] [formally in the parameter $1/N$ but actually in α in (2)].

The relationship between $\hat{\psi}$ and $\tilde{\psi}$ in (3) is simple in the approximation $\hat{\chi}_0,\hat{\chi}_0^{+}\to 1$, i.e., $\hat{\tilde{a}}_{\mathbf{p}}=\tilde{b}_{\mathbf{p}}$:

$$\hat{\tilde{\psi}}=\frac{\hat{\tilde{a}}_0}{V^{1/2}}+\frac{1}{V^{1/2}}\sum_{\mathbf{p}\neq 0}\hat{\tilde{a}}_{\mathbf{p}}e^{i\mathbf{p}\mathbf{r}}, \tag{18}$$

where the $\hat{\tilde{\alpha}}_p$ are Bogolyubov combinations of the $\hat{\tilde{a}}_p$.

From the fact that $\hat{U}$ commutes with $\hat{N}$, $\hat{\mathbf{P}}$ we have

$$\hat{N}\equiv\sum_{\mathbf{p}}\hat{a}_{\mathbf{p}}{}^{+}\hat{a}_{\mathbf{p}}=\hat{\tilde{N}}=\hat{\tilde{a}}_0{}^{+}\hat{\tilde{a}}_0+\sum_{\mathbf{p}\neq 0}\hat{\tilde{\alpha}}_{\mathbf{p}}{}^{+}\hat{\tilde{\alpha}}_{\mathbf{p}}, \tag{19}$$

$$\mathbf{P}=\sum_{p}\mathbf{p}\hat{a}_{\mathbf{p}}{}^{+}\hat{a}_{\mathbf{p}}=\sum_{p}\mathbf{p}\hat{\tilde{\alpha}}_{\mathbf{p}}{}^{+}\hat{\tilde{\alpha}}_{\mathbf{p}},$$

$$\tilde{N}_0=\langle\hat{\tilde{a}}_0{}^{+}\hat{\tilde{a}}_0\rangle=N-\sum_{p}\langle\hat{\tilde{\alpha}}_{\mathbf{p}}{}^{+}\hat{\tilde{\alpha}}_{\mathbf{p}}\rangle=N-V\int d^3p\,n_G(\varepsilon_p),$$

$$n_G(\varepsilon_p)=[e^{\varepsilon_p/T}-1]^{-1}. \tag{20}$$

We see from (19) that the "walls" $(-v\cdot\mathbf{P})$ act only on the excitations—not on the effective condensate.

The description in terms of $\hat{\tilde{\Psi}}$ is exact for the ground state (characterizing it as an effective condensate, establishing the exact meaning of the c-number field) and is the best approximation for the excited states in terms of independent one-particle excitations. It is also possible to take into account the interaction of excitations:

$$\hat{H}_1=\hat{H}-\hat{H}^{(0)},\quad \hat{H}^{(0)}=E_0+\sum_{\mathbf{p}}\varepsilon_p\hat{\tilde{\alpha}}_{\mathbf{p}}{}^{+}\hat{\tilde{\alpha}}_{\mathbf{p}}.$$

This interaction is responsible for the appearance of bound states of excitations, $\sum_{\mathbf{p}}A_{\mathbf{p}}b_{\mathbf{p}}{}^{+}b_{-\mathbf{p}}^{+}$, near points with $\varepsilon_p'=0$, etc., and for a deviation of the gas of excitations from ideal. However, since it vanishes in the limit $\mathbf{p}\to 0$, it does not generate an infrared anomaly of the anharmonicity, and by definition it does not excite quasiparticles from the effective condensate $(\hat{H}_1|\Phi\rangle=0)$.

If, in the expression for $\hat{U}$ in (17) written in the form

$$\hat{U}=e^{\hat{R}},\quad \hat{R}=N^{1/2}(\hat{b}^{(2)+}+\hat{b}_1^{(3)+}+\hat{b}_2^{(3)+}+\ldots)-\text{H.a.},$$

$$N^{1/2}\hat{b}^{(2)+}\equiv\sum_{p}f^{(2)}(p)\,b_{\mathbf{p}}{}^{+}b_{-\mathbf{p}}^{+},$$

$$N^{1/2}\hat{b}_1{}^{(3)+}\equiv\frac{1}{N^{1/2}}\sum_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0}f_1{}^{(3)}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\,\hat{b}_{\mathbf{p}_1}^{+}\hat{b}_{\mathbf{p}_2}^{+}\hat{b}_{\mathbf{p}_3}^{+}, \tag{21}$$

$$N^{1/2}\hat{b}_2{}^{(3)+}\equiv\frac{1}{N^{1/2}}\sum_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0}f_2{}^{(3)}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\,\hat{b}_{\mathbf{p}_1}^{+}\hat{b}_{\mathbf{p}_2}^{+}\hat{b}_{-\mathbf{p}_3},$$

we interpret the operators $\hat{b}^{(2)},\hat{b}_1^{(3)},\hat{b}_2^{(3)}$, ... as independent boson operators which create bound pairs, trios, etc., of above-condensate particles—more precisely, "excitations of an ideal gas" (although their commutation relations are slightly different, and operators of the type $\hat{b}_2^{(3)}$ always give us zero when they act on $|\Phi_0\rangle$)—then a state with an effective condensate will represent a coherent state of corresponding bosons with a macroscopic value (as in the original condensate) of the amplitudes. The N-independent expectation values between $|\Phi'\rangle$ of the quantities

$\langle\hat{\psi}\rangle$, $\langle\hat{\psi}\hat{\psi}\rangle$, $\langle\hat{\psi}\hat{\psi}\hat{\psi}\rangle$, $\langle\hat{\psi}\hat{\psi}\hat{\psi}^{+}\rangle,\ldots$

correspond to

$\langle\hat{a}_0\rangle\sim N^{1/2}$, $\langle\hat{b}_i^{(n)}\rangle\sim N^{1/2}$;

in this sense the effective condensate is a sort of set of original condensates: one-particle, "binary," and condensates of all "higher orders." For model (2), each of the higher-order original condensates is characterized by an intensity which is weaker, the greater the number of particles in its composite particle.

Canonical transformations dealing with $\hat{a}_0$ have been introduced previously[18] to study the relationship between a theory of the Bogolyubov type[6] and the hydrodynamic approach.[12] It was shown in Ref. 18 that even the use in $\hat{R}$ in (17) of a term with $f^{(2)}(p)$ leads to corrections to the energy and the spectrum without infrared divergences [$f^{(2)}(p)=\chi(p)$,(10), so that there are no terms $\sim(\hat{\tilde{\alpha}}_{\mathbf{p}}^{+}\hat{\tilde{\alpha}}_{-\mathbf{p}}^{+}\hat{\chi}_0^2+\hat{\chi}_0^{+2}\alpha_{\mathbf{p}}\alpha_{-\mathbf{p}})$ in $H(\tilde{\alpha},\tilde{\alpha}^{+})$]. The transformations $\hat{\alpha}_{\mathbf{p}}\to\tilde{\alpha}_{\mathbf{p}}^{<(2)}$

$$(\hat{\tilde{\alpha}}_{\mathbf{p}}^{(2)}=e^{\hat{R}^{(2)}}\hat{a}_{\mathbf{p}}e^{-\hat{R}^{(2)}}=\text{ch}\,2f^{(2)}(p)\,\hat{a}_{\mathbf{p}}-\text{sh}\,2f^{(2)}(p)\,\hat{a}_{-\mathbf{p}}^{+}\chi_0{}^2)$$

are, in contrast with $\hat{a}_p\to\hat{\alpha}_p$ in (10), nonlinear in the Fourier components of $\hat{\psi}$, so that they are capable of eliminating the infrared anomaly of the anharmonicity, according to Ref. 5. Corrections to $\hat{R}^{(2)}$ were also made in Ref. 18 in order to reach agreement with the quasiparticle operators from Ref. 12, written at a higher order of accuracy in $1/N$ than is $\hat{\tilde{\alpha}}_p^{(2)}$.

The case of a slightly inhomogeneous effective condensate does not reduce to the problem of an ideal gas [$\hat{\Psi}$, (11)] in an external field $\int U\hat{\Psi} + \hat{\Psi} d\mathbf{r}$. A change in the wave function of the effective condensate (an inhomogeneity) also changes its internal structure. The long-wave excitations retain their hydrodynamic nature: They are linear in $n_\mathbf{p}$ and $\varphi_{\mathbf{p}'}$ (but not in $\hat{\tilde{\alpha}}_\mathbf{p}$, which combine $\hat{\tilde{\alpha}}_\mathbf{p}, \hat{\tilde{\alpha}}_\mathbf{p}^+$).

2. At low values $T>0$ we have $\hat{\Psi}_T \approx \hat{\Psi}_{T=0}$, and a change in the microscopic state reduces to a transition of some of the quasiparticles from the effective condensate to "thermal" particles:

$$\langle \hat{\Psi} \rangle_{T=0} - \langle \hat{\Psi} \rangle_T \infty T^3$$

[see (20)]. If, however, at $T=0$ the wave function of the effective condensate, $\tilde{\psi} = \langle \hat{\Psi} = \langle \hat{\Psi} \rangle$ [a classical field with $H = \hat{H}^{(\mu)}(\tilde{\psi}, \tilde{\psi}^*)$], describes simultaneously a spontaneous breaking of gauge symmetry, a macroscopic motion, and long-wave excitations of the superfluid, then upon the appearance of thermal excitations these aspects of the field description correspond to noncoincident c-number fields: respectively $\langle \hat{\Psi} \rangle_T$ (the effective consendate), ψ_s (the "superfluid amplitude," which reflects the entrainment of part of the mass of the above-condensate quasiparticles as the effective condensate moves because of the "non-Galilean" nature of the spectrum), and $\tilde{\psi}_T$ (the effective field, which incorporates the hybridization of the oscillations of the "superfluid component" with the collisional sound in the gas of excitations).

Let us consider, at low T, an effective condensate which is moving at a velocity $\mathbf{v}$ in frame of reference K'. Since $\hat{\Psi}_T \approx \hat{\Psi}_{T=0}$, the mass of the quasiparticles of the effective condensate is again equal to the original mass of the particles (at $T=0$, this equality is a consequence of the principle of relativity; i.e.,

$$\mathbf{P}_0 = \int \Psi^* \frac{\hbar}{i} \nabla \Psi \, d\mathbf{r} = m\tilde{N}_0 \mathbf{v}, \quad E_0 = F_0 = \int \frac{|\nabla \Psi|^2}{2m} d\mathbf{r} = \frac{1}{2} m\tilde{N}_0 \mathbf{v}^2 \quad (\Psi = \langle \hat{\Psi} \rangle_T). \tag{22}$$

Let us assume that the gas of quasiparticles is at rest in K' (a Gibbs distribution is given in K'). If we were dealing with noninteracting particles, rather than quasiparticles, the contribution of the "above-condensate subsystem" would not depend on the motion of the condensate; in particular, its momentum would be zero. The spectrum of quasiparticles, in contrast, is tied to the rest frame of the condensate, K, so that the motion of the condensate with respect to the gas of excitations with a given T leads to a change (which depends on $\mathbf{v}$) in all the characteristics of the gas: the momentum, the energy, the free energy, the entropy, etc., even the number of quasiparticles. The corresponding equations can be written easily by noting that the only change from the case in which the Bose liquid moves as a whole is a change in the excitation distribution function:

$$n_G(\varepsilon_p) \to n_G(\varepsilon_p + \mathbf{pv}). \tag{23}$$

As a result, the gas of excitations, which is "at rest," contributes to the momentum and energy of the system, ΔP and ΔE (the volume is $V=1$). For the overall system we find

$$\mathbf{P} = \mathbf{P}_0 + \Delta \mathbf{P} = \rho_s \mathbf{v}, \quad E = E_0 + \Delta E = {}^1/_2 (\rho_s + a) v^2; \quad \rho_s = \rho - \rho_n,$$

$$\rho_n = \frac{1}{3} \int d^3p\, p^2 \left(-\frac{dn_G}{d\varepsilon_p} \right) = \int d^3p\, \tilde{\mu} n_G(\varepsilon_p),$$

$$a = \int d^3p\, \varepsilon_p \tilde{\mu} \left(-\frac{dn_G}{d\varepsilon_p} \right),$$

$$\tilde{\mu} = \tilde{m} + \frac{p}{3} \frac{d\tilde{m}}{dp}, \quad \tilde{m} = p \Big/ \frac{d\varepsilon_p}{dp}; \quad \rho = mN. \tag{24}$$

The increment in the mass of the effective condensate due to the changes in the gas of excitations changes the very nature of the temperature dependence of the mass: $(\rho - m\tilde{N}_0) \infty T^3, [\rho - (\rho_s, \rho_s + a)] \infty T^4$. In a sense, part of the mass is entrained by the effective condensate. We wish to call attention to the discrepancy between the characteristics of the inertia of the moving effective condensate in $\mathbf{P}$ and E in (24); in an arbitrary frame of reference K_0, this discrepancy leads to a coupling of the velocities of the gas of excitations, $\mathbf{v}_n$, and of the effective condensate, $\mathbf{v}_s = \mathbf{v}_n + \mathbf{v}$ (a sort of dynamic dependence of the subsystems):

$$\mathbf{P}^{(K_0)} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s, \quad E^{(K_0)} = {}^1/_2 (\rho_n + a) \mathbf{v}_n^2 + {}^1/_2 (\rho_s + a) v_s^2 - a \mathbf{v}_s \mathbf{v}_n. \tag{25}$$

This effect, like the nonadditivity of ρ_n, studied in Ref. 5,

$$\rho_n \neq \int d^3p\, p\tilde{m} n_G(\varepsilon_p),$$

results from a violation of the relativity principle in the ε_p spectrum. This violation is responsible for the entrainment effect: $\mathbf{P} \neq \mathbf{P}_0$, $E \neq E_0$. Different manifestations of the non-Galilean nature of the spectrum are seen in different orders in v: In zeroth order we find

$$m\tilde{N}_0 + \int \tilde{m} n_G(\varepsilon_p) d^3p \neq mN = \rho,$$

in first order we find

$$\int \tilde{m} n_G(\varepsilon_p) d^3p \neq \rho_n,$$

and in second order we find a change in the internal characteristics of a gas of excitations with a fixed temperature T as the condensate moves; here

$$E_0 + \Delta E \neq \rho_s v^2/2.$$

The simultaneous breaking of Galilean symmetry and gauge symmetry is a specific feature of the superfluid transition and is not seen, for example, in the case of a Lorentz-invariant Higgs condensate. A decoupling of $\mathbf{v}_s$ and $\mathbf{v}_n$ is achieved by replacing E by the free energy F:

$$F = F_0 + \Delta F = \rho_s v^2/2, \quad F^{(K_0)} = \rho_n v_n^2/2 + \rho_s v_s^2/2. \tag{26}$$

In other words, thermodynamically, $\mathbf{v}_s$ is, in accordance with a basic assumption of two-fluid hydrodynamics, an independent variable, which complements the parameters of the "local Gibbs distribution," T, ρ, and $\mathbf{v}_n$. An analogous decoupling occurs in the energy E if we fix the entropy of the gas of excitations, $S(\Delta E(S,\rho) = \Delta F(T,\rho)$, instead of T.

The renormalization of the mass of the effective condensate, $m\tilde{N}_0 \to \rho_s$, implies a renormalization of the amplitude of the c-number field which describes the superfluid motion: $|\Psi_T|^2 = \tilde{N}_0 \to |\psi_s|^2 = \rho_s/m$ (in ΔF, $|\nabla\Psi_T|^2/2m \to |\nabla\psi_s|^2/2m$): The parameter m—the coupling between $\mathbf{v}$ and the wavelength of the field—cannot change. There is no renormalization in a description of an inhomogeneous state of the effective condensate if the gas of excitations is assumed to be at rest with respect to the effective condensate (boundary effects, etc.).

To calculate the spectrum and structure of the long-wave oscillations of the effective condensate we need to consider, in addition to the entrainment, their hybridization with collisional sound in the gas of excitations ($\tilde{\psi}_T$). A corresponding calculation was carried out in Ref. 5 for low T, with damping ignored.[1)]

A microscopic description by means of an effective condensate and excitations is a more general approach than a thermodynamic (or hydrodynamic) separation into superfluid and normal components. It can describe local deviations from equilibrium of the following entities: a gas of excitations (a kinetic equation), the amplitude of the c-number field (a nonlinear field equation which uses an effective field Hamiltonian and the damping due to the hybridization of the perturbations of the field with dissipative processes in the system of excitations), and the field structure (i.e., the "composition" of the quasiparticles of the effective condensate, $\hat{\tilde{a}}_0 = \hat{U}\hat{a}_0\hat{U}^{-1}$, the relations among the densities of the original condensate—the one-particle, binary, etc.). It is the perturbations of the latter type which are characterized by an infrared anomaly of the anharmonicity, $\chi(p\to 0)\to\infty$.

3. INFRARED ANHARMONICITY ANOMALY IN BOSE SYSTEMS OF VARIOUS DIMENSIONALITIES

1. When the original condensate disappears, $\langle\psi\rangle = n_0^{1/2} = 0$, the original field theory, (1) (Refs. 6–8), cannot even be formulated, but the infrared anomaly of the anharmonicity is manifested in the "combined" variables $(\tilde{n}_L, \tilde{\varphi}_L, \psi_{sh})$:

$$\psi=\psi_L+\psi_{sh},\qquad \psi_L=\frac{1}{V^{1/2}}\sum_{|\mathbf{p}|<q_0} a_\mathbf{p} e^{i\mathbf{pr}}=\tilde{n}_L^{1/2}e^{i\tilde{\varphi}_L}, \tag{27}$$

where the diagrams are field diagrams in the region $|p|\geqslant q_0$ (the role of n_0 in these diagrams is played by $\langle\tilde{n}_L\rangle$). It is possible to eliminate all manifestations of the infrared anharmonicity anomaly by choosing q_0 appropriately?

We first consider the simplest model, (2). A specific feature of the case $n_0 = 0$ is a divergence (with decreasing q_0) $n' = n - \langle\hat{n}_L\rangle$ in the harmonic approximation:

$$n'^{(\mathrm{B})}_{T=0}=\frac{1}{V}\sum_{|p|\geqslant q_0}\frac{\varepsilon_p{}^0+nV_p-\varepsilon_p{}^\mathrm{B}}{2\varepsilon_p{}^\mathrm{B}}\sim(nV_0m)^{1/2}\int_{q_0} p^{d-2}\,dp;$$

$$\Delta n'^{(\mathrm{B})}_T=\frac{1}{V}\sum_{|p|\geqslant q_0}\frac{\varepsilon_p{}^0+nV_p}{\varepsilon_p{}^\mathrm{B}}n_G(\varepsilon_p)\sim Tm\int_{q_0} p^{d-3}\,dp, \tag{28}$$

$$n'(q_0\sim\tilde{p}_c)\sim n,\quad \tilde{p}_c\sim\frac{T}{c}e^{-n/mT}\sim\frac{T}{c}e^{-cp_0/\alpha T}\quad(d=2,T>0),$$

$$\tilde{p}_c\sim p_0e^{-n/mc}\sim p_0e^{-1/\alpha}\qquad(d=1,T=0),$$

$$\tilde{p}_c\sim\frac{mT}{n}\sim\frac{\alpha T}{c}\quad(d=1,T>0). \tag{29}$$

The choice $q_0 \gg \tilde{p}_c$ provides for the factor $\langle\tilde{n}_L\rangle$ in the diagrams the same estimate as in the case $n_0\neq 0(\langle\tilde{n}_L\rangle = n[1-O(\alpha)])$. Correspondingly, we can estimate the divergence of an arbitrary field diagram by analogy with the case $n_0\neq 0$:

$$A^{T=0}\sim A_0[\alpha(p_0/p)^{3-d}]^R;\qquad A^{T>0}\sim A_0[\alpha(p_0/p)^{3-d}(T/cp)]^R,$$

$$A_0\sim\alpha^{r/2-1}p_0^{2+d-(dr/2)-2s}\,m^{s-1}. \tag{30}$$

Here p is the lower limit of the integration; $R = (n_3/2 + n_4 - r/2 + 1)$ is the number of integrations; r, s, n_3, and n_4 are the numbers of external lines of particles and the potential and the numbers of three- and four-prong vertices, respectively; A_0 is an estimate of the "tree" diagram; and in the case $d=3$ we should make the replacement $(p_0/p)^{3-d}\to\ln(p_0/p)$. Estimate (30) incorporates the lowering of the degree of the divergence upon a summation over the directions of the lines of the particles at the three-prong vertices: the factor $(p/p_0)^{n_3}$ (Ref. 10). It can be seen from (30) that for all d there exists a definite "momentum of the infrared anomaly," p_c ($\gg\tilde{p}_c$), such that the choice $q_0\gg p_c$ preserves the small factor in the integration in the region $|p|\geqslant q_0$:

$$T=0:$$

$$p_c\sim p_0e^{-1/\alpha}(d=3);\quad \alpha p_0(d=2);\quad \alpha^{1/2}p_0(d=1); \tag{31}$$

$$T>0\quad(T\gg cp_c^{T=0}):$$

$$p_c\sim\alpha\frac{T}{c}(d=3),\ \left(\alpha\frac{T}{c}p_0\right)^{1/2}(d=2),\ \left(\alpha\frac{T}{c}p_0^2\right)^{1/3}(d=1).$$

2. We now consider the integration in the region $|p| < q_0$. The interaction of the hydrodynamic modes is weak if $q_0 \ll p_0 \sim mc$; for model (2), this condition does not contradict the requirement $p_c \ll q_0$ everywhere outside the "fluctuation region" (Section 6). In models of the low-density type, $\beta = (n/p_0^d)^{1/2} \ll 1$, where the long-wave characteristics of the system are affected substantially by the short-wave region, $p > q_0$, the choice $q_0 \ll mc$ simplifies the incorporation of this effect, allowing us to express the long-wave vertices in terms of thermodynamic derivatives.[16]

The lines of the density $\pi = \tilde{n}_L - \langle\tilde{n}_L\rangle$ and the phase $\varphi = \tilde{\varphi}_L - \langle\tilde{\varphi}_L\rangle$ are connected to the field diagrams by factors $(\langle\tilde{n}_L\rangle + \pi)^{1/2}$, which replace the contributions from the condensate lines $n_0^{1/2}$ at "incomplete" vertices of the Belyaev technique,[7] and increments in μ in the field Green's functions:[16]

$$\mu\to\mu+i\dot{\varphi}-(\nabla\varphi)^2/2m.$$

It is not difficult to see that the restriction

$$g_{\pi\pi}=-np^2/mD,\quad D=\varepsilon^2+c^2p^2$$

for $\varepsilon/c \sim p \to 0$ and a differentiation of φ at the vertices with

$$g_{\pi\varphi}=\varepsilon/D,\quad g_{\varphi\varphi}=-mc^2/nD$$

keep the integrals with $g_{ab}(p)$ small, despite the fact that each prong φ is accompanied by an additional field Green's function G_{ik}, which introduces a factor $(p_0/q_0)^2 \gg 1$. The choice $p_c \ll q_0 \ll p_0$ thus causes each integration over both the region $p > q_0$ (G_{ik}) and the region $p < q_0$ (g_{ab}) to contribute a small factor. This circumstance might appear to contradict the result $\Sigma_{12}(p) \equiv 0$ (Section 5). The situation can be explained by noting that the connection g_{ab} mentioned above implies, along with (27), the transformation[16]

$$\psi_{sh} = \psi_{sh}' e^{i\tilde{\varphi}_L}$$

[without which, the lines π and φ would have been connected in the combination

$$(\langle \tilde{n}_L \rangle + \pi)^{1/2} e^{\pm i\tilde{\varphi}_L},$$

where the phase is not differentiated, and a divergence remains]. The difference between ψ'_{sh} and ψ_{sh} is important in the case $n_0 = 0$, while at $n_0 \neq 0$ it can be ignored. The substitution

$$\psi_{sh} = \psi_{sh}' e^{i\tilde{\varphi}_L}$$

with an expansion of the exponential function, for example, in the correlation function $\langle \psi_{sh}(x)\psi_{sh}(x')\rangle$ gives us, in addition to $\langle \psi'_{sh}(x)\psi'_{sh}(x')\rangle$, a sum of terms which comprise in the Fourier representation integrals of

$$G(p+k)g_{\varphi\varphi}(k), \quad G(p+k+k')g_{\varphi\varphi}(k)g_{\varphi\varphi}(k'), \ldots$$

for $0 < |k| < |k'| < \ldots < q_0$. These integrals diverge specifically in the case $n_0 = 0$ (e.g., at $T > 0, d = 2$ we find $\int d^2k/k^2$). We can make similar arguments for

$$\langle \psi_{sh}(x)\psi_{sh}(x')\psi_{sh}(x'')\rangle,$$

etc. However, we have

$$\hat{\psi}_{sh}'^{+}\hat{\psi}_{sh}' = \hat{\psi}_{sh}^{+}\hat{\psi}_{sh},$$

so that the distribution of particles with $p > q_0$ is described adequately by the harmonic approximation.

For any n_0 we can use $\hat{\tilde{n}}_L, \tilde{\varphi}_L, \hat{\psi}'_{sh}$ (or $\hat{n}_L, \hat{\varphi}_L, \tilde{\psi}_{sh}$, where $\tilde{\psi}_{sh}$ is related to $\hat{n}_{sh} = \hat{n} - \hat{n}_L, \hat{\varphi}_{sh} = \hat{\varphi} - \hat{\varphi}_L$ in a way similar to that in which $\hat{\psi}$ is related to $\hat{n}, \hat{\varphi}$) to construct $\tilde{\psi}$ [see (3)]. In terms of $\tilde{\psi}$, the harmonic approximation is adequate. In a determination with the help of the exact quasiparticle operators (i.e., when $\tilde{n}_L, \hat{\varphi}_L$ are used at small $\mathbf{p}$),

$$|\langle \tilde{\hat{\psi}} \rangle|^2 = |\langle \hat{\Psi} \rangle|^2 = \tilde{n}_0$$

is the density of the effective condensate.

3. Estimates of the type in (29) and (31) can also be found in the more general case of a Bose system with

$$\alpha_{eff} = m p_{ph}^{d-2} V_{eff} \ll 1$$

($p_{ph} \sim mc$ is a characteristic phonon momentum, and $V_{\text{eff}} = mc^2/n$ is the effective potential). Let us consider, for example, the low-density model

$$\beta = (n/p_0{}^d)^{1/2} \ll 1$$

at $T = 0$ ($\alpha = m p_0^{d-2} V_{\text{eff}}$ is arbitrary). In an integration beyond the phonon region, the basic contribution comes from the ladder diagrams, which determine

$$V_{eff} \sim V/(1 - V\Pi) \sim |\Pi|^{-1},$$

$$|\Pi| \sim \int_{p_{ph}}^{p_0} \frac{p^{d-1}\,dp}{p^2/m} \sim m p_0 \,(d=3);$$

$$m \ln \frac{p_0}{p_{ph}} \,(d=2); \qquad \frac{m}{p_{ph}} \,(d=1)$$

(Π is the contribution of a "rung of the ladder"—an integral of the product of two Green's functions),

$$p_{ph} \sim (\mu m)^{1/2}, \qquad \mu = \Sigma_{11}^{\text{ladder}}(0) - \Sigma_{12}^{\text{ladder}}(0) \sim n V_{eff}.$$

We thus have

$$p_{ph} \sim (m n V_{eff})^{1/2}$$

$$\sim \begin{cases} (n/p_0)^{1/2} \sim \beta p_0 \sim \beta^{1/3} n^{1/3} & (d=3) \\ [n/\ln(p_0/p_{ph})]^{1/2} \sim \bar{\beta}^{1/2}\beta p_0 \sim \bar{\beta}^{1/2} n^{1/2}, & \\ \beta \ll \bar{\beta} \sim \left(\ln \frac{1}{\beta}\right)^{-1} \ll 1 & (d=2) \\ (n p_{ph})^{1/2} \sim n & (d=1) \end{cases} \qquad (32)$$

$$\alpha_{eff} \sim m p_{ph} V_{eff} \sim \frac{p_{ph}}{p_0} \sim \beta \quad (d=3),$$

$$m V_{eff} \sim \left(\ln \frac{p_0}{p_{ph}}\right)^{-1} \sim \bar{\beta} \quad (d=2); \qquad \frac{m V_{eff}}{p_{ph}} \sim 1 \quad (d=1). \qquad (33)$$

In the cases $d = 3$ and 2 we have $\tilde{p}_c, p_c \ll p_{ph}$ [see (29), (31) with $\alpha \to \beta, \bar{\beta}, p_0 \to p_{ph}$]. The reason for the "lowering" of the "smallness" of α_{eff} with decreasing d lies in the weakening of the role of large momenta, which suppress the effective interaction in the summation of the ladders (the other side of the increase in the role of small momenta).

4. NECESSARY AND SUFFICIENT EVIDENCE OF SUPERFLUIDITY

1. Neither the condensate density $n_0 \neq 0$ nor the superfluid density $\rho_s \neq 0$ determines the necessary and sufficient condition for superfluidity: Superfluidity persists at $d = 2, T > 0$ and $d = 1$, $T = 0$ if $n_0 = 0$, but it disappears if $d = 1, T > 0$, although here we have $\rho_s \neq 0$ (in the case $d = 1$, the superfluid motion is understood as being with respect to the "substrate," or the "background system," which is interacting with the gas of excitations). This fact is explained clearly by Langer's analysis,[19] which describes a Bose system in terms of a path integral over coherent states (eigenstates of $\hat{\psi}$): Continuous changes in the fluctuating field $\psi \neq 0$ cannot alter the circulation of the velocity, which has discrete values, so that the superfluid motion is maintained even if we have a vanishing expectation value $\langle \psi \rangle = 0$. The superfluidity is lost only if an important role is played by configurations for which the condition $\langle |\psi| \rangle \neq 0$ is violated (there is a disruption of the phase at the points $\langle |\psi| \rangle = 0$). In the one-dimensional case (a closed contour of length L), the appearance of even one such excitation (which is unavoidable for any $T > 0, L \to \infty$) makes a preservation of the

superfluid motion impossible. A superfluidity at $d=1$, $T>0$ would contradict the result of Ref. 20 for the exactly solvable model without a phase transition at all $T>0$ (see also Ref. 16).

The concept of an effective field $\tilde{\psi}$ (or $\hat{\Psi}$) indicates that a necessary and sufficient indication of superfluidity is an effective condensate. We have $|\langle\hat{\Psi}\rangle|^2=\tilde{n}_0\neq 0$ if and only if the system is a superfluid system, $\tilde{n}_0 = L^{-d}\langle\hat{\tilde{a}}_0^+\tilde{a}_0\rangle = n-\tilde{n}'$ [see (20)], but the number density of phonons, $\tilde{n}'=\int n_G(\varepsilon_p)d^dp$, is finite in the cases $d=2$, $T>0$ and $d=1, T=0$ and divergees logarithmically in the case $d=1, T>0$.

In terms of $\hat{\Psi}$ we thus see why, "at the microscopic level," a transition from $T=0$ to small $T>0$ for a two-dimensional Bose liquid essentially means a slight change in state (in particular, superfluidity is preserved), while the change in the case of a one-dimensional system is a radical change: While the effective condensate changes only insignificantly in the former case, it disappears completely in the latter case. The superfluidity emerges as a manifestation of the macroscopic (c-number) field $\langle\hat{\Psi}\rangle=\tilde{n}_0^{1/2}\sim n^{1/2}$, which is responsible for the macroscopically filled "quasiparticle" level $\tilde{N}_0\sim N\to\infty$, as a consequence of an effective long-range order (ELO)—even in the case with $\langle\hat{\Psi}\rangle=n_0^{1/2}=0$, when the original long-range order (OLA) is disrupted by phase fluctuations.

As was mentioned in Ref. 5, the infrared anomaly of the anharmonicity is a consequence of specifically the Goldstone theorem, not of the degeneracy with respect to φ itself (this degeneracy is not present in cases of an external field $h_1\hat{a}_0^+\hat{a}_0$, with a singular potential, e.g., $V_p\sim 1/p^2$). In this connection we wish to emphasize that the anomaly is a consequence of a spontaneous breaking of a global, not local, continuous symmetry, where the Goldstone excitations acquire a gap. Correspondingly, in the local case (a superconductor or a Higgs condensate) there should be no disruptions of the original long-range order (the original condensate) (except in the case $d=1$, $T>0$, where nonperturbative irregularities are important). We can assume that the "transverse" oscillations of ψ (the source of the infrared anomaly) are established entirely by the gauge, which fixes the phase φ, transforming into longitudinal gauge bosons with $m\neq 0$. In the case of superconductivity, however, it is necessary to take into account the circumstance that the system as a whole is neutral and that there is a breaking of not only the local gauge symmetry but also the global Galilean symmetry. As a result, a gap-free mode (sound) appears, as one of the pole components of $\langle\varphi\varphi\rangle$. The answer to the question of whether the original condensate (the original long-range order) is preserved depends on the choice of gauge: affirmative if the phase φ is fixed, and negative if the longitudinal component of A is fixed, e.g., $A_{\parallel}=0$ (in this case, we can introduce an effective condensate and an effective long-range order in the standard way).

2. We also wish to call attention to the definite role played by the original condensate in the condition for superfluidity. Let us consider a Bose liquid with an increasing but finite volume L^d. If $n_0\neq 0$ ($d=3, T\geqslant 0; d=2, T=0$) the condensate level $p_{\min}\sim 2\pi\hbar/L$ is distinguished from its neighbors in that it is filled more rapidly with increasing volume L^d:

$$N_0\equiv N_{p_{\min}}\propto L^d,\quad N^{(T=0)}_{p\sim p_{\min}}\propto L,\quad N^{(T>0)}_{p\sim p_{\min}}\propto L^2 \tag{34}$$

[the result of the harmonic approximation, $N^{T=0}_{p\to 0}\infty 1/p, N^{T>0}_{p\to 0}\infty 1/p^2$—see (28)—is exact in this case]. In the case $d=2, T>0$ and $d=1, T=0$, on the other hand, the condensate level is not singled out:

$$N_0\equiv N_{p_{\min}}\sim N_{p\sim p_{\min}}\infty L^2\ (d=2, T>0);\ \infty L\ (d=1, T=0) \tag{35}$$

[as long as the condition $L\ll\hbar/\tilde{p}_c$, (29) holds, we can use the, harmonic approximation, (28), for N_p]. In the case $d=1, T>0$ the number of particles in the neighboring levels increases under these conditions even more rapidly ($\propto L^2$) than in the condensate ($\propto L$). Interestingly, the relation between the particle filling numbers of the condensate level and of the nearest above-condensate levels for an interacting Bose system in the harmonic approximation, (34), (35), at $T>0$ is the same for an ideal gas, despite the important difference between the spectra of the two systems. The role of the spectrum can be seen in terms of $\hat{\Psi}$ in the case $d=2, T>0, L^d\to\infty$: The gas of particles loses its condensate ($N_0^{(T>0)}=0$), while the gas of quasiparticles retains its condensate ($\tilde{N}^{T<T_c}\approx N$).

At small momenta, $p<\tilde{p}_c$, the phase fluctuations which destroy the condensate also change the asymptotic behavior of the Green's functions:[14,21,16]

$$\begin{aligned}
&G_{11}(r,\tau=0)\infty r^{-\delta},\quad \delta=mT/2\pi n;\\
&\quad N_p\infty 1/p^{2-\delta}\quad (d=2,\ T>0);\\
&G_{11}(r,\tau=0)\infty r^{-\gamma},\quad \gamma=mc/2\pi n,\\
&\quad N_p\infty 1/p^{1-\gamma}\ (d=1,\ T=0);\\
&G_{11}(r,\tau=0)\infty e^{-ar},\quad a=mT/2n;\\
&N_p\infty a/(p^2+a^2)\quad (d=1,\ T>0).
\end{aligned} \tag{36}$$

With increasing $L^d\to\infty$ (beginning at $L\gtrsim\hbar/\tilde{p}_c$), we thus find that (35) is replaced by

$$N_0\sim N_{p\sim p_{\min}}\infty L^{2-\delta}\ (d=2,\ T>0);\ \infty L^{1-\gamma}\ (d=1, T=0). \tag{37}$$

We see that the number of particles in the lowest energy state (the "condensate"), which again is undistinguished from its neighbors, retains an important feature of an ordinary condensate: It is "macroscopic," increasing without bound with increasing volume (although slightly more slowly). This circumstance refines the conclusion that the condensate disappears [that $\lim_{L\to\infty}(n_0=N_0/L^d)$ is zero]. In the case $d=1$, $T>0$, on the other hand, the condensate disappears completely and the number of particles in the low-lying levels in "microscopic"—it does not increase with the "volume" L of the Bose system:

$$N_0\sim N_{p\sim p_{\min}}\infty 1/a. \tag{38}$$

The macroscopic nature of the filling of the condensate level in the cases $d=2, T>0$ and $d=1, T=0$ justifies the existence of a phase operator $\hat{\varphi}$ for a Bose system [it appears in $\tilde{\psi}$ in (3)], whose "correct" definition is related to the assumption that N_0 is "macroscopic."[13] At small values $T>0$ ($d=2$), the definition of $\hat{\varphi}$ and $\tilde{\psi}$ differs only very slightly from that in the case $T=0$ ($d=2$): $\hat{\Psi}_T \approx \Psi_T = 0$. The generalization of $\hat{\varphi}$ (Ref. 13) ($T=0, d=3$) to the case $T=0$, $d=2$ is obvious. In the nonsuperfluid case, $d=1$, $T>0$, we cannot introduce a phase $\hat{\varphi}$ or thus $\tilde{\psi}$, $\hat{\Psi}$. Relations (36)–(38) explain the meaning of the relationship between superfluidity and the nature of the decrease in the field correlation function over distance: A slow (nonexponential) decrease leads to a filling of the condensate level which increases with the volume (in this sense, the filling is macroscopic), and this result is sufficient for the appearance of an effective long-range order and superfluidity.

3. At $d=2$, $T>0$ and $d=1$, $T=0$, the infrared anomaly of the anharmonicity in $G_{11} = \langle \hat{\psi}\hat{\psi}^+ \rangle$ leads to the disappearance of the original long-range order in the correlation function $\langle \hat{\psi}(\mathbf{r})\hat{\psi}+(0)\rangle$ [see (36)]. The absence of an infrared anomaly from

$$\tilde{G}_{11} = \langle \hat{\tilde{\psi}}(\mathbf{r})\hat{\tilde{\psi}}^+(0)\rangle \qquad (\tilde{G}_{11} \infty \langle \hat{\psi}(\mathbf{r})\hat{\psi}^+(0)\rangle_{\mathbf{B}})$$

means

$$\langle \hat{\tilde{\psi}}(\mathbf{r})\hat{\tilde{\psi}}^+(0)\rangle = |\hat{\psi}|^2 + \langle \hat{\tilde{\psi}}'\hat{\tilde{\psi}}'^+\rangle,$$

where $|\tilde{\psi}|^2$ is the effective long-range order, and $\langle \tilde{\psi}'(\mathbf{r})\,\tilde{\psi}'^+(0)\rangle$ has the form of the solution of the Laplace equation in a d-dimensional space at $T>0$ or a $(d+1)$-dimensional space at $T=0$. At $d=2$, $T>0$ and $d=1$, $T=0$ the fluctuations $\langle \tilde{\psi}'^+\tilde{\psi}'\rangle \sim n'^{\mathbf{B}}$ in (28) diverge, but the transformation $\tilde{\psi} \rightarrow \hat{\Psi}$, (11), (18), causes them to converge:

$$\langle \hat{\Psi}'\hat{\Psi}'\rangle = \tilde{n}' = \sum_p \tilde{N}_p, \qquad \tilde{N}_p^{T>0} = n_G(\varepsilon_p) \infty T/cp$$

(at $T=0$, the field $\hat{\Psi}$ does not fluctuate at all:

$$\tilde{N}_p^{T=0} = 0, \qquad \hat{\Psi}|\Phi'\rangle = \Psi|\Phi'\rangle).$$

In the case $d=2$, $T>0$ we have

$$\langle \hat{\Psi}(\mathbf{r})\hat{\Psi}^+(0)\rangle = |\Psi|^2 + \langle \hat{\Psi}'(\mathbf{r})\hat{\Psi}'^+(0)\rangle,$$
$$|\Psi|^2 = |\tilde{\psi}|^2, \qquad \langle \hat{\Psi}'(\mathbf{r})\hat{\Psi}'^+(0)\rangle \sim 1/r.$$

In the case $d=1$, $T>0$ we have neither an original long-range order nor an effective long-range order: $\tilde{n}'$ diverges. If we formally set $\langle \hat{\Psi}_{T>0} = \hat{\Psi}_{T=0}$, we find

$$\langle \hat{\Psi}'(\mathbf{r})\hat{\Psi}'^+(0)\rangle_{T>0} \sim \ln r.$$

This analysis of superfluidity can be generalized to the case of a crystalline state. Two types of ordering—crystalline (a correlation in the positions of the particles) and superfluid (a coherence of the wave functions)—constitute the "particle-wave alternatives" for the state of a Bose system in the low-temperature limit.[5] An infrared anomaly of the anharmonicity also arises in the case of a crystal: This is a general property of systems with a spontaneously broken global continuous symmetry. The replacement in a crystal of the original order parameter

$$\rho(\mathbf{r}) = \sum_b \rho_b \exp[i\mathbf{b}(\mathbf{r}-\mathbf{u}(\mathbf{r}))]$$

by some effective parameter $R(\mathbf{r})$ [which is linear in $u(\mathbf{r})$] eliminates the infrared anomaly and preserves the effective long-range order in the cases $d=2$, $T>0$ and $d=1$, $T=0$. The meaning of $\hat{R}(\mathbf{r})$ (an analog of $\tilde{\psi}$) is less "physical" than that of $\tilde{\psi}$: It simply reflects the regularity of the lattice of mean positions of the atoms. In no case does the infrared anomaly of the anharmonicity disrupt the internal structure of the state, for which the smallness of the fluctuations of the gradient of the degeneracy parameter is important: $\langle \mathbf{v}^2\rangle_T \infty \rho_n$ for a superfluid and $\langle (\mathbf{u}_i - \mathbf{u}_i + 1)^2\rangle$ for a crystal. The parameter of the effective long-range order of a crystal (an analog of an effective condensate, $\tilde{\psi} = \Psi$) at $T=0$ is naturally normalized by $n = N/V$, under the assumption that the ordering extends to all the particles, as in the case of superfluidity. In each case the parameter of the effective long-range order at $T=0$ is thus independent of the intensity of the zero-point vibrations, although the latter may greatly reduce the parameter of the original long-range order ($d=3$, $T \geqslant 0$; $d=2$, $T=0$). In other words, in terms of the effective long-range order there is no substantial difference between systems with a slight anharmonicity corresponding directly to the field and particle semiclassical behavior[5] [the model with $\alpha \ll 1$(2), a crystal with a small de Boer parameter, $\Lambda \ll 1$] and to the general case of coherent and crystalline ordering. Thermal phonons play different roles for these two types of ordering: The phonons lower the coherent ordering (they "deplete" the effective condensate) but not the crystalline ordering. In a crystal, the parameter of the effective long-range order reduces the thermal defectons (defects): vacancies and interstitials. However, the parameter of the effective long-range order at $T=0$ is equal to n even in the case of a crystal with a low-lying band of vacancions, where the number of particles may turn out to be lower than the number of sites because of corrections to the band approximation. Only when a superfluid component appears in the crystal do the two parameters of the effective long-range order at $T=0$ "take their shares" of the total number of degrees of freedom. Finally, we note that a superfluid is similar to a normal Fermi liquid at $T=0$ in the sense that in both cases, in terms of quasiparticles, an analogy with an ideal gas becomes apparent. The number of quasiparticles in a Bose condensate and in the Fermi filling is equal to the total number of particles.

5. DISTINCTIVE FEATURES OF THE INFRARED ANOMALY IN A BOSE SYSTEM WITHOUT A CONDENSATE

1. We can show that in the case $n_0 = 0$ the infrared anomaly of the anharmonicity which stems from the phase degeneracy "distorts" the field characteristics of the system (it introduces a qualitative distinction from the harmonic approximation), not only at small momenta, $p \lesssim p_c$, but also at large momenta, $p \gg p_c$. A direct calculation in terms of the variables n, φ (in the case $d=2$, $T>0$, for example) indi-

cates that the Green's function $G_{12}(p)$ vanishes:

$$\langle\psi(r,\tau)\psi(r',\tau')\rangle=\langle[n(r,\tau)n(r',\tau')]^{1/2}\times\exp\{i[\varphi(r,\tau)+\varphi(r',\tau')]\}\rangle$$
$$\sim n\exp\{-{}^1/_2\langle[\varphi(r,\tau)+\varphi(r',\tau')]^2\rangle\};$$
$$\frac{1}{2}\langle[\varphi(r,\tau)+\varphi(r',\tau')]^2\rangle$$
$$\sim\frac{T}{S}\sum_{\varepsilon,p}\frac{m}{n(p^2+\varepsilon^2/c^2)}\{1+\cos[p(r-r')-\varepsilon(\tau-\tau')]\}. \quad (39)$$

The expression in the argument diverges logarithmically (because of the term with $\varepsilon = 0$), so that we have $\langle\psi(r,\tau)\psi(r',\tau')\rangle = 0$. The result is physically understandable: The same factor which disrupts the single-particle condensate, i.e., the long-wave phase fluctuations (note, for example, the vanishing of the expectation value

$$\langle\psi\rangle=n_0^{1/2}e^{i\langle\varphi\rangle}=\langle n^{1/2}e^{i\varphi}\rangle\approx n^{1/2}\exp\{-{}^1/_2\langle[\varphi-\langle\varphi\rangle]^2\rangle\}e^{i\langle\varphi\rangle}$$

in the case $d = 2$, $T > 0$ by virtue of the logarithmic divergence of the argument

$$\frac{1}{2}\langle[\varphi-\langle\varphi\rangle]^2\rangle=\frac{T}{S}\sum_{\varepsilon,p}\frac{m}{n(p^2+\varepsilon^2/c^2)}\Big),$$

rules our the existence of "binary" and other "higher-order" condensates. Consequently, if the infrared anomaly leads to the equality $\Sigma_{12}(0) = 0$ in the case $n_0\neq 0$, in the case $n_0 = 0$ we have

$$\Sigma_{12}(p)\equiv 0. \quad (40)$$

At the same time, for $G_{11} = -\langle\psi\psi^*\rangle$ in the region $p\gg p_c$ the harmonic approximation is justified [the expression analogous to (39) with a phase difference in place of the sum converges]. This result is understandable: $\hat\psi_{sh}^+\hat\psi_{sh} = \hat\psi_{sh}'^+\hat\psi_{sh}'$ (Section 3).

Using $G_{11}(p,\varepsilon)$ and $G_{12}(p,\varepsilon) = 0$ we can easily find the susceptibilities $\chi_\parallel$ and $\chi_\perp$ to perturbations of the field ψ (Ref. 5). In the case $n_0 = 0$, $\chi_\parallel$ ($\mathbf{p}\to 0,\varepsilon = 0$) not only diverges (otherwise, we would have $\chi_\parallel^B = -1/4mc_B^2$) but also (in contrast with $\chi_\parallel^{(n_0\neq 0)}$) agrees entirely with $\chi_\perp$ (which is understandable in view of the disappearance of the condensate, $\langle\psi\rangle = N_0/L^2\to 0$, which distinguishes $\chi_\parallel$ from $\chi_\perp$):

$$\chi_\parallel,\chi_\perp=G_{11}^{(d=2,\,T>0)}\quad(\mathbf{p}\to 0,\ \varepsilon=0)$$
$$\sim\begin{cases}\chi_\perp^B(p,0)\propto 1/p^2 & (\tilde p_c\ll p\ll mc)\\ 1/p^{2-\delta} & (p\lesssim\tilde p_c)\end{cases}. \quad (41)$$

2. The inadmissibility of the approximate identification of $\tilde\psi_{sh}, \hat\psi_{sh}'$ with $\hat\psi_{sh}$ in the case $n_0 = 0$ characterizes a "stiffening" of the restrictions on the choice of suitable variables. This stiffening also has some other aspects. In the case $n_0 = 0$, we cannot restrict the expression for the suitable normal modes in terms of the field modes to the lowest-order corrections to the linear terms, as we may in the case $n_0\neq 0$ (Ref. 5). If we carry out a series expansion in the expression $G_{11}\sim n\exp\{-\frac{1}{2}\langle[\varphi(\mathbf{r},\tau)-\varphi(\mathbf{r}',\tau')]^2\rangle\}$ and switch to the Fourier representation,

$$G_{11}(p)\sim ng_{\varphi\varphi}(p)-\frac{n}{2!}\int d^Dq g_{\varphi\varphi}(p+q)g_{\varphi\varphi}(q)$$
$$+\frac{n}{3!}\int d^Dq d^Dq' g_{\varphi\varphi}(p+q+q')$$
$$\times g_{\varphi\varphi}(q)g_{\varphi\varphi}(q')+\dots\ (D=1+d), \quad (42)$$

we find that in the case $n_0\neq 0$ the divergence of the terms in the limit $p\to 0$ decreases rapidly with the index of the term of the expansion: The first term (the pole term) is proportional to $1/p^2$; the second is proportional to $\ln p$ for $d = 3$, $T = 0$ or to $1/p$ for $d = 3$, $T > 0$ or $d = 2$, $T = 0$; the third is proportional to $\ln p$ for $d = 3$, $T > 0$ or $d = 2$, $T = 0$; the other terms converge. The addition to ψ (ψ^*) in $G_{11} = -\langle\psi\psi^*\rangle$ of certain expressions of second and third orders in ψ, ψ^* thus eliminates all the divergences except the pole term (all manifestations of the infrared anomaly of the anharmonicity). In the case $n_0 = 0$ ($d = 2$, $T > 0$; $d = 1$, $T = 0$), on the other hand, the divergence of all the terms in (42) in the limit $p\to 0$ is identical (the expansion becomes formal), and it is not possible to eliminate the divergences by corrections of any sort to ψ, ψ^* in G_{11} which are of finite order in the fields. The approximate phase operator proposed in Ref. 22 and the approximate phonon operator proposed in Ref. 21 are suitable.

A specific feature of the case $n_0 = 0$ is also seen in the nature of the permissible approximations for $\hat U = e^{\hat R}$, (17). While in the case $n_0\neq 0$ we can restrict the expression for $\hat R$ to the lowest powers of $\hat\psi$, in the present case this is, in a sense, not legitimate: The low-order approximations correspond to the incorporation of the infrared anomalies of pairs, trios, etc., but all the infrared anomalies "disappear", $N_0^{(i)}/L^d\to 0$, and in order to obtain an effective condensate with $\tilde N_0\sim L^d$ we would have to use the entire series in (17).

6. FLUCTUATION REGION FOR A TWO-DIMENSIONAL BOSE SYSTEM

In contrast with the case $d = 3$, where model (2) loses its small parameter only near T_c ($\tau\equiv T_c - T/T_c\ll 1$, in the case $d = 2$ this happens as early as $T\sim T_c$ ($\tau\sim 1$). The field diagrams in the short-wave region, $p > q_0$, lose their small parameter if the lower limit of the integrations, q_0, turns out to be on the order of p_c in (31). On the other hand, the anharmonicity of the hydrodynamic modes in the region $p < q_0$ becomes important when q_0 reaches the characteristic hydrodynamic momentum $p_0(T) = mc(T)$; here $c(T)\sim[n_0(T)V_0/m]^{1/2}$, where $n_0(T)$ is the number density of particles of the original condensate or of the effective condensate. Assuming $n_0(T)\sim n\tau$, i.e., $c(T)\sim c\tau^{1/2}$, $p_0(T)\sim p_0\tau^{1/2}$, we find the size of the "fluctuation region" from the condition $p_0(T)\sim p_c$:

$$(d=3)\qquad p_0\tau^{1/2}\sim\alpha\frac{T_c}{c}\sim\alpha\frac{n^{2/3}}{mc}\sim\alpha^{1/3}p_0,\qquad\tau\sim\alpha^{2/3}\ll 1;$$

$$(d=2)\qquad p_0\tau^{1/2}\sim\Big(\alpha\frac{T_c}{c}p_0\Big)^{1/2}\sim\Big(\alpha\frac{n}{mc}p_0\Big)^{1/2}\sim p_0,\qquad\tau\sim 1.$$

This result agrees with the special role played by the "nonperturbative" irregularities (vortices) in the case $d = 2$: Regardless of the strength of the interaction, the vortices dissociate, disrupting the superfluidity, at $T_c^{KT} = \pi n/2m \sim T_c$ (Ref. 17).

In a certain sense, the two types of infrared anomalies of the anharmonicity—that due to the phase degeneracy and that due to the increase in the fluctuation amplitude toward T_c—are independent: The analog of a nondegenerate Bose system, the real field $\hat{\Phi}$, has the same fluctuation region. Here $\tilde{\Phi} = \hat{\Phi} = \Phi_0 + \hat{\Phi}'$. At $T \sim T_c$, we can use the substitution $\hat{\Phi}' \to \Phi'$. The Hamiltonian is analogous to that used in the microscopic theory of phase transitions,[23]

$$H = \int \left[\frac{C}{2} (\nabla\Phi)^2 + \frac{a}{2}\Phi^2 + \frac{b}{4}\Phi^4 - h\Phi \right] dr,$$

where

$$a \equiv a_1\tau \sim -n_0(T)\,V_0 \sim -n\tau V_0, \quad b \sim V_0, \quad C \sim 1/m, \quad T_c \sim n^{2/d}/m.$$

In the case $d = 2$, as in the case $d = 3$ (Ref. 5), we can define a number Gi by examining the ratio of the fluctuations of the order parameter in a "volume" with a linear dimension r_c to the square of the expectation value of the parameter $\Phi_0 = (|a|/b)^{1/2}$:

$$\langle (\Phi')^2 \rangle / \Phi_0^2 \equiv (Gi/\tau)^{1/2} (d=3),$$

$$\mathrm{Gi}/\tau (d=2) \; (\langle (\Phi')^2 \rangle \sim T_c/C r_c^{d-2}).$$

By analogy with Ref. 23, we find

$$\mathrm{Gi} \sim T_c^2 b^2 / C^3 |a_1| \sim \alpha^{2/3} \;\; (d=3), \quad T_c b / C |a_1| \sim 1 \;\; (d=2)$$

{the definition of r_c [$r_c^{-1} = \varkappa = (|a|/C)^{1/2}$] follows from the asymptotic behavior of the correlation function,

$$\langle \Phi'(r)\,\Phi'(0) \rangle \sim (T/C)\, e^{-\varkappa r}/r \;\; (d=3);$$

$$(T/C)\, e^{-\varkappa r} (\varkappa r)^{-1/2} \;\; (d=2)\}.$$

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[1)]It is difficult to carry out a quantitative calculation of all of the effects for a strongly interacting Bose system at large T: The interaction of excitations changes the distribution of excitations, and it also contributes directly to the entertainment (since it is non-Galilean) and complicates the hybridization.

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