Nonlinear motions of a plasma across a magnetic field

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The dynamics of nonlinear magnetosonic oscillations is considered. Model equations are derived and used to show that hydrodynamic wave breaking may be important in systems at high Mach numbers. In this case a magnetosonic shock wavefront may have a "flickering" structure in which regions of hydrodynamic wave breaking form and decay. The density of reflected ions is much higher inside these regions, the plasma is more turbulent, and energy is dissipated faster. Elsewhere, the plasma can be described by the standard theory for collisionless shock wavefronts of laminar structure. Novel classes of solutions are found which describe stationary collisionless shock waves in the ion-kinetic approximation.

The basic physics of collisionless shock waves (CSW) was developed by Sagdeev in Refs. 1 and 2. According to this theory, two competing effects—dispersion and nonlinearity—ensure that the profile of the shock wavefront is stationary. However, an oscillatory structure is found if weak damping is included in the model.[2] Leter work has been concerned with finding a self-consistent oscillatory structure with allowance for specific plasma-turbulence mechanisms and for an effective collision frequency related to the plasma turbulence. At the same time, Gurevich and Pitaevskiĭ[3,4] studied nonlinear one-dimensional plasma motion in the absence of a magnetic field and discovered some novel effects such as kinetic wave breaking, which produces several peaks of particle-velocity distribution function and runaway of soliton-like structures from the shock wavefront. In the present paper we study some general features of the dynamics of nonlinear magnetosonic waves and analyze some properties of quasistationary structures.

STATEMENT OF THE PROBLEM

We consider a plasma in a magnetic field H such that

$$n_e m_e c^2 \ll H^2/8\pi \ll n_i m_i c^2,$$

where n_e and n_i are the electron and ion densities and m_e, m_i are the corresponding masses. We will analyze nonlinear plasma motions with plasma velocity $V \sim V_A \ll c$ and characteristic times $\tau \sim \omega_{pi}^{-1} \ll \omega_{Hi}^{-1}$. Here $\omega_{pi} = (4\pi n_i e^2/m_i)^{1/2}$ and $\omega_{Hi} = eH/m_i c$ are the plasma and gyrofrequencies of the ions, and $V_A = H/(4\pi n_i m_i)^{1/2}$ is the Alfven velocity. In addition, we assume that $8\pi n_i T_i/H^2 \ll \omega_{Hi}^2/\omega_{pi}^2$, where T_i is the ion temperature. We can then neglect the magnetic field and the ion kinetic viscosity in the equations of motion for the ions, i.e., we use for the ions the "collisionless hydrodynamic" approximation. Under these assumptions the electrons can be described in the hydrodynamic approximation with their inertia neglected, and the Lorentz force in the ion equations of motion can be discarded. The system of equations describing the plasma motion is then given by[1]

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial X}\left(c n_e \frac{E}{H} \right) = 0, \qquad (1)$$

$$\frac{\partial E}{\partial X} = -\frac{1}{c}\frac{\partial H}{\partial t}, \qquad (2)$$

$$\frac{\partial H}{\partial X} = -\frac{4\pi e}{c}\left(\int f V_y d^3V - \frac{n_e c}{H}\frac{\partial \Phi}{\partial X} + \frac{2T_e c n_e}{n_0 e H}\frac{\partial n_e}{\partial X} \right) - \frac{1}{c}\frac{\partial E}{\partial t}, \qquad (3)$$

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial X} - \frac{e}{m_i}\frac{\partial \Phi}{\partial X}\frac{\partial f}{\partial u} + \frac{e}{m_i}E\frac{\partial f}{\partial V_y} = 0, \qquad (4)$$

$$\frac{\partial^2 \Phi}{\partial X^2} = 4\pi e\left(n_e - \int f d^3V \right), \qquad (5)$$

where all variables vary along the X axis, and the field H and the solenoidal component of the electric field E are directed along the Z and Y axes, respectively. We use the adiabatic approximation with exponent $\gamma = 2$ to describe the electron pressure across the magnetic field lines, and denote by zero subscripts quantities that are unperturbed, say, in the limit $X \to -\infty$, for which $n_e = n_i = n_0$, $H = H_0$, $\Phi = 0$. We can use our assumptions to simplify (1)–(5) considerably. First of all, (1) and (2) imply that

$$\frac{\partial}{\partial t}\left(\frac{n_e}{H} \right) + c\frac{E}{H}\frac{\partial}{\partial X}\left(\frac{n_e}{H} \right) = 0.$$

We note next that in terms of the vector potential $\mathbf{A} = (0, A, 0)$, where $H = \partial A/\partial X$ and $E = c^{-1}\partial A/\partial t$, we have $n_e = \psi(A)\partial A/\partial X$, where the function $\psi(A)$ is arbitrary. Throughout the following we will assume that the density-field ratio is "uniformly frozen-in", i.e., $n_e/H = n_0/H_0$. In view of our above assumptions, we can neglect in (3) the displacement current $c^{-1}\partial E/\partial t$ and the ion current; we then obtain

$$(1+\beta_e)(H-H_0) = \frac{4\pi e n_0}{H_0}\Phi, \qquad \beta_e = 8\pi n_0 T_e/H_0^2.$$

The electron density thus depends linearly on the potential Φ:

$$n = n_0(1 + 4\pi e n_0 \Phi/H_0^2(1+\beta_e)). \qquad (6)$$

Since the field E does not appear explicitly in Eqs. (3), (5) and can easily be found from (2) if H (or Φ) is known, we can eliminate it also from Eq. (4) after integrating f over $dV_y dV_z$. The integral

$$F = \int f dV_y dV_z$$

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then satisfies the equation

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial X}-\frac{e}{m_i}\frac{\partial \Phi}{\partial X}\frac{\partial F}{\partial u}=0, \tag{7}$$

where $n_i = \int F du$. Equation (5) thus reduces to the Helmholtz equation with right-hand side

$$\frac{\partial^2\Phi}{\partial X^2}-\frac{16\pi^2e^2n_0^2}{H_0^2(1+\beta_e)}\Phi=4\pi e(n_0-n_i) \tag{8}$$

and Φ can be related to the density by the integral equation

$$\Phi(X,t)=2\pi e\lambda^2\int_{-\infty}^{+\infty}(n_i(\xi)-n_0)\exp\left[-\frac{|\xi-X|}{\lambda}\right]d\xi,$$

where

$$\lambda^2=H_0^2(1+\beta_e)/16\pi^2e^2n_0^2=V_A^2(1+\beta_e)/\omega_{pi}^2.$$

The irrotational component of the electric field is given by

$$E_x=-\frac{\partial\Phi}{\partial X}=-2\pi e\lambda\int_{-\infty}^{+\infty}\frac{\partial n_i}{\partial\xi}\exp\left[-\frac{|\xi-X|}{\lambda}\right]d\xi. \tag{9}$$

Physically, the above description says that because the electron density is linear in Φ, the electric field is uniquely determined by the ion density n_i and depends at any point on the density distribution in a small neighborhood $\Delta X\sim\lambda$; the only role of the electrons is to produce in the plasma the familiar Debye screening (with "effective Debye radius" equal to λ), while the sum of the electron and magnetic-field pressures divided by n_0 plays the role of the electron temperature. The entire system (1)–(5) has thus been reduced to the pair of equations (7), (8). Since the hydrodynamic approximation correctly describes many of the general dynamic properties of nonlinear motion, we will derive the hydrodynamic system of equations that correspond to (7), (8). First we observe that since the characteristic times $\tau\sim\omega_{pi}^{-1}\ll\omega_{Hi}^{-1}$, the transverse motion of the ions is one-dimensional; the ion pressure can therefore be described adiabatically with adiabatic exponent $\gamma=3$. The "collisionless hydrodynamic" equations for the plasma then take the form

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial X}=-\frac{2\pi e^2\lambda}{m_i}\int_{-\infty}^{+\infty}\frac{\partial n_i}{\partial\xi}\exp\left[-\frac{|\xi-X|}{\lambda}\right]d\xi-\frac{3T_in_i}{m_in_0^2}\frac{\partial n_i}{\partial X}, \tag{10}$$

$$\frac{\partial n_i}{\partial t}+\frac{\partial}{\partial X}(n_iV)=0, \tag{11}$$

where V is the ion velocity. It will be helpful to switch to dimensionless variables in (10), (11) and (7), (8) by defining

$$\varphi=e\Phi/V_A^2(1+\beta_e)m_i,\quad x=X/\lambda,\quad \tau=\omega_{pi}t,$$
$$w=u/\lambda\omega_{pi},\quad v=V/\lambda\omega_{pi},\quad \eta=n_i/n_0.$$

System (11), (12) then becomes

$$\eta_\tau+(v\eta)_x=0, \tag{12}$$

$$v_\tau+vv_x=-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\partial\eta}{\partial\xi}e^{-|\xi-x|}d\xi-\beta\eta\eta_x, \tag{13}$$

$$\varphi_{xx}=1+\varphi-\eta,\quad \beta=3T_i/\lambda^2\omega_{pi}^2m_i. \tag{14}$$

We note that if the radius of convergence of the Taylor series expansion of $\eta(x,\tau)$ about $x=x_0$ is infinite, then E_x can be expressed as a finite sum of derivatives of the density,

$$E_x=-\sum_{n=1}^{\infty}\frac{\partial^{2n-1}\eta}{\partial x^{2n-1}}. \tag{15}$$

On the other hand, if the gradient is $\partial\eta/\partial x>1$ in a neighborhood Δx of some point x_0, the approximation

$$E_x=-\delta\eta e^{-|x-x_0|}-\int_{-\infty}^{x_0-\Delta x}\frac{\partial\eta}{\partial\xi}e^{-|\xi-x|}d\xi-\int_{x_0+\Delta x}^{\infty}\frac{\partial\eta}{\partial\xi}e^{-|\xi-x|}d\xi \tag{16}$$

is valid near x_0; here $\delta\eta\simeq\eta(x_0+\Delta x)-\eta(x_0-\Delta x)$, where Δx is the characteristic length of the region in which $\partial\eta/\partial x>1$. Physically, (16) says that an "isolated" abrupt density gradient will locally enhance the electric field; for example, the local field $E\approx-\delta\eta\exp(-|x|)$ for a step-function change $\delta\eta$ in the density.

STATIONARY NONLINEAR WAVES

We begin our analysis of the solutions of Eqs. (12)–(14) (nonlinear waves) by studying solutions that depend on the single variable $\chi=x-v_\Phi\tau$. In this case we readily find two integrals of the motion, which correspond to particle flux and energy conservation, respectively:

$$(v-v_\Phi)\eta=M,$$
$$\tfrac{1}{2}(v-v_\Phi)^2+\varphi+\tfrac{1}{2}\beta\eta^2=\tfrac{1}{2}M^2+\tfrac{1}{2}\beta,$$

where we assume that $\eta\to1$ as $\chi\to\infty$, M is the nominal Mach number, and $\varphi(-\infty)\to0$. If we assume that $\beta<M^2$ then we find the expression

$$\eta=\tfrac{1}{2}\{[(I+1)^2-2\varphi/\beta]^{1/2}-[(I-1)^2-2\varphi/\beta]^{1/2}\},$$

for η in terms of φ, where $I=M/\beta^{1/2}$. Setting $\varphi_\chi=0$ for $\varphi=0$, we then get the nonlinear-oscillator equation[1]

$$\frac{\varphi_x^2}{2}-\frac{\varphi^2}{2}-\varphi-\frac{\beta}{6}\left[(I+1)^2-\frac{2\varphi}{\beta}\right]^{3/2}+\frac{\beta}{6}(I+1)^3+\frac{\beta}{6}\left[(I-1)^2-\frac{2\varphi}{\beta}\right]^{3/2}-\frac{\beta}{6}(I-1)^3=0.$$

The nonlinear-wave solutions can be found by analyzing the Sagdeev potential[2] and the energy levels in a potential well for a potential of the form

$$U(\varphi)=-\frac{\varphi^2}{2}-\varphi-\frac{\beta}{6}\left[(I+1)^2-\frac{2\varphi}{\beta}\right]^{3/2}+\frac{\beta}{6}(I+1)^3+\frac{\beta}{6}\left[(I-1)^2-\frac{2\varphi}{\beta}\right]^{3/2}-\frac{\beta}{6}(I-1)^3.$$

For $\varphi>0$, U is defined only on the interval $0\leqslant\varphi\leqslant(1/2)\beta(I-1)^2$; moreover, since we have chosen φ_χ to vanish for $\varphi=0$, the well will exits only if $\partial^2U/\partial\varphi^2|_{\varphi=0}<0$, i.e., if $M^2 > 1+\beta$. In order for a solution to exist such that $\varphi_\chi = 0$ for $\varphi = 0$, we must have

$$U(\varphi_{max}) \geqslant 0, \quad \varphi_{max} = \frac{\beta}{2}(I-1)^2,$$

which implies that

$$\frac{\beta^2}{8}(I-1)^4 - \frac{\beta}{2}(I+1)^2 + \frac{5}{6}\beta - \frac{4}{3}\beta I^{3/2} \leqslant 0.$$

This condition becomes particularly simple if $M \gg \beta^{1/2}$; in this case it is equivalent to $M \leqslant 2$, as was first noted by Sagdeev.[1] We stress here that there is a critical Mach number above which the solution is such that the potential φ is smoother than the density (and hence the velocity). Assume that

$$\varphi = \beta(I-1)^2/2 - \alpha(\chi-\chi_0)^2 + o[(\chi-\chi_0)^2],$$
$$\alpha = \frac{1}{2}[M^{1/2}\beta^{-1/4} - \beta(I-1)^2/2 - 1]$$

near a maximum point χ_0 (here $\eta = M^{1/2}\beta^{-1/4}$ and $\varphi = \varphi_{max}$). The E vanishes at χ_0 and

$$\partial\eta/\partial\chi = -(\alpha/2\beta)^{1/2}\,\mathrm{sign}\,(\chi-\chi_0),$$

which shows that the density crests for these nonlinear solutions become sharper. The electron and ion densities may differ greatly; for example, for $M \gg \beta^{1/2}$ we have

$$\eta_e = 3, \quad \eta_i = 2^{1/2}\beta^{-1/4}.$$

We will now use the Vlasov kinetic equation to analyze the nonlinear stationary waves by making use of Eqs. (7), (8), which take the dimensionless form

$$\frac{\partial F}{\partial \tau} + w\frac{\partial F}{\partial x} - \frac{\partial \varphi}{\partial x}\frac{\partial F}{\partial w} = 0, \tag{17}$$

$$\varphi_{xx} = 1 + \varphi - \eta, \tag{18}$$

$$\eta = \int F\,dw. \tag{19}$$

As before, we seek solutions that depend on x,τ through the single variable $z = x - u_0\tau$. The problem then reduces to the stationary Bernstein-Greene-Kruskal equations[5] if we pass to a coordinate system moving at velocity u_0. The distribution function depends only on a single variable (the energy) in the integral of motion (the velocity u can be neglected). If we choose the potential $\varphi(z)$ so that φ_{zz} is some arbitrary function of φ and specify the velocity distribution of the untrapped ions, we can then derive an Abel integral equation for the reflected (or trapped) ions. The solution of the Abel equation is well known and the problem is exactly the same as for a dipole sheet,[6] except that in our case the electron density is specified *ab initio*. In order to analyze a solution that describes a collisionless shock wave, we choose the profile

$$\varphi(z) = \varphi_0(1 + \mathrm{th}\,z/L).$$

Then

$$\varphi_{zz} = -\frac{2}{L^2}(\varphi-\varphi_0)\left[1 - \frac{(\varphi-\varphi_0)^2}{\varphi_0^2}\right].$$

If the distribution function for the untrapped ions is taken to be of the form

$$F_f(W) = \frac{n_0\theta(v)[\theta(2\varphi_0+T-W) - \theta(W-2\varphi_0)]}{2^{1/2}[(2\varphi_0+T)^{1/2} - 2^{1/2}\varphi_0^{1/2}]},$$

where $\theta(v) = 1$ for $v \geqslant 0$ and 0 otherwise, one can show that the quasineutrality condition for $x \to +\infty$ (where $\varphi = 2\varphi_0$) implies that

$$n_0 = (1+2\varphi_0)[(2\varphi_0+T)^{1/2} - (2\varphi_0)^{1/2}]/T^{1/2}.$$

Our model thus yields the expression

$$n = \frac{1+2\varphi_0}{T^{1/2}}[(2\varphi_0+T-\varphi)^{1/2} - (2\varphi_0-\varphi)^{1/2}]$$

for the untrapped-ion density; the particle flux is constant and equal to

$$nv = (1+2\varphi_0)(T/2)^{1/2}.$$

In this case we define the Mach number M as the particle flux divided by the number of untrapped ions as $\varphi \to 0$, i.e., as the Mach number for the ions in the incident flux; we thus have

$$M = [(2\varphi_0+T)^{1/2} + 2^{1/2}\varphi_0^{1/2}]/2^{1/2}.$$

We define the temperature of the untrapped ions as

$$\bar{\varepsilon} = \frac{1}{n}\int_{2\varphi_0}^{2\varphi_0+T}\frac{1}{2}(v-\bar{v})^2 f(W)\frac{dW}{2^{1/2}(W-\varphi)^{1/2}},$$

which turns out to be

$$\bar{\varepsilon} = \frac{1}{12}(4\varphi_0+T-2\varphi) - \frac{1}{6}[(2\varphi_0+T-\varphi)(2\varphi_0-\varphi)]^{1/2}.$$

It increases monotonically from $(\varphi_0/3)\,[1 + T/4\varphi_0 - (1+T/2\varphi_0)^{1/2}]$ at $\varphi = 0$ to $T/12$ at $\varphi = 2\varphi_0$.

The distribution function of the reflected ions is readily found to be[7]

$$F_{tr}(W) = \frac{1}{2^{1/2}\pi}\left\{\frac{1+2\varphi_0}{T^{1/2}}\arccos\frac{(2\varphi_0-W)^{1/2}}{(2\varphi_0+T-W)^{1/2}} - 2(2\varphi_0-W)^{1/2} + \frac{8}{5L^2\varphi_0^2}(4W^2 - 6W\varphi_0 + \varphi_0^2)(2\varphi_0-W)^{1/2}\right\}.$$

The system has then three free parameters, φ_0, T, and L, subject to the single requirement that $F_{tr}(W) > 0$ for $0 < W < 2\varphi_0$. This inequality restricts the range of L values for a specified potential φ_0. This restriction can also be regarded from another viewpoint as saying that each given width L corresponds to an interval of admissible φ_0 values (i.e., Mach numbers). We note one more curious fact that the ion density may depend nonmonotonically on φ if $L < 2$; in this case it has a minimum

$$N_{min} = 1 + \varphi_0\left[1 - \frac{4}{L^2}\left(\frac{L^2+2}{6}\right)^{3/2}\right],$$

$$\varphi_{min} = \varphi_0\left(1 - \left(\frac{L^2+2}{6}\right)^{1/2}\right)$$

and a maximum

$$N_{max}=1+\varphi_0\left[1+\frac{4}{L^2}\left(\frac{L^2+2}{6}\right)^{1/2}\right],$$

$$\varphi_{max}=\varphi_0\left(1+\left(\frac{L^2+2}{6}\right)^{1/2}\right).$$

The density profile and the potential φ have then the form shown schematically in Fig. 1.

It is easy to see the cause of the density dip on the leading edge of the shock wavefront and of the peak on the trailing edge. The condition $L<2$ means that the oscillations are determined almost entirely by the potential, so that departures from quasineutrality should be important; in fact, they are responsible for the dip and subsequent peak of the ion density. Many other types of solution, including solitons, periodic waves, etc., also exist for our model. We note that the lack of a one-to-one correspondence between the Mach number and the width of the front is probably due to our neglect of plasma turbulence. Our result that there are no solutions with $L<L_{cr}$ (where L_{cr} is determined by the equality $F_{tr}(W)=0$) should therefore be interpreted as asserting the nonexistence of stationary solutions with Mach numbers $M>M_{cr}$. One must analyze the time-dependent equations in order to find the behavior in this case.

DYNAMICS OF NONLINEAR MOTION

We have shown that in the hydrodynamic approximation, the density profile of nonlinear waves with $M=M_{cr}$ becomes sharper at the point where φ reaches a maximum. Whitham[8] has observed that this sharpening implies that hydrodynamic wave breaking (or in other words, a gradient catastrophe) may occur in the system. Since this process can give rise to interesting observable singularities, we will analyze the general dynamic properties of these types of nonlinear motion. We start with the case $L\gg1$; since $\beta<\omega_{Hi}^2/\omega_{Pi}^2\ll1$, we may also neglect the ion pressure. If we set $E=-\partial\eta/\partial x$ in accordance with (14), the system (12), (13) can be written as

$$\frac{\partial l}{\partial\tau}+\frac{3l+r}{4}\frac{\partial l}{\partial x}=0, \tag{20}$$

$$\frac{\partial r}{\partial\tau}+\frac{l+3r}{4}\frac{\partial r}{\partial x}=0 \tag{21}$$

in terms of the Riemann invariants $l=v+2\eta^{1/2}$, $r=v-2\eta^{1/2}$. The standard method for solving such systems is the following. Using a hodograph transformation we interchange the dependent and independent variables. Introducing next a function $W(l,r)$ such that

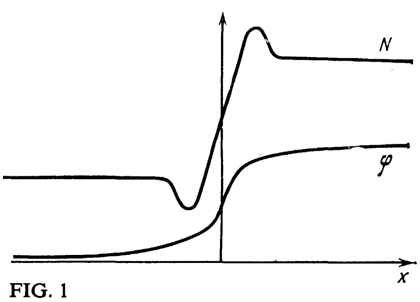

FIG. 1

$$x=\frac{l+3r}{2(l-r)}W_r-\frac{3l+r}{2(l-r)}W_l,$$

$$\tau=\frac{2}{l-r}(W_r-W_l),$$

we obtain the Euler-Poisson equation

$$W_{lr}-\frac{1}{2(l-r)}(W_l-W_r)=0, \tag{22}$$

whereby the Cauchy problem of the initial system is reduced to the Cauchy problem in the Riemann method for Eq. (22) (see, e.g., Ref. 9). However, even in the simplest cases the solution is too elaborate to yield much insight. We will therefore not strive for completeness but merely state the main facts that will be required (detailed derivations may be found in Refs. 10 and 11).

Simple waves, i.e., those for which one of the Riemann invariants is constant, are the simplest type of nonlinear motion described by Eqs. (16), (17). As an example, consider a shock wave in which[8]

$$\eta=\eta(\xi),\quad v=2(\eta(\xi))^{1/2}-2,$$
$$x=\xi+(3\eta^{1/2}(\xi)-2)t. \tag{23}$$

Since the velocity increases with density, all such waves break, i.e., η_x becomes infinite within a finite time.

A more general result states that a gradient catastrophe (i.e., wave breaking) is impossible only if $\partial l_0/\partial x>0$ and $\partial r_0/\partial x>0$ for all x [here $l_0(x)=l(x,0)$ and $r_0(x)=r(x,0)$ are the values at the initial instant]. On the other hand, suppose that $\partial l_0/\partial x<0$ at some point x_0 and let C^+ be the characteristic curve emanating from x_0 on which $dx/dt=(3l+r)/4$; then the derivative $\partial l/\partial x$ will be negative at all times t on C^+ and will reach $-\infty$ within a finite time; a similar assertion holds for $\partial r/\partial x$. A detailed proof can be found in Ref. 11.

In terms of the physical variables v and η, this means that if

$$\frac{\partial v(x,0)}{\partial x}+\frac{1}{\eta^{1/2}(x,0)}\frac{\partial\eta(x,0)}{\partial x}<0$$

or

$$\frac{\partial v(x,0)}{\partial x}-\frac{1}{\eta^{1/2}(x,0)}\frac{\partial\eta(x,0)}{\partial x}<0,$$

then one of these quantities (whichever has a negative derivative) will reach $-\infty$ within a finite time, meaning that regions of multiple flow will form.

It follows in particular, that in order for wave breaking not to occur one must have $\partial d/\partial x>0$ everywhere, which is more an exception than the rule. We note that Refs. 10 and 11 have given a detailed analysis of the interaction among l-waves, r-waves and shock waves for polytropic gases whose equations of motion coincide with (20) and (21).

The next step is to allow for the finite ion pressure, i.e., the fact that $\beta\neq0$. Here we will assume that $L\gg1$, so that Eqs. (12), (13), expressed in terms of invariants under the same assumptions as above [i.e., $E=-\partial\eta/\partial x+o(L^{-2})$], take the following form:

$$\{v+\eta^{1/2}(1+\beta\eta)^{1/2}+\beta^{-1/2}\ln[(\beta\eta)^{1/2}+(1+\beta\eta)^{1/2}]\}_\tau$$
$$+[v+\eta^{1/2}(1+\beta\eta)^{1/2}]\{v+\eta^{1/2}(1+\beta\eta)^{1/2}+\beta^{-1/2}\ln[(\beta\eta)^{1/2}+(1+\beta\eta)^{1/2}]\}_x=0, \quad (24)$$
$$\{v-\eta^{1/2}(1+\beta\eta)^{1/2}-\beta^{-1/2}\ln[(\beta\eta)^{1/2}+(1+\beta\eta)^{1/2}]\}_\tau$$
$$+[v-\eta^{1/2}(1+\beta\eta)^{1/2}]\{v-\eta^{1/2}(1+\beta\eta)^{1/2}-\beta^{-1/2}\ln[(\beta\eta)^{1/2}+(1+\beta\eta)^{1/2}]\}_x=0. \quad (25)$$

The simple waves in this case are given by

$$\eta=\eta(\xi),\quad v=\eta^{1/2}(1+\beta\eta)^{1/2}+\beta^{-1/2}\ln[(\beta\eta)^{1/2}+(1+\beta\eta)^{1/2}],$$
$$x=\{\beta^{-1/2}\ln[(\beta\eta)^{1/2}+(1+\beta\eta)^{1/2}]\}\tau+\xi.$$

They also break, i.e., η_x becomes infinite in a finite time. Moreover, since $\beta \ll 1$, the ion pressure can be described in an approximation linear in $\delta\eta$ if $\eta < 1/\beta$, so that in this case the gradient-catastrophe theorem applies without modification.

As long as we consider the dispersionless case, we cannot reach any conclusions regarding the dynamics of actual systems. However, hydrodynamic wave breaking is also known to occur in systems with nonlinear dependence of the frequency or the wave vector.

We examine next an equation that describes waves closely resembling simple Riemann waves. The dispersion equation for oscillations described by Eqs. (12), (13) reads

$$\omega^2=\frac{k^2}{1+k^2}+\beta k^2, \quad (26)$$

which after expansion in powers of the wave vector k yields

$$\omega=k(1+\beta)^{1/2}\left[1-\frac{k^2}{2(1+\beta)}\right] \quad (27)$$

for small k. In order to calculate the nonlinear term in the equation, we will assume that v and η are related by

$$v=\eta^{1/2}(1+\beta\eta)^{1/2}+\beta^{-1/2}\ln[(\beta\eta)^{1/2}+(1+\beta\eta)^{1/2}], \quad (28)$$

just as for the case of a simple Riemann wave. Substituting $v(\eta)$ from (28) into (24), setting $\eta=1+\delta\eta$ where $\delta\eta<1$, expanding in powers of $\delta\eta$, and adding a third-derivative term corresponding to the expansion (27), we get the standard Korteweg–de Vries equation

$$\delta\eta_\tau+(1+\beta)^{1/2}\delta\eta_x+\frac{3+4\beta}{2(1+\beta)^{1/2}}\delta\eta\delta\eta_x+\frac{1}{2(1+\beta)^{1/2}}\delta\eta_{xxx}=0. \quad (29)$$

This equation has been analyzed extensively. However, since ω grows as k^3 for $k>1$ whereas $\omega\sim k$ in our original system, it will be helpful to consider a somewhat different model equation corresponding to an expansion of ω of the form

$$\omega=k(1+\beta)^{1/2}\left[1-\frac{k^2}{2(1+\beta)(1+k^2)}\right].$$

Although this expression is not valid for $k>1$, it correctly reflects the qualitative $\omega(k)$ dependence both for $k\ll1$ and for $k>1$. In this case the equation for nonlinear waves takes the form

$$\delta\eta_\tau+(1+\beta)^{1/2}\delta\eta_x+\frac{3+4\beta}{2(1+\beta)^{1/2}}\delta\eta\delta\eta_x$$
$$+\frac{1}{4(1+\beta)^{1/2}}\int\left(\frac{\partial\,\delta\eta}{\partial\xi}-\frac{\partial\,\delta\eta}{\partial x}\right)e^{-|\xi-x|}d\xi=0. \quad (30)$$

This equation was used by Whitham[8] and Seliger[12] as a model to describe waves in shallow water; they showed that it also describes hydrodynamic wave breaking, although in the latter case a wave-breaking threshold must be reached. The proof is due to Seliger[12] and the result can be states as follows. Let G and g be defined by

$$G(\tau)=\max_x \eta_x(x,\tau),\quad g(\tau)=\min_x \eta_x(x,\tau);$$

then if the profile is asymmetric enough so that

$$G(0)+g(0)<-2/(3+4\beta),$$

g will blow up within a finite time, i.e., it tends to $-\infty$ as $\sim(\tau_0-\tau)^{-1}$. Similarly, one can show easily that for the second Riemann wave [described by Eq. (24)] $G(\tau)$ tends to $+\infty$ within a finite time if $G(0)+g(0)>2/(3+4\beta)$. The self-similar solution

$$\delta\eta=\frac{1}{(\tau_0-\tau)^\alpha}\delta\eta\left[\frac{x}{(\tau_0-\tau)^{1-\alpha}}\right]$$

is readily found to exist if we pass to a moving coordinate system (chosen so that the terms containing $\delta\eta_x$ disappear); moreover, the last term on the right in (30) is negligible during the last stage of wave breaking. Since this term describes the electric field E, this means that the behavior of the wave breaking is independent of E, which can therefore be neglected in the analysis of the generic singularities. That is, we may assume that the particles move freely and do not interact (if we neglect the pressure), or we may use the gas-dynamic approximation. We stress that wave breaking has a threshold, and in the above analysis the threshold is due the difference between the characteristic gradients. It must be noted however, that wave breaking can actually occur under less restrictive conditions, for the following two reasons. As we showed in the long-wavelength limit, nonlinear waves have the fundamental property that their wavefronts become steeper. The particle motion is therefore influenced only by the density distribution in a region of diameter $\sim\lambda$; this implies that the gradient difference used in the proof actually corresponds physically to a gradient threshold. The second reason is that when the full system of equations is considered, initial conditions with subthreshold gradients may still give rise to a profile that satisfies the conditions of the theorem at some time; thereafter, the dispersion will no longer be able to suppress the nonlinearity.

A kinetic process resembling one-dimensional hydrodynamic wave breaking was investigated by Gurevich and Pitaevskiĭ (Ref. 4, p. 30). In this case the analogy with hydrodynamics is complete—the nonuniqueness of the velocity profile for finite x in hydrodynamics corresponds to a bifurcation of the distribution function. This bifurcation can be described as follows: if we choose the line l in the (x,v) plane so that $f(v,x\in l)=f_{\max}(v)$ for a given x, then as the parameter τ varies the curve of type[1] in Fig. 2 will turn into a curve of type[2], i.e., a curve with nonunique projection on the x axis. Since we have already seen that the electric field dur-

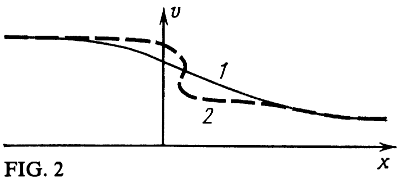

FIG. 2

ing the onset of "multi-flow" motion has no influence on the qualitative form of the singularities generated during the late stage of wave breaking, we can ignore pressure effects and analyze "generic" hydrodynamic "catastrophes" by assuming that the particles move freely (Refs. 13, 14). The trajectory is then given by the expression

$$\mathbf{r}=\mathbf{a}+\mathbf{v}(\mathbf{a})\tau, \tag{31}$$

where $\mathbf{v}(\mathbf{a})$ is the spatial velocity distribution at time zero. The velocity distribution at time τ is given by $\mathbf{v}(\mathbf{a}(\mathbf{r},\tau))$, where the solution $\mathbf{a}$ of Eq. (30) is in general not single-valued. We have

$$n(\mathbf{a},\tau)=n(\mathbf{a},0)/\det(\delta_{ik}+\tau\Omega_{ik}(\mathbf{a})),$$
$$\Omega_{ik}=\partial v_i(\mathbf{a})/\partial a_k.$$

If we expand $\mathbf{r}$ near a singularity in powers of $\mathbf{a}$ (we can arrange that the singularity occurs at $\mathbf{a}=\mathbf{r}=0$ by a suitable choice of the coordinate system), we find that

$$r_i=(\delta_{ik}+\tau\Omega_{ik}(0))a_k+{}^1/_2B_{ikl}a_ka_l$$
$$+{}^1/_3C_{iklm}a_ka_la_m.$$

The singularity occurs at the points at which the Jacobian

$$\det\frac{\partial r_i}{\partial a_k}=\det[\delta_{ik}+\tau\Omega_{ik}(0)+B_{ikl}a_ka_l+C_{iklm}a_la_m]$$

vanishes, and according to Ref. 15 it is generic, i.e., cannot be removed by a small change (jiggling) of the initial conditions. It has the form of a disk

$$\left(\frac{z}{R_z}\right)^2+\left(\frac{y}{R_y}\right)^2+\left(\frac{x}{a}\right)^{2/3}<\frac{\tau-\tau_c}{\tau_0}\quad(0<\tau-\tau_c\ll\tau_0),$$

where $\tau_0=a/v$ is the characteristic wave-breaking time and R_z,R_y are characteristic gradients on the profile along the z and y axes.

The interior corresponds to a "fold" singularity in the $v(x)$ profile, while the boundary is a "cusp" singularity (see Fig. 3). The bifurcation makes the velocity profile multiple-valued in the interior and the derivatives $\partial\eta/\partial d$ and $\partial v/\partial x$ infinite on the boundaries. Moreover, in the "ballistic" approximation (free particle motion), the density also becomes infinite on the boundary; however, an analysis of the stationary solutions with the finite pressure taken into account shows that $\partial\eta/\partial x$ becomes infinite while the density remains finite. In the kinetic description, the derivatives

$$\partial\left(\int F\,du\right)\Big/\partial x,\quad \partial\left(\int Fu\,du\right)\Big/\partial x$$

become infinite. Since $\partial\eta/\partial x$ is infinite on the boundary, the local electric field will have according to (16) a spike at the point x_0 (see Fig. 4):

$$E_x=-\partial\varphi/\partial x=-\delta\eta\exp(-|x-x_0|).$$

The electron current thus increases appreciably near x_0, the ions near the boundary are acted upon by an accelerating electric field, so that the ion boundary is accelerated and their density behind it drops, i.e., the density spike may separate from the "fold." Since the ion distribution function within such a region has several peaks, various types of plasma oscillations may build up (magnetosonic, ion-sound, modified Buneman instability). This elongates the "tails" of the energetic electrons parallel to the magnetic lines of force[16] and scatters the ions, whose diffusion tends to form a plateau in the ion distribution function (this is confirmed by numerical simulation and by experimental data[17,18]). Macroscopically, this will cause rapid local heating of the ions and deceleration of the overturned ion flux. We can approximately describe how these effects alter the behavior and structure of the density profile by adding to the right-hand side of (29), a term $-\nu_{\rm eff}\delta\eta$ that accounts for the momentum lost by the overturned ions. This is legitimate if $\delta\eta$ for the reflected ions is much greater than $\delta\eta$ in the original solution without wave breaking. Equation (29) thus takes the form

$$N_\tau+(1+\beta)^{1/2}N_x+\frac{3+4\beta}{2(1+\beta)^{1/2}}NN_x+\frac{1}{2(1+\beta)^{1/2}}N_{xxx}=-\nu_{eff}N. \tag{32}$$

Here N describes the flux density of the overturned ions. Detailed solutions of (32) have been given in the literature.[19,20] According to Ref. 20, the single-soliton solution found by perturbation theory has a trailing "shelf" and the soliton amplitude decays as $\exp(-\nu_{\rm eff}\tau)$. In the limit $x\to\infty$ the shelf is a train of long, dispersing waves of small amplitude $\sim\nu_{\rm eff}$; nevertheless, it may contain up to one-third of all the particles in the soliton. Plasma turbulence and the associated deceleration of the overturned ion flux thus tends to flatten out the characteristic gradients. The same is true if the magnetic field is taken into account—the dependence of the Larmor radii of the ions on their velocities causes a dispersive broadening of the density profile and

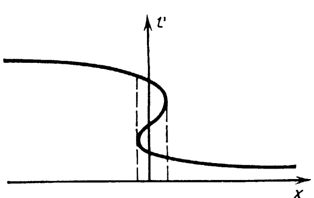

FIG. 3

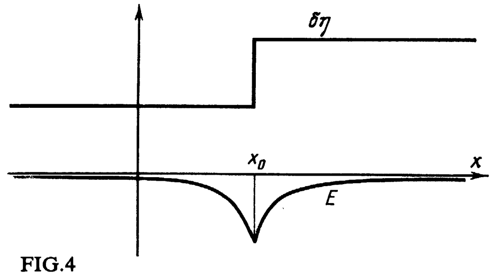

FIG.4

smooths out the gradients. (Of course, our analysis permits us to consider only the qualitative behavior of the system). After the smoothing has occurred the wave-breaking may be repeated again, with the result that the edge of the collisionless shock wave takes on a "flickering" structure. Disk-shaped wave-breaking regions may "flare up" and die out at different points along the front and may be accompanied by macroscopic changes (an increase in the plasma turbulence and in the number of reflected ions, electron acceleration, rapid heating of the oncoming flux in narrow, local regions, etc.). The regions of rapid heating border on regions of quasilaminar flow in which the qualitative structure of the wavefront is given by the solutions of Ref. 1. In closing, we note that the behavior described above is quite general and is not confined to fast magnetosonic waves—it also holds for ion-sound and other types of plasma oscillations which exhibit hydrodynamic nonlinearity and have dispersions that saturate.

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