Interaction of electromagnetic waves in metals

N. M. Makarov, I. V. Yurkevich, and V. A. Yampol'skiĭ

Institute of Radiophysics and Electronics, Ukrainian Academy of Sciences

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The interaction of two electromagnetic waves with greatly differing frequencies is investigated. The dependence of the surface impedance at the low frequency on the amplitude $\mathscr{H}$ of the high-frequency signal and on the external magnetic field strength h_0 is investigated. The cases of weak and strong nonlinearity are analyzed with asymptotic accuracy. It is shown that the nonlinear impedance is the product of the quasilinear low-frequency impedance and a certain factor that contains practically the entire dependence on $\mathscr{H}$ and h_0. This factor can cause the nonlinear impedance to increase or decrease by many times in relatively narrow ranges of $\mathscr{H}$ and h_0. At high high-frequency-field amplitudes the low-frequency impedance exhibits hysteresis as a function of the external magnetic field h_0. The results agree with the main conclusions of the experiments of Dolgopolov, Murzin, and Chuprov [Sov. Phys. JETP **51**, 166 (1980)].

1. INTRODUCTION

We investigate here theoretically the interaction of two electromagnetic waves (one of high and the other of low frequency) that penetrate into a normal metal. The quantities representing the behavior of this interaction are taken to be the dependences of the low-frequency surface impedance on the amplitude $\mathscr{H}$ of the high-frequency signal and on an external constant and uniform magnetic field h_0.

Electromagnetic-wave interaction in a metal is essentially a nonlinear effect having no analog in the linear regime. No theory has been developed for this effect so far. There is only one known experimental study[1] of the influence of two interacting waves on the low-frequency impedance. It follows unequivocally from this study that its results cannot be interpreted in terms of the known concepts concerning the dependence of the surface impedance on the magnetic field. The results of Ref. 1, in particular, contradict the heretofore prevalent opinion that the nonlinear dependence of the surface impedance on the amplitude of an external wave can be qualitatively analyzed by replacing, in the known linear-theory expression for the impedance, the constant magnetic field by the wave amplitude or by some alternating and nonuniform wave field averaged over time and space.

It is shown in the present paper that the low-frequency impedance of a metal in which two waves interact nonlinearly is equal to the quasilinear impedance at this frequency, multiplied by some factor. The quasilinear impedance depends little on the alternating-signal amplitude, just as the linear impedance depends little on the constant magnetic field. The second factor is determined essentially by the interaction conditions, and indeed contains in practice the entire dependence on the high-frequency signal $\mathscr{H}$ and on the constant field h_0. This dependence is found to be significant even at low amplitudes $\mathscr{H}$ (up to the threshold for the excitation of current states[2]). The variation of the impedance is then on the order of the impedance itself in relatively small ranges of the amplitude $\mathscr{H}$ and of the external field h_0.

Our paper consists of five sections. In the second we derive, on the basis of transparent and quite general arguments, expressions for the low-frequency impedance under the conditions of nonlinear interaction of the waves. The third and fourth sections are devoted to an asymptotically accurate solution of our problem in two actual physical situations: in the case of weak nonlinearity, when the wave interaction can be analyzed by successive approximation, and in the strong-nonlinearity regime, when the metal is in a current state. The results of these two sections confirm first of all the validity of the more general equations of the second section. In addition, analysis of these actual examples permits a more detailed study of the behavior of the impedance and leads to a number of nontrivial conclusions (fifth section).

The results of the paper agree with the main experimental conclusions of Ref. 1.

2. STATEMENT OF PROBLEM. PHYSICAL ANALYSIS OF THE PHENOMENON

1. Consider a semi-infinite metal in an external constant and uniform magnetic field $\mathbf{h}_0$ parallel to the metal surface. The x axis is directed inward perpendicular to the surface, and the z axis is parallel to the vector $\mathbf{h}_0$ (Fig. 1). Let two monochromatic waves, so polarized that their magnetic-component vectors are collinear with the magnetic field $\mathbf{h}_0$, be incident on the metal surface. The incident-wave amplitudes are $\mathscr{H}$ and $\tilde{\mathscr{H}}$, and the respective frequencies are ω and $\tilde{\omega}$. We consider the case when the anomalous skin effect sets in at both frequencies and the following conditions are satisfied:

$$\tilde{\omega} \ll \omega \ll \nu, \tag{2.1}$$

where ν is the electron-relaxation frequency.

The electromagnetic field of the wave in the metal

$$\mathbf{H}(x,t) = \{0, 0, H(x,t)\}, \qquad \mathbf{E}(x,t) = \{0, E(x,t), 0\} \tag{2.2}$$

is determined by the Maxwell equations:

$$-\frac{\partial H(x,t)}{\partial x} = \frac{4\pi}{c} j(x,t), \qquad \frac{\partial E(x,t)}{\partial x} = -\frac{1}{c}\frac{\partial H(x,t)}{\partial t}. \tag{2.3}$$

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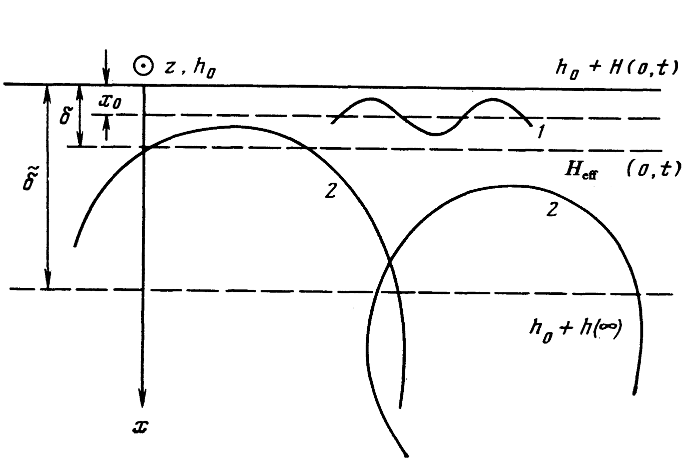

FIG. 1. Coordinate frame and electron trajectories in an alternating magnetic field: 1—trapped electrons; 2—Larmor electrons.

One boundary condition for the Maxwell equations at $x = 0$ is written, to the accuracy with which the impedance is known, in the form

$$H(0,\ t)=2\mathscr{H}\cos\omega t+2\tilde{\mathscr{H}}\cos\tilde{\omega}t. \qquad (2.4)$$

The second boundary condition is the requirement that the magnetic field of the wave be bounded as $x\rightarrow\infty$.

Of greatest interest is the effect of the high-frequency signal of large amplitude $\mathscr{H}$ on the metal's electrodynamic properties resulting from the presence of the low-frequency signal. Such a property is the low-frequency impedance

$$Z_{\tilde{\omega}}(\mathscr{H},\tilde{\mathscr{H}},h_0)=\frac{4\pi}{c}\frac{E_{\tilde{\omega}}(0)}{H_{\tilde{\omega}}(0)}=\frac{4\pi}{c}\frac{E_{\tilde{\omega}}(0)}{\tilde{\mathscr{H}}}, \qquad (2.5)$$

where $E_{\tilde{\omega}}(0)$ and $H_{\tilde{\omega}}(0)=\tilde{\mathscr{H}}$ are respectively the amplitudes of the first harmonics of the low-frequency electric and magnetic fields on the metal surface [they are proportional to $\exp(-i\tilde{\omega}t)$]:

$$E_{\tilde{\omega}}(0)=\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}dt\,E(0,t)\,e^{i\tilde{\omega}t}\,\frac{\sin\varepsilon t}{t}. \qquad (2.6)$$

The surface impedance (2.5) has the same physical meaning as in the linear situation. Its real part is proportional to the wave reflection coefficient at the frequency $\tilde{\omega}$, and the imaginary part is proportional to the phase shift in the reflected signal.

The distinguishing features of the interaction of the incident waves manifest themselves in the way the impedance depends on the amplitude $\mathscr{H}$ of the high-frequency signal. The dependence on the amplitude $\mathscr{H}$ is due to nonlinear effects of the low-frequency wave.

The nonlinearity mechanism is connected with the time dependence of the distribution of the magnetic field $H(x,t) + h_0$, the sum of the wave field $H(x,t)$ and of the external field h_0. The total magnetic field determines the electron trajectories and hence the time dependence of the metal's conductivity. Obviously, the behavior of the magnetic field in the metal is an extremely important factor in this magnetodynamic mechanism of the nonlinearity.

2. It can be seen from (2.5) that to calculate the impedance we must find the distribution of the low-frequency electric field. Since the electromagnetic field incident on the metal contains two frequencies, it is natural to expect two characteristic distance scales to appear in the metal, $\delta\sim(c^2l/3\pi^2\omega\sigma_0)^{1/3}$ and $\tilde{\delta}\sim(c^2l/3\pi^2\tilde{\omega}\sigma_0)^{1/3}$, corresponding to the skin-layer depths at the frequencies ω and $\tilde{\omega}$. It follows then from the condition $\tilde{\omega}\ll\omega$ that

$$\tilde{\delta}\gg\delta. \qquad (2.7)$$

Here c is the speed of light, l the electron mean free path, and σ_0 the static conductivity of a bulky sample.

The electromagnetic field in the high-frequency skin layer has two components: one of low frequency, and the other of high frequency modulated by the low one. On the other hand, at distances of order δ from the metal surface, only a low-frequency signal is present, together with a time-independent magnetic field that in general differs from h_0. When solving the problem in the two regions of space, it is necessary to "join" these solutions in the transition region $\delta\ll x\ll\tilde{\delta}$. In other words, the asymptotic value of the electromagnetic field of the high-frequency skin layer serves at $x\gg\delta$ as an effective boundary condition for the low-frequency skin-layer field. We begin therefore by finding the wave field in the high-frequency skin layer. We consider first the "single-frequency" problem with only one high-frequency incident wave. Since the conductivity depends on the wave's magnetic field and hence also on the time, a rectified current and a resulting constant nonuniform magnetic field $h(x)$ are produced in the sample. The field $h(x)$, equal to zero on the metal surface, varies with the distance δ and has a value h_∞ at $x\gg\delta$. At $2\mathscr{H}>|h_0|$, in particular, the so-called "current state" sets in and leads, under certain conditions (see Refs. 2–5) to hysteresis of the field h_∞ as a function of the external field h_0.

We consider now the "two-frequency" problem. We seek the distribution of the fields in the high-frequency skin layer in an adiabatic approximation whose validity is ensured by the conditions (2.1). This means that the skin layer δ is formed within times much shorter than the period $2\pi/\tilde{\omega}$ of the low-frequency signal. To determine the electromagnetic field in the $x\lesssim\delta$ region, the "slow time" can therefore be regarded as an external parameter. We represent the electric component $E(x,t)$ of the wave as a sum of two terms:

$$E(x,\ t)=\langle E(x,\ t)\rangle+E_1(x,\ t), \qquad (2.8)$$

where the angle brackets denote averaging over the "fast time," i.e., integration over the period $2\pi/\omega$ of the high-frequency wave; this integration does not affect the dependence on $\tilde{\omega}t$.

Assume that the current produced in the skin layer δ by the low-frequency electric field $\langle E(x,t)\rangle$ is weak compared with the current due to the high-frequency perturbation $E_1(x,t)$ of the field. The Maxwell equations for the region $x \lesssim \delta$ contain therefore only the high-frequency electric field $E_1(x,t)$, and the only difference between the two- and single-frequency problems is in the boundary condition at $x=0$. Whereas in the single-frequency problem the total magnetic field at the boundary was $h_0 + 2\mathscr{H}\cos\omega t$, in the two-frequency problem considered the total magnetic field at the boundary is

$$h_0+2\mathscr{H}\cos\omega t+2\tilde{\mathscr{H}}\cos\tilde{\omega}t.$$

This means that the field distribution in the high-frequency skin layer is the same as it would be in the single-frequency problem, but the role of h_0 is now assumed by $h_0 + 2\tilde{\mathscr{H}}\cos\tilde{\omega}t$. Therefore, whereas in the single-frequency problem the magnetic field at distances $x \gg \delta$ was $h_0 + h_\infty(h_0)$, in the two-frequency problem the total magnetic field in the space $\delta \ll x \ll \tilde{\delta}$ assumes the value

$$h_0+2\tilde{\mathscr{H}}\cos\tilde{\omega}t+h_\infty(h_0+2\tilde{\mathscr{H}}\cos\tilde{\omega}t).$$

The subsequent behavior of the low-frequency signal is thus determined by the solution of the Maxwell equation (2.3) with the "effective" boundary condition for the total magnetic field

$$H_{\text{eff}}(0,t)=h_0+2\tilde{\mathscr{H}}\cos\tilde{\omega}t+h_\infty(h_0+2\tilde{\mathscr{H}}\cos\tilde{\omega}t). \quad (2.9)$$

Equation (2.9) can be interpreted physically. Just as in the single-frequency problem the constant boundary field h_0 is transformed at a distance on the order of δ into a field $h_0 + h_\infty(h_0)$, in the two-frequency problem the low-frequency boundary field $h_0 + 2\tilde{\mathscr{H}}\cos\tilde{\omega}t$ is transformed over the same distance into $H_{\text{eff}}(0,t)$. Although this transformation of the low-frequency signal is effected over a distance which is negligible compared with the thickness of the low-frequency layer, it turns out to be significant and plays a major role in the nonlinear-interaction problem. The last term of (2.9), which describes the transformation of the low-frequency signal in a skin layer δ, determines in the final analysis the basic dependence of the surface impedance on the amplitude $\mathscr{H}$ and on the constant field h_0.

In contrast to the magnetic field, the low-frequency electric field $\langle E(x,t)\rangle$ is not significantly altered in the high-frequency skin layer. Solution of the electrodynamic problem in the region $x\sim\tilde{\delta}$ yields therefore the field $E_{\tilde{\omega}}(0,t)$ that enters in the definition (2.5) of the surface impedance. Indeed, it can be seen from the second Maxwell equation for the fields averaged over the fast time

$$\frac{\partial\langle E(x,t)\rangle}{\partial x}=-\frac{1}{c}\frac{\partial\langle H(x,t)\rangle}{\partial t} \quad (2.10)$$

that the $\langle E(x,t)\rangle$ changes over a distance of order δ by an amount $\langle H\rangle\tilde{\omega}\delta/c$. A similar estimate in the region $x\sim\tilde{\delta}$ yields

$$\langle E\rangle\sim\langle H\rangle\tilde{\omega}\tilde{\delta}/c.$$

It follows from these estimates that the characteristic scale of the average-electric-field variation is $\tilde{\delta}$ and that the field behavior is determined by the solution of the Maxwell equations in the low-frequency region, with boundary condition (2.9). The value $E_{\text{eff}}(0,t)$ of the high-frequency field on the "effective boundary" ($\delta \ll x \ll \tilde{\delta}$) coincides with its value $\langle E(0,t)\rangle$ on the real boundary ($x=0$), accurate to terms of order $\delta/\tilde{\delta}$. Equation (2.5) can be rewritten to the same accuracy in the form

$$Z_{\tilde{\omega}}(\mathscr{H},\tilde{\mathscr{H}},h_0)=\frac{4\pi}{c}\frac{E_{\text{eff}}^{\tilde{\omega}}(0)}{\tilde{\mathscr{H}}}, \quad (2.11)$$

where $E_{\text{eff}}^{\tilde{\omega}}(0)$ is the amplitude of the first harmonic of the electric field at the effective boundary.

3. The final expression for the surface impedance can be easily obtained if the expansion

$$h_\infty(h_0+2\tilde{\mathscr{H}}\cos\tilde{\omega}t)\approx h_\infty(h_0)+\frac{\partial h_\infty}{\partial h_0}2\tilde{\mathscr{H}}\cos\tilde{\omega}t. \quad (2.12)$$

is valid. The effective boundary condition (2.9) then takes the form

$$H_{\text{eff}}(0,t)=h_0+h_\infty(h_0)+(1+\partial h_\infty/\partial h_0)2\tilde{\mathscr{H}}\cos\tilde{\omega}t. \quad (2.13)$$

Condition (2.13) means that the low-frequency skin layer contains a constant and uniform magnetic field h_0+h_∞, and that a monochromatic wave of frequency $\tilde{\omega}$ and amplitude

$$\tilde{\mathscr{H}}(1+\partial h_\infty/\partial h_0)$$

is incident on the effective boundary of the metal. The ratio of the first $\tilde{\omega}$-harmonics of the electric and magnetic fields $E_{\text{eff}}^{\tilde{\omega}}(0)$ and $H_{\text{eff}}^{\tilde{\omega}}(0)$ is therefore determined by the solution of the low-frequency "single-wave" problem and is given by

$$E_{\text{eff}}^{\tilde{\omega}}(0)/H_{\text{eff}}^{\tilde{\omega}}(0)=\frac{c}{4\pi}Z_{\tilde{\omega}}\left(0,\left(1+\frac{\partial h_\infty}{\partial h_0}\right)\tilde{\mathscr{H}},h_0+h_\infty\right), \quad (2.14)$$

where the right-hand side contains the metal impedance, which is generally speaking nonlinear at the frequency $\tilde{\omega}$ in the external magnetic field h_0+h_∞ if the incident wave has a frequency $\tilde{\omega}$ and an amplitude $(1+\partial h_\infty/\partial h_0)\tilde{\mathscr{H}}$. According to (2.13) we have then

$$H_{\text{eff}}^{\tilde{\omega}}(0)=(1+\partial h_\infty/\partial h_0)\tilde{\mathscr{H}}. \quad (2.15)$$

Substituting (2.14) and (2.15) in (2.11) we obtain

$$Z_{\tilde{\omega}}(\mathscr{H},\tilde{\mathscr{H}},h_0) =\left(1+\frac{\partial h_\infty}{\partial h_0}\right)Z_{\tilde{\omega}}\left(0,\left(1+\frac{\partial h_\infty}{\partial h_0}\right)\tilde{\mathscr{H}},h_0+h_\infty\right). \quad (2.16)$$

This equation agrees fully with the statement made in Ref. 1 that the experimentally observed difference between the nonlinear and linear impedances is proportional at small $\tilde{\mathscr{H}}$ and $\tilde{\omega}$ to the derivative of the magnetic moment of the sample with respect to the external field h_0.

Equation (2.16) allows us to determine the impedance, in the interaction between low- and high-frequency electromagnetic waves, from the known solutions of the corresponding single-frequency problems. The solution of the high-frequency problem specifies the value of $h_\infty(h_0)$, while that of the low-frequency problem yields the impedance at the frequency $\tilde{\omega}$ for $\mathscr{H}=0$. We note that the major part of the dependence of the impedance (2.16) on the high-frequency-wave amplitude $\mathscr{H}$ and on the external field h_0 is contained in the first factor. According to Refs. 2–4, the quantity $(1+\partial h_\infty/\partial h_0)$, and hence the impedance (2.16), can increase (or decrease) by several times when $\mathscr{H}$ and h_0 are varied. Moreover, the increase of the impedance may turn out to be quite appreciable, for if the amplitude $\mathscr{H}$ is large enough the $h_\infty(h_0)$ curve has a critical point $h_0=h_0^*$ at which the derivative $\partial h_\infty/\partial h_0$ becomes infinite. The impedance increase that occurs when h_0 approaches h_0^*, however, is bounded by the fact that the expansion (2.12) and the expression (2.16) for the impedance are not valid in the immediate vicinity of the critical point.

4. If the expansion (2.12) is not valid, equations for the impedance can be obtained only in the approximation linear in the low-frequency amplitude, when the influence of the low-frequency signal field on the electron trajectories in the skin layer $\tilde{\delta}$ can be neglected. This approximation is valid if one of the conditions

$$|H_{\text{eff}}^{\tilde{\omega}}(0)|\ll\tilde{g},\quad |H_{\text{eff}}^{\tilde{\omega}}(0)|\ll|H_{\text{eff}}^{0}| \tag{2.17}$$

is satisfied. Here

$$g=\frac{8cp_F\tilde{\delta}}{el^2},\quad H_{\text{eff}}^{0}=h_0+\frac{1}{2\pi}\oint d\varphi h_\infty(h_0+2\mathscr{H}\cos\varphi), \tag{2.18}$$

$$H_{\text{eff}}^{\tilde{\omega}}(0)=\tilde{\mathscr{H}}+\frac{1}{2\pi}\oint d\varphi\cos\varphi h_\infty(h_0+2\tilde{\mathscr{H}}\cos\varphi),$$

where $-e$ and p_F are the charge and Fermi momentum of the electron.

The first inequality of (2.17) expresses the weakness of the low-frequency magnetic field compared with the characteristic value $\tilde{g}$ at which the electron-path length in the skin layer $\tilde{\delta}$ is equal to the mean free path l. When the second inequality of (2.17) is satisfied, the electron orbit is formed by the constant and uniform magnetic field H_{eff}^0, which is the zeroth harmonic of the field $H_{\text{eff}}(0,t)$ [see (2.9)].

Under conditions (2.17), the Maxwell equations in the skin layer $\tilde{\delta}$ are linear in the low-frequency field and can be separately written for each of the harmonics contained in (2.9). The ratio of the first $\tilde{\omega}$ harmonics of the electric and magnetic fields at the effective boundary is determined by the linear-theory impedance $Z_{\tilde{\omega}}^{\text{lin}}(H_{\text{eff}}^0)$ at the frequency $\tilde{\omega}$ in the external constant and uniform magnetic field H_{eff}^0:

$$E_{\text{eff}}^{\tilde{\omega}}(0)/H_{\text{eff}}^{\tilde{\omega}}(0)=\frac{c}{4\pi}Z_{\tilde{\omega}}^{\text{lin}}(H_{\text{eff}}^0). \tag{2.19}$$

Substituting (2.18) in (2.10) we get

$$Z_{\tilde{\omega}}(\mathscr{H},\tilde{\mathscr{H}},h_0)=\frac{H_{\text{eff}}^{\tilde{\omega}}(0)}{\tilde{\mathscr{H}}}Z_{\tilde{\omega}}^{\text{lin}}(H_{\text{eff}}^0). \tag{2.20}$$

Just as the main dependence of the impedance on $\mathscr{H}$ and h_0 was contained in the factor $(1+\partial h_\infty/\partial h_0)$ of (2.16), the principal role is played here by the factor $H_{\text{eff}}^{\tilde{\omega}}(0)/\tilde{\mathscr{H}}$.

Although the conditions under which (2.16) and (2.20) hold are different, they have a common region of validity. The two equations coincide in the regime linear in the high-frequency field when the expansion (2.12) is valid.

We point out that expressions (2.16) and (2.20) for the low-frequency regime have much in common. They were derived using neither the anomaly of the skin effect at the two frequencies, nor the relations between the frequencies of the incident waves and the electron relaxation frequency. As written, expressions (2.16) and (2.20) are valid in a wide frequency range (at $\tilde{\omega}\ll\omega$). The various physical situations differ only in the explicit forms of the quantities

$$h_\infty(h_0),\ Z_{\tilde{\omega}}^{\text{lin}}(H_{\text{eff}}^0),\quad Z_{\tilde{\omega}}(0,(1+\partial h_\infty/\partial h_0)\tilde{\mathscr{H}},h_0+h_\infty).$$

To analyze the behavior of the low-frequency impedance, we consider in the sections that follow two actual physical situations. In Sec. 3 we analyze the case of weak nonlinearity in a weak external magnetic field h_0, when the influence of the total magnetic field on the electron trajectories is treated as a perturbation. Section 4 is devoted to the situation when the high-frequency field amplitude is so high that a current results. The expressions for the impedances in these two cases are obtained independently of the results of Sec. 2, by an asymptotically accurate solution of the problems. This approach is by itself of interest for several reasons. First, the two-frequency problem can be solved by a unified approach, without dividing it into two single-frequency problems. Second, the asymptotically accurate expressions obtained for the high-frequency expressions in the two actual cases confirm the validity of the more general expressions obtained in Sec. 2.

3. WEAK NONLINEARITY

Weak nonlinearity implies that the amplitudes $\mathscr{H}$ and $\tilde{\mathscr{H}}$ are small compared with the characteristic field h at which the length $L\sim[4cp_F\delta/e(\mathscr{H}+\tilde{\mathscr{H}})]^{1/2}$ of the electron arc in the skin layer δ is equal to the mean free path

$$\mathscr{H},\ \tilde{\mathscr{H}}\ll g,\quad g=8cp_F\delta/el^2,\quad \delta=(c^2l/3\pi^2\omega\sigma_0)^{1/3}. \tag{3.1}$$

We assume in addition that the external constant field is also weak:

$$|h_0|\ll g. \tag{3.2}$$

If the inequalities (3.1) and (3.2) hold, the electron trajectories are almost straight lines. The magnetic field gives rise only to small corrections to the current density and to the surface impedance. According to the results of Ref. 6, the Fourier transform of the current density

$$j(k,t)=2\int_0^\infty dxj(x,t)\cos kx \tag{3.3}$$

can be expressed as a sum of two terms:

$$j(k,t)=j_0(k,t)+\Delta j(t), \tag{3.4}$$

where j_0 is the anomalous skin current in the linear regime in the absence of a magnetic field (see, e.g., Ref. 7). The nonlinear correction Δj is smaller than j in terms of the parameter $(H(0,t)+h_0)^2/g^2$:

$$\Delta j(t)=-9\frac{\sigma_0}{l}\delta^2\left(\frac{H(0,t)+h_0}{g}\right)^2\zeta(\rho)E(0,t), \tag{3.5}$$

where

$$\zeta(\rho)=2-(1-\rho)^2\sum_{n=1}^{\infty}\frac{\rho^{n-1}}{n^2};$$

ρ is the probability of specular electron reflection from the sample boundary.[8]

The solution by perturbation theory of the Maxwell equations (2.3) in the k-representation at the current density (3.4), (3.5) yields

$$[Z_{\tilde{\omega}}(\mathscr{H},\tilde{\mathscr{H}},h_0)-Z_{\tilde{\omega}}^{\text{lin}}(0)]/Z_{\tilde{\omega}}^{\text{lin}}(0)$$

$$\equiv \Delta Z_{\tilde{\omega}}(\mathscr{H},\tilde{\mathscr{H}},h_0)/Z_{\tilde{\omega}}^{\text{lin}}(0)=-\frac{2\pi}{c\tilde{\mathscr{H}}}\Delta j_{\tilde{\omega}}. \quad (3.6)$$

Here $Z_{\tilde{\omega}}^{\text{lin}}$ is the surface impedance of the metal in the linear regime at the frequency $\tilde{\omega}$ in the absence of a magnetic field; $\Delta j_{\tilde{\omega}}$ is the amplitude of that term of $\Delta j(t)$ which is proportional to $\exp(-i\tilde{\omega}t)$.

As a result, the relative change of the low-frequency impedance is of the form

$$\Delta Z_{\tilde{\omega}}(\mathscr{H},\tilde{\mathscr{H}},h_0)/Z_{\tilde{\omega}}^{\text{lin}}(0)=\frac{12\sqrt{3}}{\pi}\frac{\sin^2(\pi z_0/3)}{\sin^2(\pi z_0/2)}\zeta(\rho)$$

$$\times\left\{\mathscr{H}^2+\frac{1}{2}\left(\frac{\tilde{\omega}}{\omega}\right)^{2/3}[e^{-i\pi/3}(h_0^2+2\mathscr{H}^2+2\tilde{\mathscr{H}}^2)+e^{i\pi/3}\tilde{\mathscr{H}}^2]\right\}g^{-2}, \quad (3.7)$$

where $\cos\pi z_0=\rho$.

We separate in (3.7) the term $\Delta Z_{\tilde{\omega}}(\mathscr{H})$ that describes the low-frequency-impedance correction necessitated by the interaction of the two waves (and dependent on the amplitude $\mathscr{H}$ of the high-frequency field):

$$\Delta Z_{\tilde{\omega}}(\mathscr{H})/Z_{\tilde{\omega}}^{\text{lin}}(0)=9\frac{\sigma_0}{l}\delta^2|Z_{\omega}^{\text{lin}}(0)|\zeta(\rho)$$

$$\times\left\{1+\left(\frac{\tilde{\omega}}{\omega}\right)^{2/3}e^{-i\pi/3}\right\}\left(\frac{\mathscr{H}}{g}\right)^2. \quad (3.8)$$

Note that (3.8) is valid at any ratio of the frequencies ω and $\tilde{\omega}$.

We now compare the asymptotically accurate expression (3.8) with expression (2.16) obtained in Sec. 2. The validity of (2.16) in the case of weak nonlinearity is obvious, since at $\mathscr{H}\ll g$ or $h_0\ll g$ the induced field h_∞ (or h_0) is a linear function of h_0 (Ref. 5):

$$h_\infty(h_0)=9\frac{\sigma_0}{l}\delta^2|Z_{\omega}^{\text{lin}}(0)|\zeta(\rho)\left(\frac{\mathscr{H}}{\sigma}\right)^2h_0. \quad (3.9)$$

Rewriting (2.16) in a form similar to (3.8), taking (3.9) into account, and neglecting terms of order $(\mathscr{H}/g)^4$, we obtain

$$\Delta Z_{\tilde{\omega}}(\mathscr{H})/Z_{\tilde{\omega}}^{\text{lin}}(0)=9\frac{\sigma_0}{l}\delta^2|Z_{\omega}^{\text{lin}}(0)|\zeta(\rho)\left(\frac{\mathscr{H}}{g}\right)^2. \quad (3.10)$$

Comparison of (3.10) and (3.8) shows that in the region where (2.16) [and hence also (3.10)] is valid, $\tilde{\omega}\ll\omega$, the equation obtained in Sec. 2 for the low-frequency impedance coincides with that obtained by the asymptotically exact solution of the problem.

4. CURRENT STATES

1. We consider in this section the case when the incident-wave amplitudes satisfy the inequality

$$2\mathscr{H}>|h_0|+2\tilde{\mathscr{H}}. \quad (4.1)$$

In addition, we assume the amplitude $\tilde{\mathscr{H}}$ to be so small that the low-frequency-signal propagation in the region $x\gg\delta$ is described by the linear-theory equations. It suffices for this purpose that the amplitude of the magnetic component of the low-frequency signal be much less at $x\gg\delta$ than $|h_0+h(\infty)|$, where $h(\infty)$ is the magnetic field induced in the interior of the sample. The electron trajectories in the low-frequency skin layer are formed then by the constant magnetic field $h_0+h(\infty)$, and the conductivity in the region $\delta\ll x\ll\tilde{\delta}$ is independent of time.

Condition (4.1) means that the period $2\pi/\omega$ of the high-frequency wave includes a time interval during which the sign of the total magnetic field $h_0+2\mathscr{H}\cos\omega t+2\tilde{\mathscr{H}}\cdot\cos\tilde{\omega}t$ on the metal surface differs from that of the field $h_0+h(\infty)$ outside the high-frequency layer:

$$(2\mathscr{H}\cos\omega t+h_0+2\tilde{\mathscr{H}}\cos\tilde{\omega}t)/(h_0+h(\infty))<0. \quad (4.2)$$

In these time intervals direction of the magnetic field in the skin layer δ changes as a function of position, and a class of "trapped" electrons is produced in the sample. Their trajectories twist around the plane $x=x_0(t)\lesssim\delta$ in which the field $H(x,t)+h_0$ reverses sign (see Fig. 1). The trapped electrons make a large contribution to the conductivity of the high-frequency skin layer δ and dominate in the formation of the current states. Besides the trapped electrons, the Larmor and surface electrons also contribute to the conductivity of the skin layer δ. The Larmor electrons move during the greater part of the time along a circular orbit in the constant magnetic field $h_0+h(0)$ and interact with the high-frequency-wave field for only a short time $\Delta t\sim L/v$ (v is the electron Fermi velocity).

The situation is different in the low-frequency skin layer. Its conductivity is determined only by the contribution of the Larmor and surface electrons, which is independent of time.

2. To derive an expression for the current densities of all the electron groups, it is convenient to represent the Fourier transform of the electric field

$$\mathscr{E}(k,t)=2\int_0^\infty dx E(x,t)\cos kx \quad (4.3)$$

in accord with (2.8) in the form of a sum of two terms

$$\mathscr{E}(k,t)=\langle\mathscr{E}(k,t)\rangle+\mathscr{E}_1(k,t). \quad (4.4)$$

Here $\langle\mathscr{E}(k,t)$ is the Fourier transform, averaged over the fast time, of the electric field characterized by values $k\sim\delta^{-1}$. The characteristic values of k for the high-frequency field $\mathscr{E}_1(k,t)$ are of order δ^{-1}.

According to the model of Ref. 5, the connection between the current density $j(k,t)$ and the electric field $\mathscr{E}(k,t)$ for diffusely reflected electrons takes the form

$$j(k,t)=\frac{3\pi}{4}\frac{\sigma_0}{kl}[1-e^{-2\nu T_a}]^{-1}\{\langle\mathscr{E}(k,t)\rangle+S(k,t)\mathscr{E}_1(k,t)\}, \quad (4.5)$$

where

$$S(k,t)=1+\alpha(t)F(kx_0)\theta_-, \quad \alpha=\frac{1-\exp(-2\nu T_a)}{1-\exp(-2\nu T_b)}, \quad (4.6)$$

$$\theta_-=\theta\left(-\frac{2\mathscr{H}\cos\omega t+h_0+2\tilde{\mathscr{H}}\cos\tilde{\omega}t}{h_0+h(\infty)}\right),$$

$$2T_a=\frac{\pi mc}{e\,|\,h_0+h(\infty)\,|},\quad 2T_b=2\pi\left[\frac{mc}{evH'(x_0,t)}\right]^{1/2}. \tag{4.7}$$

Here $2T_a$ and $2T_b$ are the periods of motion of the Larmor and trapped electrons, and $\theta(x)$ is the Heaviside unit step function, equal to zero at $x<0$ and to unity at $x>0$. The function $F(kx_0)$ is of order $(kx_0)^2$ at $kx_0\lesssim 1$ and tends to $(x_0/\delta)^2$ as $kx_0\to 0$, while α is the ratio of the conductivities of the trapped and Larmor electrons in the high-frequency electric field.

The structure of the expression for the current density $j(k,t)$ is quite clear. The term proportional to $\langle\mathscr{E}(k,t)$ in (4.5) is the current density in the low-frequency skin layer $\tilde{\delta}$, and the second term is the current density in the high-frequency skin layer δ. The function θ_- in $S(k,t)$ takes into account the fact that the trapped electrons exist only during the time interval defined by (4.2). The quantity $\alpha(t)>1$ describes the decrease of the period of the trapped electrons compared with the Larmor ones. The function $F(kx_0)$ reflects the character of the spatial dispersion (the k-dependence) of the trapped-electron conductivity.

3. The Maxwell equations (2.3) with current density (4.5) and with allowance for the boundary condition (2.4) are written in the form

$$4\mathscr{H}\cos\omega t+4\tilde{\mathscr{H}}\cos\tilde{\omega}t-kH(k,t)$$
$$=\frac{3\pi^2\sigma_0}{c^2k^2l}[\mathrm{i}-e^{-2\nu T_a}]^{-1}\left\{\frac{\partial\langle H(k,t)\rangle}{\partial t}+S(k,t)\frac{\partial H_1(k,t)}{\partial t}\right\},$$
$$k\mathscr{E}(k,t)=\frac{1}{c}\frac{\partial H(k,t)}{\partial t}, \tag{4.8}$$

where

$$H(k,t)=2\int_0^\infty dx H(x,t)\sin kx,$$
$$H(k,t)=\langle H(k,t)\rangle+H_1(k,t). \tag{4.9}$$

We divide the first Maxwell equation in (4.8) by $S(k,t)$ and average it over the fast time. To find the low-frequency impedance we need know only the first $\tilde{\omega}$ harmonic of the electric field, $\mathscr{E}_{\tilde{\omega}}(k)=-i\tilde{\omega}H_{\tilde{\omega}}(k)/ck$, with characteristic values $k\sim\tilde{\delta}^{-1}$. We therefore write down the result of averaging for $k\delta\ll 1$:

$$4\tilde{\mathscr{H}}\cos\tilde{\omega}t+2h_\infty(h_0+2\tilde{\mathscr{H}}\cos\tilde{\omega}t)-k\langle H(k,t)\rangle=\frac{3\pi^2\sigma_0}{c^2k^2l}[1-e^{-2\nu T_a}]^{-1}\frac{\partial\langle H(k,t)\rangle}{\partial t}. \tag{4.10}$$

To obtain (4.10) we have used the relation

$$2\mathscr{H}\langle S^{-1}(0,t)\cos\omega t\rangle/\langle S^{-1}(0,t)\rangle=h_\infty(h_0+2\tilde{\mathscr{H}}\cos\tilde{\omega}t). \tag{4.11}$$

Equation (4.11) is valid in the adiabatic approximation and is a generalization of the equality

$$2\mathscr{H}\int_0^{2\pi/\omega}dtS^{-1}(0,t)\cos\omega t\Big/\int_0^{2\pi/\omega}dtS^{-1}(0,t)=h_\infty(h_0),$$

obtained in Ref. 5 for $\tilde{\mathscr{H}}=0$, to the two-wave problem. We point out that the induced field $h_\infty(h_0)$ of the single-particle problem generally speaking is unequal to the field $h(\infty)$ induced when the waves interact. We obtain $h(\infty)$ by separating the zeroth harmonic from (4.10):

$$h(\infty)=\frac{1}{2\pi}\oint d\varphi h_\infty(h_0+2\tilde{\mathscr{H}}\cos\varphi). \tag{4.12}$$

We note that the total magnetic field $h_0+h(\infty)$ in the interior of the metal coincides with the result (2.18) for the field $H^0_{\rm eff}$.

The first $\tilde{\omega}$ harmonics of the fields are determined from (4.10) and from the second Maxwell equation of (4.8):

$$H_{\tilde{\omega}}(k)=2\tilde{\mathscr{H}}\Phi(\mathscr{H},\tilde{\mathscr{H}},h_0)\left\{k-i\frac{3\pi^2\sigma_0\tilde{\omega}}{c^2k^2l}[1-e^{-2\nu T_a}]^{-1}\right\}^{-1}$$
$$\mathscr{E}_{\tilde{\omega}}(k)=-\frac{i\tilde{\omega}}{ck}H_{\tilde{\omega}}(k), \tag{4.13}$$

where

$$\Phi(\mathscr{H},\tilde{\mathscr{H}},h_0)=1+\frac{\tilde{\mathscr{H}}^{-1}}{2\pi}\oint d\varphi\cos\varphi h_\infty(h_0+2\tilde{\mathscr{H}}\cos\varphi). \tag{4.14}$$

From this we obtain for the impedance (2.5)

$$Z_{\tilde{\omega}}(\mathscr{H},\tilde{\mathscr{H}},h_0)=\Phi(\mathscr{H},\tilde{\mathscr{H}},h_0)\frac{16\pi}{3\sqrt{3}}e^{-i\pi/3}[1-e^{-2\nu T_a}]^{1/3}\frac{\tilde{\omega}\tilde{\delta}}{c^2}. \tag{4.15}$$

The factor Φ coincides with $H^{\tilde{\omega}}_{\rm eff}(0)/\tilde{\mathscr{H}}$ [see (2.18)], while the second factor of (4.15) is the linear impedance of a metal in an external magnetic field $h_0+h(\infty)$. Expression (2.20) for $Z_{\tilde{\omega}}$, obtained from general considerations of the wave interaction, agrees thus with the asymptotically correct expression (4.15). The condition that $\tilde{\mathscr{H}}$ be small, which is necessary for (4.15) to be valid, also coincides with Eqs. (2.17) that bound the region where (2.20) is valid.

5. ANALYSIS OF RESULTS

Let us study the salient features of the behavior of the low-frequency impedance of a metal in the current state. We analyze the function Φ, which is the one containing the principal dependence of the impedance (4.15) on the high-frequency field amplitude $\mathscr{H}$ and on the external constant magnetic field h_0. The function $h_\infty(h_0)$ needed for this analysis was obtained in Ref. 4 for various values of the nonlinearity parameter $b=(g/2\mathscr{H})^{1/2}$. For simplicity, we assume that $\tilde{\mathscr{H}}\ll\mathscr{H}$.

The characteristic h_0 interval over which the function $h_\infty(h_0)$ changes is $\mathscr{H}$. Exceptions are the vicinities of the singular points $h_0=h_0^*$, at which $|\partial h_\infty/\partial h_0|=\infty$. Far from these points, the expression (4.14) for Φ can be simplified by expanding h_∞ in powers of $\tilde{\mathscr{H}}$:

$$\Phi(\mathscr{H},\tilde{\mathscr{H}},h_0)=1+\partial h_\infty/\partial h_0. \tag{5.1}$$

The quantity $\partial h_\infty/\partial h_0$ can be an arbitrary function of $\mathscr{H}$ and h_0. There exists, in particular, a region in which

$$\partial h_\infty/\partial h_0<-1. \tag{5.2}$$

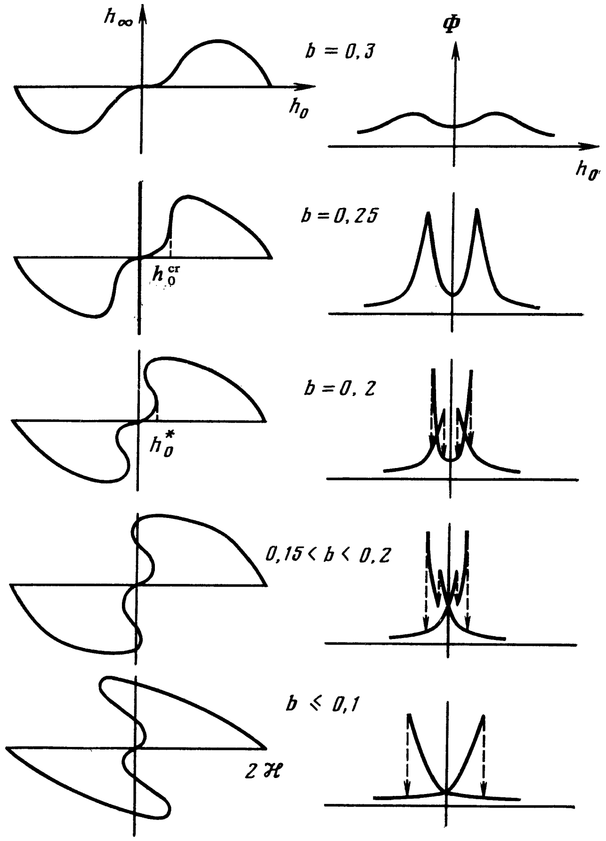

FIG. 2.

If (5.2) is satisfied the real part of the surface impedance becomes negative. This means that the amplitude of the wave of frequency $\tilde{\omega}$ reflected from the metal boundary exceeds the amplitude $\mathscr{H}$ of the incident wave. In other words, the wave interaction can cause an appreciable energy transfer from the high- to the low-frequency field. In this case, however, the question of the stability of such states arises. The answer to this question calls for a special investigation outside the scope of the problems solved in the present paper.

As h_0 approaches the singular point h_0^* the impedance increases abruptly. In the vicinity $|h_0 - h_0^*| \ll 2\mathscr{H}$ Eq. (5.1) no longer holds for the function Φ. An analysis of the exact expression (4.14) shows that the value of Φ at the maximum point is of order

$$\Phi_{\max} \sim \left(\frac{\mathscr{H}}{\tilde{\mathscr{H}}}\right)^{1-\alpha} \operatorname{sign}\frac{\partial h_\infty}{\partial h_0}. \qquad (5.3)$$

Here α characterizes the singularity of the function $h_\infty(h_0)$ at the point $h_0 = h_0^*$:

$$\frac{h_\infty(h_0) - h_\infty(h_0^*)}{\mathscr{H}} = \left|\frac{h_0 - h_0^*}{\mathscr{H}}\right|^\alpha . \qquad (5.4)$$

At the values of $\mathscr{H}$ corresponding to the onset of hysteresis of the induced field h_∞, and $h_\infty(h_0)$ curve has two singular points $h_0^* \pm h_0^{cr}$ (Fig. 2), for which $\alpha = 1/3$. In all other cases $\alpha = 1/2$.

It can be seen from (5.3) that as the ratio $\tilde{\mathscr{H}}/\mathscr{H}$ decreases the maximum value of the high-frequency impedance increases ($Z_{\tilde{\omega}}^{\max} \propto \Phi_{\max}$). This increase is described by Eq. (5.3) until $|Z_{\tilde{\omega}}|$ becomes of the order of the impedance $Z_{vac} = 4\pi/c$ of the vacuum. Our results are not valid if $|Z_{\tilde{\omega}}| \gtrsim Z_{vac}$, since they were obtained by using the boundary condition (2.4) under the assumption that $|Z_{\tilde{\omega}}| \ll Z_{vac}$.

A characteristic feature of the low-frequency impedance is its hysteresis as a function of the external magnetic field h_0. Figure 2 shows schematically a set of hysteresis loops of $\Phi(\mathscr{H}, \mathscr{H}, h_0)$ for different values of the amplitude $\mathscr{H}$ (of the nonlinearity parameter b). For the sake of clarity we show alongside the $\Phi(h_0)$ curves schematic plots of $h_\infty(h_0)$ at the same values of the parameter b. We note that the "pre-hysteresis" $\Phi(h_0)$ curve corresponding to $b = 0.25$ agrees with the experimental curve of Ref. 1. The dependences, observed in Ref. 1, of the signal amplitude ($\propto Z_{\tilde{\omega}}^{\max}$) on $\mathscr{H}, \tilde{\mathscr{H}}, \omega$, and the temperature agree qualitatively with our result (4.15).

In conclusion, we consider one more possible formulation of the wave-interaction problem. It is easier to have in experiment a situation in which the nonlinearity is due to the large amplitude of the low-frequency signal ($\tilde{\mathscr{H}} \gg \tilde{g}$), and the amplitude of the high-frequency wave is small ($\mathscr{H} \ll g$). The high-frequency impedance is then the linear impedance at the frequency ω in a magnetic field $h_0 + 2\tilde{\mathscr{H}}\cos\tilde{\omega}t$, averaged over the period of the low-frequency signal:

$$Z_\omega(\tilde{\mathscr{H}}, h_0) = \frac{1}{2\pi}\oint d\varphi\, Z_\omega^{\text{lin}}(h_0 + 2\tilde{\mathscr{H}}\cos\varphi). \qquad (5.5)$$

Such a situation is of less physical interest than the case considered in the present paper. Indeed, as can be seen from (5.5), the impedance $Z_\omega(\tilde{\mathscr{H}}, h_0)$ depends little on the amplitude $\tilde{\mathscr{H}}$.

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