Near zone of an antenna in a magnetoactive plasma

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The structure of the electromagnetic field and of the plasma density in the near zone of an antenna located in a magnetized plasma is investigated both in a linear approximation and with allowance for the ponderomotive force exerted by the electromagnetic field. An expression is obtained for the distortion of the external magnetic field by the antenna field. It is shown that the plasma can be either expelled from or drawn into the strong-field region by the ponderomotive force. A threshold-dependent effect is pointed out and investigated. It implies that when the antenna current exceeds a certain critical value the nonlinearity qualitatively alters the near-zone structure and produces at a certain distance from the antenna a narrow region in which the field and density have very large gradients.

1. INTRODUCTION

In the antenna near zone, i.e., a region whose dimensions are small compared with radiation wavelength λ, the electric field intensities attain their maximum values. For electromagnetic waves generated in a magnetoactive plasma, this region can be quite large (of order 10^2 m) for waves of low frequency, i.e., lower than the electron gyrofrequency. Since the fields in the near zone are strong, the major role should be assumed by nonlinear effects due to alteration of the plasma state in the external field. At any rate, the role of these effects and of their influence on the impedance and on other antenna characteristics must be estimated first in the near zone.

The present study was prompted by an attempt to estimate the role of nonlinear effects of electromagnetic-wave emission in experiments carried out in laboratory and in outer-space plasma. We confine ourselves here to nonlinearities due to the ponderomotive forces exerted by a field of the form

$$\tfrac{1}{2}\mathbf{E}(\mathbf{R})e^{-i\omega t}+\text{c.c.} \tag{1.1}$$

on a plasma.[1)] A number of linear-approximation topics of importance to the nonlinear theory are also dealt with. The basic equations are considered and analyzed in Sec. 2. The starting point is a system consisting of the Maxwell and hydrostatic equations, with allowance made for the ponderomotive forces. These equations are applied in Sec. 3 to one of the nonlinear effects, viz., the distortion of an external magnetic field by the electric field (1.1). The near zone of a long (compared with the "dispersion length") electric red antenna is considered in Sec. 4. The equations discussed describe not only the near-zone structure but also the so-called "resonance" cones in which the radiation field is electrostatic. The resonance cones (more accurately, layers) that extend into the wave zone are of importance in understanding the near-zone structure and are considered in the present paper from just this standpoint. Investigation of the nonlinear structure of the near zone shows that the plasma can not only be expelled from but also drawn into the strong-field region by the ponderomotive force (Sec. 4.2). Section 5 deals with the near zone of a magnetic antenna (a current-carrying loop of finite radius, with its plane perpendicular to the external field). It is shown that there are no resonance cones even for a "point" source (Sec. 5.3). Nonlinear effects are investigated in Sec. 5.3. It is shown, in particular, that these effects have a threshold. It is found that when an antenna threshold current I_0 is exceeded there exists on the z axis passing through the center of the loop a point z_0 at which the field derivative is $dE/dz=\infty$. The field itself becomes discontinuous at the point z_0 (if dispersion is disregarded). Estimates show that the current attained in experiments (e.g., in Refs. 2–4), is either close to I_0 or exceeds it.

2. BASIC EQUATIONS AND THEIR ANALYSIS

In the near zone one can neglect the displacement currents, so that the basic equations for the field take the form

$$\operatorname{rot}\mathbf{H}=-(4\pi/c)\mathbf{j}, \tag{2.1}$$

$$\operatorname{rot}\mathbf{E}=-(1/c)\partial\mathbf{H}/\partial t, \tag{2.2}$$

$$\operatorname{div}\mathbf{D}=4\pi\rho. \tag{2.3}$$

Here $\hat{\varepsilon}=(\varepsilon_{ik})$ is the plasma dielectric tensor and depends on the plasma density $N(\mathbf{R})$ and on the external magnetic field $\mathbf{B}(\mathbf{R})$; $\mathbf{j}$ and ρ are the current and charge densities in the antenna. Introducing the vector potential

$$\mathbf{H}=\operatorname{rot}\mathbf{A},\qquad \operatorname{div}\mathbf{A}=0, \tag{2.4}$$

we get from (2.1)

$$\Delta\mathbf{A}=-(4\pi/c)\mathbf{j}. \tag{2.5}$$

Equations (2.1) and (2.5) do not contain ε_{ik}. The magnetic field in the near zone is therefore determined by the same equations as the static field in a vacuum. The plasma properties, however, particularly its anisotropy and nonlinearity, play a substantial role in the equation for $\mathbf{E}$. Substituting (2.4) in (2.2) and assuming that the time dependence is determined by the phase factor $\exp(-i\omega t)$, we obtain

$$\mathbf{E}=i(\omega/c)\mathbf{A}-\nabla\psi, \tag{2.6}$$

where ψ is a scalar potential whose equation is obtained by substituting (2.6) in (2.3):

$$\operatorname{div}(\hat{\varepsilon}\nabla\psi)=i(\omega/c)\operatorname{div}(\hat{\varepsilon}\mathbf{A})-4\pi\rho. \tag{2.7}$$

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This equation must be considered jointly with the material equations. We assume here that the plasma is collisionless and "cold," i.e.,

$$8\pi NT/B^2 \ll 1 \tag{2.8}$$

(we assume for simplicity that $T_{\parallel} = T_{\perp}$). We can then use the magnetohydrodynamic equations, which take for time-independent source amplitudes the form

$$\nabla p + (1/4\pi)[\mathbf{B}\,\mathrm{rot}\,\mathbf{B}] = \mathbf{f}. \tag{2.9}$$

Here $p = 2NT$, $T = (T_e + T_i)/2$ (we assume that $T = \mathrm{const}$), and **f** is the volume density of the ponderomotive force exerted by the antenna on the plasma. Under these assumptions, this force can be chosen in the form (see Refs. 5 and 6 and the literature cited there)

$$\mathbf{f} = (1/16\pi)\{(\varepsilon_{ij} - \delta_{ij})\nabla(E_i^* E_j) + M_i \nabla B_i + [\mathbf{B}\,\mathrm{rot}\,\mathbf{M}]\}, \tag{2.10}$$

where **M** is the density of the magnetic moment induced by the oscillating electromagnetic field,

$$\mathbf{M} = \frac{1}{16\pi}\frac{\partial \varepsilon_{ij}}{\partial \mathbf{B}} E_i^* E_j. \tag{2.11}$$

We now designate the density and the external magnetic field at large distances from the antenna by N_0 and $\mathbf{B}_0$, respectively, and introduce the relative quantities

$$\mathbf{b} = (\mathbf{B} - \mathbf{B}_0)/B_0, \qquad \nu = (N - N_0)/N_0. \tag{2.12}$$

Analysis of Eq. (2.9) shows that condition (2.8) makes b small:

$$b \sim (8\pi NT/B_0^2)|\nu| \ll |\nu| \leqslant 1, \tag{2.13}$$

so that the terms that contain **b** can be linearized in (2.9) [Eq. (2.13) will be corroborated below by the results of the corresponding solution]. As a result we have

$$\nabla p + (\rho_0 c_A^2/B_0)[\mathbf{B}_0[\nabla \mathbf{b}]] = \mathbf{f}, \tag{2.14}$$

where $\rho_0 = (zm_e + m_i)N_0 \approx m_i N_0$ and $c_A^2 = B_0^2/4\pi\rho_0$. We emphasize that (2.14) was derived without assuming that ν is small.

Since b_1 is small, we can put $\mathbf{B} = \mathbf{B}_0$ in the expressions for ε_{ij}. The nonzero components ε_{ik} are thus $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon$, $\varepsilon_{xy} = -\varepsilon_{yx} = -ig$, $\varepsilon_{zz} = \eta$, where

$$\varepsilon = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2(N)}{\omega_{c\alpha}^2 - \omega^2}, \qquad g = -\sum_\alpha \frac{\omega_{p\alpha}^2(N)\,\omega_{c\alpha}}{\omega(\omega_{c\alpha}^2 - \omega^2)},$$
$$\eta = 1 - \sum_\alpha \omega_{p\alpha}^2(N)/\omega^2. \tag{2.15}$$

The summation here is over the particle species ($\alpha = e, i$), $\omega_{p\alpha}(N)$ is the plasma frequency at the density N, and $\omega_{c\alpha}$ is the cyclotron frequency at $B = B_0$ ($\mathbf{B}_0$ is assumed directed along the z axis).

We assume hereafter that the antenna axis coincides with z and introduce the cylindrical coordinates r, φ, and z, assuming that all the derivatives with respect to φ are zero. Then

$$(\varepsilon_{ij} - \delta_{ij})E_i^* E_j = (\varepsilon - 1)(E_r^* E_r + E_\varphi^* E_\varphi) + ig(E_\varphi^* E_r - E_\varphi E_r^*) + (\eta - 1)E_z^* E_z, \tag{2.16}$$

$$M_z = \frac{1}{16\pi}\left\{\frac{\partial \varepsilon}{\partial B_0}(|E_r|^2 + |E_\varphi|^2) + i\frac{\partial g}{\partial B_0}(E_\varphi^* E_r - E_\varphi E_r^*)\right\}, \tag{2.17}$$
$$M_r = M_\varphi = 0,$$

$$f_r = \frac{1}{16\pi}\left[(\varepsilon_{ij} - \delta_{ij})\frac{\partial}{\partial r}(E_i^* E_j) + B_0 \frac{\partial M_z}{\partial r}\right], \tag{2.18}$$

$$f_z = \frac{1}{16\pi}(\varepsilon_{ij} - \delta_{ij})\frac{\partial}{\partial z}(E_i^* E_j). \tag{2.19}$$

Substituting (2.19) in the z component of (2.14) and recognizing that the quantities $\varepsilon_{ij} - \delta_{ij}$ are proportional to the plasma density N, we obtain[2)]

$$N = N_0 \exp\{(32\pi N_0 T)^{-1}[(\varepsilon_0 - 1)(|E_r|^2 + |E_\varphi|^2) + (\eta_0 - 1)|E_z|^2 + ig_0(E_\varphi^* E_r - E_r^* E_\varphi)],$$
$$\varepsilon_0 = \varepsilon(N_0), \quad \eta_0 = \eta(N_0), \quad g_0 = g(N_0). \tag{2.20}$$

Substituting (2.18) in the r component of (2.14) and taking (2.20) into account, we get

$$\partial b_z/\partial r - \partial b_r/\partial z = (4\pi/B_0)\partial M_z/\partial r. \tag{2.21}$$

With this equation we can, in principle, express **b** in terms of the antenna electric field **E** (see Sec. 3).

The study of the antenna near zone reduces thus to solution of Eq. (2.7) jointly with (2.5), (2.15), and (2.20). An important feature of Eq. (2.7) is that the coefficients ε and η can differ in sign and in frequency dependence. This circumstance manifests itself substantially even in the linear approximation, i.e., when it is assumed in (2.15) that $N = N_0$. In this case we can in fact rewrite (2.7) in the form

$$\frac{\varepsilon_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \eta_0 \frac{\partial^2 \psi}{\partial z^2} = \frac{i\omega}{c}\mathrm{div}(\hat{\varepsilon}A) - 4\pi\rho. \tag{2.22}$$

Assuming that (2.5) has been solved, we can regard the right-hand side of (2.22) as known. Equation (2.22) is an inhomogeneous equation which is elliptic if the signs of ε_0 and η_0 coincide and parabolic if they differ. The latter occurs if

$$\max(\omega_{pe}, \omega_{ce}) < \omega < \omega_{UH} \quad (\varepsilon < 0,\ \eta > 0), \tag{2.23a}$$

$$\omega_{LH} < \omega < \min(\omega_{ce}, \omega_{pe}) \quad (\varepsilon > 0,\ \eta < 0), \tag{2.23b}$$

$$0 < \omega < \omega_{ci} \quad (\varepsilon > 0,\ \eta < 0), \tag{2.23c}$$

where ω_{UH} and ω_{LH} are the upper and lower hybrid frequencies:

$$\omega_{UH}^2 = \omega_{pe}^2 + \omega_{ce}^2, \quad \omega_{LH}^2 = \omega_{ce}^2(\omega_{pi}^2 + \omega_{ci}^2)/(\omega_{pe}^2 + \omega_{ce}^2).$$

In the remaining frequency bands, however ε and η are of like sign, which can be easily determined by using (2.15). Relations (2.23) are known to be the conditions for the existence of electrostatic waves. It can be easily seen that the latter propagate precisely along the characteristic directions of Eq. (2.22). If the right-hand side of (2.22) contains a σ function or its derivatives (point sources), the characteristics originating at the sources form resonance cones. These cones (including the cases when account is taken of the spatial dispersion, which we neglect here) have been extensively studied (see, e.g., Refs. 7–9 and the literature cited therein). All these factors, which become much more complicated when allowance is made for nonlinear effects, must be taken into consideration when Eqs. (2.7) and (2.22) are solved and investigated, and their manifestations in various cases will be discussed, in particular, in the sections that follow.

3. DISTORTION OF EXTERNAL MAGNETIC FIELD BY THE ANTENNA-INDUCED FIELD

From the equation div $\mathbf{B}=0$, i.e.,

$$(1/r)\partial(rb_r)/\partial r+\partial b_z/\partial z=0,$$

it follows that

$$b_r=-\partial\chi/\partial z, \quad b_z=(1/r)\partial(r\chi)/\partial r, \tag{3.1}$$

where $\chi(r,z)$ is some function. Substituting (3.1) in (2.21) we obtain

$$\frac{\partial^2\chi}{\partial z^2}+\frac{\partial^2\chi}{\partial r^2}+\frac{1}{r}\frac{\partial\chi}{\partial r}-\frac{1}{r^2}\chi=\frac{4\pi}{B_0}\frac{\partial M_z}{\partial r}. \tag{3.2}$$

This equation yields $\chi(r,z)$ if the electric field excited by the antenna is known. To solve (3.2) we take the Hankel transform with respect to r and the Fourier trransform with respect to z:

$$\tilde\chi(p,s)=\int_{-\infty}^{\infty}dze^{-ipz}\int_0^{\infty}drrJ_1(rs)\chi(r,z),$$

$$\chi(r,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dpe^{ipz}\int_0^{\infty}dssJ_1(rs)\tilde\chi(p,s), \tag{3.3}$$

where J_1 is a Bessel function and $\tilde\chi(p,s)$ is the corresponding transform. Substituting (3.3) in (3.2) and using the equation for $J_1(rs)$ we obtain ultimately

$$\chi(r,z)=-\frac{4}{B_0}\int_{-\infty}^{\infty}dz_1\int_0^{\infty}dr_1r_1^{1/2}Q_{1/2}\left(1+\frac{(r_1-r)^2+(z_1-z)^2}{2rr_1}\right)\times\frac{\partial M_z(r_1,z_1)}{\partial r_1} \tag{3.4}$$

where $Q_{1/2}(x)$ is a Legendre function of the second kind. It is useful here to bear in mind the relations

$$Q_{1/2}(x)=(\pi/4\sqrt2)x^{-3/2}F(5/4;3/4;2;x^{-2})$$
$$=x\left(\frac{2}{1+x}\right)^{1/2}K\left(\left(\frac{2}{1+x}\right)^{1/2}\right)$$
$$-[2(1+x)]^{1/2}E\left(\left(\frac{2}{1+x}\right)^{1/2}\right), \tag{3.5}$$

where F is a hypergeometric function and K and E are complete elliptic integrals. It follows from (3.5), in particular, that

$$Q_{1/2}(x)\sim x^{-3/2}\ (x\gg1),\quad Q_{1/2}(x)\sim\ln[32(x-1)^{-1/2}]\ (x\to1).$$

Equation (3.4) determines in principle the distortion of an external magnetic field in the near zone of an antenna. This equation leads, in particular, to the estimate (3.13). Although $b\ll1$ in a "cold" plasma, this quantity may be of interest for the analysis of geomagnetic perturbations produced by an antenna in space plasma, as well as for the analysis of the generation of geomagnetic pulsations if the radiated electromagnetic waves are amplitude-modulated. We intend to consider these questions, which are outside the scope of the present article, elsewhere.

4. NEAR ZONE OF A LONG ELECTRIC-ROD ANTENNA

By way of one important example we consider a long electric-rod antenna consisting of two close ideally conducting rods of length L oriented along the z axis. If the wavelength is $\lambda\gg2\pi L$, the linear charge density at each point of the antenna, except at the end points of the rods, can be regarded as constant; we can then write for the volume density ρ:

$$\rho=q(z)\delta(x)\delta(y)\exp(-i\omega t), \tag{4.1}$$
$$q(z)=\sigma\quad(0<z<L),\quad q(z)=-\sigma\quad(-L<z<0),$$

where $\sigma=$ const stands for the linear-charge-density amplitude. This expression can be deduced from Gauss's theorem. We note also that in an isotropic medium (4.1) represents the limiting case of the known current distribution over an ideally conducting rod antenna (see, e.g., Ref. 11):

$$i(z)=i_0\exp(-i\omega t)\sin k(L-|z|)/\sin kL\quad(|z|<L),$$
$$i(z)=0\quad(|z|>L).$$

By substituting this expression in the continuity equation $\partial i/\partial z=i\omega q(z)$ and putting $kL\ll1$ we arrive at (4.1).

We will be considering only the region outside the transition layer that screens the antenna charge (the structure of this layer at $B_0=0$ was investigated in Ref. 1). We neglect also effects due to the plasma-particle collisions with the antenna. Furthermore, since we use expressions (2.15), i.e., we neglect spatial dispersion, the antenna dimensions must be regarded as much larger than the Debye and Larmor radii.

4.1. Linear theory

We consider first the rod-antenna electric field (4.1) in the linear approximation, i.e., using Eq. (2.22). It is easily verified that in the near zone we can neglect the first term of the right-hand side of (2.22). Introducing the quantity

$$\gamma^2(\omega)=-\varepsilon_0(\omega)/\eta_0(\omega) \tag{4.2}$$

and taking the Fourier transform of (2.22) with the $\mathbf{A}$ term omitted, we get

$$\psi=-(\sigma/\eta\gamma^2)[F_+(r,z)+F_-(r,z)], \tag{4.3}$$

$$F_\pm(r,z)=-\frac{i}{\pi}\int_0^{\infty}dqqJ_0(qr)\times\int_{-\infty}^{\infty}dp\frac{\exp[ip(z\pm L)]-\exp[ipz]}{p(p^2-\gamma^2q^2)}, \tag{4.4}$$

where J_0 is a Bessel function. If ε_0 and η_0 are of like sign, the denominator in (4.4) does not vanish. If the signs of ε_0 and η_0 are opposite (this is precisely the case considered below), the integrand has a pole that must be integrated around in accord with the condition that the field is adiabatically switched on at $t = -\infty$. This is equivalent to assuming that ω has an infinitesimal positive imaginary part that must tend to zero after we calculate (4.4).[12]

Recognizing that at $\omega = \omega_0 + i\alpha(\alpha \ll \omega_0)$ we have

$$\operatorname{Im}\gamma(\omega) = \alpha\gamma(\omega_0)(\varepsilon_0'/\varepsilon_0 - \eta_0'/\eta_0)/2, \quad \varepsilon_0' = d\varepsilon_0/d\omega_0 > 0,$$

$$\eta_0' = d\eta_0/d\omega_0 > 0,$$

we obtain at $\gamma(\omega_0) > 0$

$$\operatorname{sign}\operatorname{Im}\gamma(\omega) = \operatorname{sign}\varepsilon_0(\omega_0) = -\operatorname{sign}\eta_0(\omega_0). \quad (4.5)$$

Evaluating the integral with respect to p in (4.4) yields then

$$F_\pm(r,z) = \frac{z\pm L}{|z\pm L|}\int_0^\infty \frac{dq}{q} J_0(qr)\exp\left\{ i\gamma\frac{\varepsilon}{|\varepsilon|}|z\pm L|\right\} - \frac{z}{|z|}\int_0^\infty \frac{dq}{q} J_0(qr)\exp\left\{ i\gamma\frac{\varepsilon}{|\varepsilon|}|z|\right\}. \quad (4.6)$$

To regularize the integrals in (4.6) we make the substitution $J_0 \to J_\nu$ with $\nu > 0$, and let next $\nu \to 0$. The $\sim \nu^{-1}$ terms are then cancelled if

$$\operatorname{sign}(z\pm L) = \operatorname{sign} z. \quad (4.7)$$

If (4.7) is satisfied we have

$$F_\pm(r,z) = \ln[\beta_0 + (\beta_0^2-1)^{1/2}] - \ln[\beta_{\pm1} + (\beta_{\pm1}^2-1)^{1/2}] \quad (z>0), \quad (4.8a)$$

$$F_\pm(r,z) = -\ln[-\beta_0 + (\beta_0^2-1)^{1/2}] + \ln[-\beta_{\pm1} + (\beta_{\pm1}^2-1)^{1/2}] \quad (z<0), \quad (4.8b)$$

where $\beta_n = \gamma(z+nL)/r$, $n = 0, \pm 1$. If furthermore $|\beta_n| < 1$, it must be assumed that in (4.8)

$$\ln[\pm\beta_n + (\beta_n^2-1)^{1/2}] = -i\operatorname{sign}\eta\arccos(\pm\beta_n)\,(|\beta_n|<1). \quad (4.9)$$

If (4.7) is not satisfied, the integral representation (4.6) does not hold for $F_\pm(r,z)$. At such values of z the functions $F_\pm(r,z)$ can be obtained by analytically continuing the functions (4.8) from the regions where the conditions (4.7) hold. It follows that (4.8) and (4.9) are valid also outside the regions (4.7).

Equations (4.3), (4.8) and (4.9) describe fully the electric field in the rod antenna near zone. The potential $\psi(r, z)$ has a substantially different behavior in the different sectors indicated in Fig. 1. The field is thus real in the sectors $A_\pm$ (and is determined by the sum of expressions (4.8) with $\beta_n > 1$). In sectors $B_\pm$ the function $\psi(r,z)$ acquires an imaginary part. As the straight line $\gamma z = \pm r \mp L$ is approached in sectors $C_\pm$, the real part of ψ vanishes, so that the potential in the sector $D_+ + D_-$ is purely imaginary. The transition of $\psi(r,z)$ from sector to sector is continuous, but the derivatives of ψ, i.e., the components of $\mathbf{E}(r,z)$, become infinite on the boundaries of the sectors that are separated from one another by the cones $\gamma z = \pm r + C$, $C = 0, \pm L$. It can easily be verified that the latter are the characteristics of Eq. (2.22). The singularities on the sector boundaries are due to the end points $z = 0$ and $z = \pm L(r=0)$, whose influence "persists" along the characteristics.

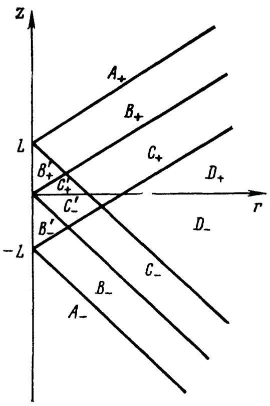

FIG. 1. Diagram of the various characteristic regions for the near zone of an electric rod antenna.

We consider next the potential $\psi(r,z)$ at large and small distances from the rod antenna. Thus, at $R \gg L$ and $\gamma^2 z^2 - r^2 \gg L^2$ we have from (4.3), (4.8), and (4.9)

$$\psi = -(p/\eta_0)\gamma z(\gamma^2 z^2 - r^2)^{-3/2}, \quad \gamma|z|>r, \quad (4.10a)$$

$$\psi = i(p/|\eta_0|)\gamma z(r^2 - \gamma^2 z^2)^{-3/2}, \quad \gamma|z|<r, \quad (4.10b)$$

where $p = \sigma L^2$ is the amplitude of the antenna dipole moment.

Equations (4.10) determine the potential of a point dipole. In this approximation, both the field intensity and the potential $\psi(r,z)$ have singularities on the resonance cone $r = \pm\gamma z$, with ψ real and purely imaginary inside and outside the cone, respectively. For a rod of finite size the cone spreads to form a conical layer of thickness $2L$ (which can be called the resonance layer), within which the field decreases very slowly with increasing distance from the source ($\psi \sim R^{-1/2}$) and changes gradually from real (in the A region) to purely imaginary (in the D region). This situation is typical not only of rod antennas but also of other sources.[9] Allowance for the spatial dispersion which is completely neglected in our case) leads to the oscillating resonance-layer fine structure considered for certain sources in Refs. 8 and 9. We note that since the field decreases very slowly in the resonance layer, its structure is preserved only in the wave zone. This is precisely the region where the electric field is irrotational also at large distances (electrostatic waves).

Expressions (4.10) can be represented in the form $\psi = 4\pi p(\partial/\partial z)G(\mathbf{R})$, where $G(\mathbf{R}-\mathbf{R}')$ is the Green's function of (2.22) (i.e., $G(\mathbf{R})$ defines the potential of a single point charge located at $R = 0$):

$$G(R) = (\operatorname{sign}\eta/4\pi)|\varepsilon_0\eta_0|^{-1/2}(\gamma^2 z^2 - r^2)^{-1/2} \quad (|\gamma z|>r), \quad (4.11a)$$

$$G(R) = (i/4\pi)|\varepsilon_0\eta_0|^{-1/2}(r^2 - \gamma^2 z^2)^{-1/2} \quad (|\gamma z|<r). \quad (4.11b)$$

Expressions (4.11) coincide with the Green's function obtained in Ref. 9 for Eq. (2.22) by another method. At $\gamma^2 < 0$ Eq. (4.11) leads to the Green's function for the case $\varepsilon\eta > 0$ (Ref. 12, §13). The transition from $\varepsilon\eta > 0$ to $\varepsilon\eta < 0$ (just as from $|\gamma z| > r$ to $|\gamma z| < r$) is thus via a corresponding analytic continuation. The problem, however, is how to choose correctly the cuts and sheets in the complex planes of the branching functions.

We consider now the field at short distances from the rod antenna, where the conditions $\gamma|z \pm L| \gg r$, $\gamma|z| \gg r$ are satisfied. In this case it follows from (4.3) and (4.8) that

$$E_r \approx \text{sign}\, z(2\sigma/\varepsilon_0 r) \gg E_z. \qquad (4.12)$$

It was recognized in the derivation of (12) that, e.g., at $L > z > 0$ (region B'_+ in Fig. 1) and at $r \ll \gamma(L-z)$ we have

$$\gamma(z-L)+[\gamma^2(z-L)^2-r^2]^{1/2}=-r^2/2\gamma(L-z)+o(r^2).$$

The results (4.12) agree with those obtained from Gauss's theorem in the calculation of the flux of the induction D through a cylindrical surface of small radius surrounding an antenna section in the region where the influence of the points $z = L$ and $z = 0$ can be neglected.

4.2. Some nonlinear effects

Applying the Gauss theorem as before, but using the expression for $\varepsilon(N)$, we readily obtain an approximate expression for the field at short distance from the antenna, with allowance for the density change due to the ponderomotive force. We then obtain in lieu of (4.12)

$$\varepsilon(N)E_r \approx 2\,\text{sign}\, z(\sigma/r), \qquad E \approx |E_r|. \qquad (4.13)$$

Here $\varepsilon(N)$ is defined in (2.15). Recognizing that $\varepsilon(N) = 1 + (\varepsilon_0 - 1)N/N_0$ and introducing the new quantities

$$[(\varepsilon_0-1)/32\pi N_0 T]^{1/2}E=\Xi, \qquad \sigma[8\pi N_0 T(\varepsilon_0-1)]^{-1/2}=r_0 \qquad (4.14)$$

(we assume that $\varepsilon_0 > 1$, as is the case when $\omega < \omega_{ce}$), we obtain for the field an equation in the form

$$[(\varepsilon_0-1)^{-1}+\exp \Xi^2]\Xi=r_0/r, \qquad (4.15)$$

which can be easily solved graphically. This equation is valid because of the inequality $r \ll L$. Equation (4.15) becomes particularly simple in the important case $\varepsilon_0 \gg 1$ which occurs at $\omega_{pe}^2 \gg \omega_{ce}^2$. We have then in place of (4.15)

$$\Xi \exp \Xi^2=r_0/r. \qquad (4.16)$$

The linear equation (4.12) follows from (4.15) at $r \gg r_0$. At $r \ll r_0$ we have

$$\Xi^2 \approx \ln(r_0/r)-(1/2)\ln\ln(r_0/r)+\ldots \qquad (4.17)$$

As $r \to 0$ the nonlinearity thus leads to a slower growth of the field than in (4.12).

It follows also from (4.15) that at small r the density behaves as

$$N(r)=N_0 \exp \Xi^2 \sim N_0 r_0/r,$$

i.e., it increases with decreasing r. The ponderomotive force thus draws the plasma into the stronger-field region. The reasons are that the principal role is played in this region by the radial component of the electric field and that $\varepsilon_0 > 1$ (i.e., $\omega < \omega_{ce}$). In the other regions the plasma can be either drawn in or expelled.

Let us estimate, for example, the change of the density at large distances, where the dipole approximation (4.10) can be used. In this case we have ($\varepsilon_0 \gg 1$, $|\eta_0| \gg 1$)

$$\frac{N-N_0}{N_0} \approx -\frac{\gamma^2 p^2}{32\pi N_0 T|\eta_0|}\frac{(r^2-\gamma^2 z^2)(r^2-4\gamma^2 z^2)}{|r^2-\gamma^2 z^2|^{5/2}} \qquad (4.18)$$

(the validity of this equation is restricted by the condition $|N - N_0| \ll N_0$). We conclude from (4.18) that

$$N-N_0>0 \quad \text{при} \quad \gamma^2 z^2<r^2<4\gamma^2 z^2, \qquad (4.19)$$

and that $N - N_0 < 0$ in the remaining regions. We plan to investigate the stability of configurations containing regions with both $N \lessgtr N_0$ in another paper.

5. MAGNETIC ANTENNA

Assume that the antenna is a circular loop of radius a in a plane perpendicular to $\mathbf{B}_0$, carrying a current I. Equation (2.7) takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\varepsilon\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\eta\frac{\partial\psi}{\partial z}\right)=\frac{\omega}{cr}\frac{\partial}{\partial r}(rgA_\varphi); \qquad (5.1)$$

it is recognized that $A_r = A_z = 0$ in this case. Then[12]

$$A_\varphi=\frac{4I}{cs}\left(\frac{a}{r}\right)^{1/2}\left[\left(1-\frac{s^2}{2}\right)K-E\right], \qquad (5.2)$$

where s is the modulus of the complete elliptic integrals $K(s)$ and $E(s)$,

$$s^2=4ar/[(a+r)^2+z^2]. \qquad (5.3)$$

As before, we neglect spatial dispersion and also the collisions between the plasma particles and the antenna.

5.1 Magnetic dipole (linear approximation)

Equation (5.2) leads at $r^2 + z^2 \gg a^2$ to the following formula for the magnetic-dipole vector potential:

$$A_\varphi=Mr(r^2+z^2)^{-3/2}, \qquad M=\pi a^2 I/c. \qquad (5.4)$$

Equation (5.1) can in this case be written in the linear approximation, i.e., for $N = N_0$, in the form

$$\frac{\gamma^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)-\frac{\partial^2\psi}{\partial z^2}$$
$$=\frac{\alpha}{|z|^3}\left[3\left(1+\frac{r^2}{z^2}\right)^{-5/2}-\left(1+\frac{r^2}{z^2}\right)^{-3/2}\right], \qquad (5.5)$$

where γ is defined in (4.1),

$$\alpha=\omega M g_0/c|\eta_0|, \qquad g_0=g(N_0). \qquad (5.6)$$

(For the sake of argument it is assumed in (5.5) and elsewhere that $\varepsilon > 0$ and $\eta < 0$. This frequency range includes, in particular, whistlers with $\omega < \omega_{ce}$ and waves of lower frequency.)

Equation (5.5) contains only one dimensional parameter α (with units of electric charge). A solution that decreases as $R \to \infty$ is therefore sought for it in the self-similar form

$$\psi=\alpha|z|^\nu f(t), \qquad t=r^2/\gamma^2 z^2. \qquad (5.7)$$

Substituting it in (5.5), we obtain $\nu = -1$, and for $f(t)$ we get the inhomogeneous hypergeometric equation

$$t(1-t)f''+[1-{}^5/_2t]f'-{}^1/_2f={}^1/_4[3(1+\gamma^2t)^{-5/2}-(1+\gamma^2t)^{-3/2}]+\delta(z)|z|[f(t)+2tf'(t)]. \quad (5.8)$$

The general solution of (5.8) at $z\neq 0$ can be written in the form

$$f(t)=C(1-t)^{-1/2}-(1+\gamma^2)^{-1}(1+\gamma^2t)^{-1/2}, \quad (5.9)$$

where the first term is the general solution of the homogeneous equation (C is an arbitrary constant), and the second is the particular solution of the inhomogeneous equation (5.8) without the δ-function term (the latter solution was chosen not to have singularities at $t=0,\ 1,\ \infty$). We consider now the term with the δ function. Substituting (5.9) in it, we get

$$f(t)+2tf'(t)=C(1-t)^{-3/2}-(1+\gamma^2)^{-1}(1+\gamma^2t)^{-3/2}.$$

We see hence that if $C\neq 0$ this term leads to a non-integrable singularity at the origin if $r\to 0$ along some line, e.g., $r^2=\gamma^2z^2-\text{const}\ z^{\mu+2}$, $\mu>4/3$. We therefore put $C=0$ in (5.9). The solution accordingly takes the form

$$\psi=-\alpha(1+\gamma^2)^{-1}R^{-1}, \quad R^2=r^2+z^2. \quad (5.10)$$

Substituting (5.10) and (5.4) in (2.6) we obtain the following expressions for the electric-field components in the near zone of a magnetic-dipole antenna:

$$E_r=-\alpha(1+\gamma^2)^{-1}rR^{-3}, \quad E_z=-\alpha(1+\gamma^2)^{-1}zR^{-3},$$
$$E_\varphi=i(\omega/c)MrR^{-3}. \quad (5.11)$$

As for the magnetic-field intensity, it is of the same form as for a static magnetic dipole.

We emphasize that, in contrast to an electric dipole, the potential (5.10), and accordingly the intensity (5.11), have no singularities on the resonant cone.

5.2 Magnetic antenna of finite radius (linear approximation)

We subdivide the inner area of the current loop into elements ds ($\int\int ds=\pi a^2$). The field produced by the current-carrying loop can then be represented as a superposition of fields produced by elementary magnetic dipoles with moments $dM=(I/c)ds$. We can write accordingly for the potential

$$\psi=-\frac{\alpha}{\pi a^2(1+\gamma^2)}\iint\frac{dx'\,dy'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}, \quad (5.12)$$

where the integration is over the area bounded by the current. Assuming

$$dx'dy'=r'dr'd\varphi,$$
$$(x-x')^2+(y-y')^2+z^2=R^2-r'^2-2rr'\cos\varphi,$$

where R is the distance from the origin (from the center of the circle), and integrating first with respect to r' and next with respect to φ, we obtain as the end result

$$\psi=-\frac{\alpha}{(1+\gamma^2)\pi a^2}\left\{\frac{8a}{s}\left(\frac{r}{a}\right)^{1/2}E-2\pi z-\frac{2R^2}{a}\left(\frac{a}{r}\right)^{1/2}\times s\left[\left(1+\frac{ar}{R^2}\right)K-2\frac{ar}{R^2}\frac{K-E}{s^2}\right]+\frac{(R-r)(R-a)s}{a}\left(\frac{a}{r}\right)^{1/2}\Pi\left(\frac{2r}{R+r},s\right)+\frac{(R+r)(R+a)s}{a}\left(\frac{a}{r}\right)^{1/2}\Pi\left(-\frac{2r}{R-r},s\right)\right\}, \quad (5.13)$$

Here K and E are complete elliptic integrals of the first and second kind with the modulus s given by (5.3), and $\Pi(\nu,s)$ is a complete elliptic integral of the third kind:

$$\Pi(\nu,s)=\int_0^{\pi/2}\frac{d\varphi}{(1+\nu\sin^2\varphi)(1-s^2\sin^2\varphi)^{1/2}}.$$

It is more convenient to use another expression, which is obtained by integrating first with respect to φ:

$$\psi=-\frac{4\alpha}{(1+\gamma^2)\pi a^2}\int_0^a\frac{dr_1r_1K(s_1)}{(R^2+r_1^2+2rr_1)^{1/2}}, \quad (5.14)$$

while s_1 is obtained from (5.3) with a replaced by r_1. As $a\to 0$ and at constant M, (5.10) can be easily obtained from (5.14), which is a simple integral representation of (5.13). It follows from (5.13) or (5.14) that at $r=0$

$$\psi=-2\alpha(1+\gamma^2)^{-1}[(z^2+a^2)^{1/2}+z]^{-1},$$
$$E_z=-2\alpha(1+\gamma^2)^{-1}a^{-2}[1-z(z^2+a^2)^{-1/2}], \quad E_r=0. \quad (5.15)$$

5.3. Nonlinear effects

We present first a simple expression for $\Delta N=N-N_0$ at $|\Delta N|\ll N_0$. Assuming that the dipole-approximation equations can be used for the electric field and substituting these equations in (2.20), we get

$$\Delta N/N_0\approx -D^2(1+\gamma^2)^{-2}(a/R)^4, \quad (5.16)$$

$$D=\frac{\pi eg_0I}{2c^2|\eta_0|(2m_eT)^{1/2}}, \quad (5.17)$$

where $\sin\theta=r/R$. It can be seen from (5.16) that, in contrast to the electric dipole, $N<N_0$ for a magnetic dipole at all θ.

We consider now the nonlinear self-consistent solution. In the general case this problem calls for the use of rather complicated numerical methods. The situation is somewhat simpler when $|\varepsilon|\gg 1$, $|\eta|\gg 1$. In this case we can assume $\varepsilon\approx(N/N_0)\varepsilon_0$, $\eta\approx(N/N_0)\eta_0$. Substituting these in (5.1) and taking into account (2.20) and also the fact that $\eta<0$ and $\varepsilon<0$, we obtain the following equations:

$$\gamma^2\frac{\partial(rE_r)}{r\partial r}-\frac{\partial E_z}{\partial z}+\frac{\partial\ln N}{\partial r}\left(\gamma^2E_r+\frac{g_0\omega}{|\eta_0|c}A_\varphi\right)-\frac{\partial\ln N}{\partial z}E_z=-\frac{\omega g_0}{c|\eta_0|}\frac{\partial(rA_\varphi)}{r\partial r}, \quad \frac{\partial E_r}{\partial z}=\frac{\partial E_z}{\partial r}; \quad (5.18)$$

$$\ln(N/N_0)\approx(32\pi N_0T)^{-1}|\eta_0|[\gamma^2|E_r|^2-|E_z|^2+\gamma^2(\omega/c)^2A_\varphi^2+2\,\mathrm{Re}\,(\omega g_0/c|\eta_0|)A_\varphi E_r]. \quad (5.19)$$

Equation (5.18) has thus a cubic nonlinearity that is contained, however, in the terms with the higher-order derivatives. These equations are still quite complicated. There is, however, an interesting case when simple analytic results can be obtained and shed light on the character of the nonlinear effects in our problem. Consider, in fact, the field near the z axis (i.e., $r \ll z$) and assume that $\gamma^2 \ll 1$; this is the case when $(\omega/\omega_{ce})^2 \ll 1$. We take into account here the finite radius of the loop.

Expanding all the functions in (5.1) in powers of r/z, we have

$$A_\varphi = Mr(a^2+z^2)^{-3/2}+O(r^2), \quad E_r=O(r), \quad \partial E_z/\partial z=O(r), \tag{5.20}$$

$$N=N_0\exp[-\mu^2(z)]+O(r^2), \tag{5.21}$$

$$\mu=-(\omega_{pe}/\omega)(32\pi N_0 T)^{-1/2}E_z(z,0). \tag{5.22}$$

Substituting (5.20) and (5.21) in (5.18) and retaining only the terms of first order in γ^2, we obtain for $\mu(z)$ the equation[3)]

$$(1-2\mu^2)\,\partial\mu/\partial z=-2Da^2(a^2+z^2)^{-3/2}, \tag{5.23}$$

where D is defined in (5.17). Solving (5.23) under the condition $\mu(z)\to 0(z\to\infty)$, we get

$$\mu-{}^2/_3\mu^3=2D[1-z(z^2+a^2)^{-1/2}]. \tag{5.24}$$

At large distances from the antenna, when the cubic term can be neglected, (5.24) leads to (5.16) if $\gamma^2 \ll 1$.

We investigate next the polynomial $P(\mu)=\mu-2/3\mu^3$ (Fig. 2). Its roots are $\mu=0$ and $\pm(3/2)^{1/2}$. The roots of $P'(\mu)$ are $\pm\mu_+$ where $\mu_+=(1/2)^{1/2}\approx 0.71$, so that $P(\mu_+)=(2/9)^{1/2}\pm 0.47$. It follows from the foregoing that (5.24) has three real roots (two positive and one negative) if $2D[1-z(z^2+a^2)^{-1/2}]<(2/9)^{1/2}$. One of the positive roots determines the $\mu(z)$ branch that vanishes as $z\to\infty$ and determines therefore the desired solution that goes over into a linear expression at large z. We call this the physical branch. If $2D<(2/9)^{1/2}$, i.e.,

$$D<D_0=(18)^{-1/2}\approx 0.24, \tag{5.25}$$

$\mu(z)$ increases monotonically, on approaching the antenna, from zero at $z=\infty$ to $\mu_{\max}\approx 2D+16/3D^3$ at $z=0$. If, however, $D>D_0$ there exists, as the antenna is approached, a point z_0 such that $\mu(z_0+0)=\mu_+=(1/2)^{1/2}$, $P(\mu_+)=(2/9)^{1/2}$. The physical branch of $\mu(z)$ merges at the point z_0 with the other positive branch. From the condition that the roots merge, we obtain the following expression for z_0:

$$z_0=a(D-D_0)[(2D-D_0)D_0]^{-1/2}. \tag{5.26}$$

In the vicinity of z_0, the positive roots of (5.24) are approximately equal to

$$\mu\approx\mu_+\pm(0{,}40/D)(2D-D_0)^{3/4}[(z-z_0)/a]^{1/2}, \tag{5.27}$$

with the minus sign corresponding to the physical branch. It can be seen from (5.27) that $d\mu/dz\to\infty$ as $z\to z_0+0$. At

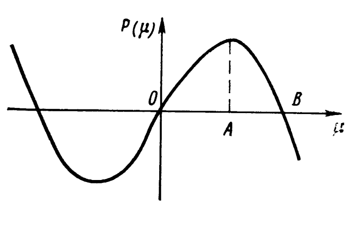

FIG. 2. Plot of the polynomial $P(\mu)=\mu-2/3\mu^3$: $OA=\mu_+\approx 2^{-1/2}\approx 0.71$, $P(\mu_+)\approx(2/9)^{1/2}\approx 0.47$, $OB=(3/2)^{1/2}\approx 1.22$.

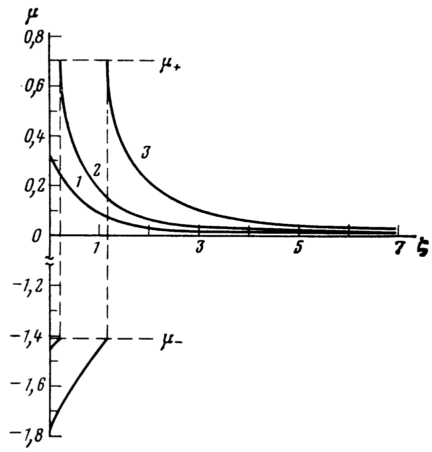

FIG. 3. Plots of the solutions $\mu(\zeta)$ of Eq. (5.24) ($\zeta=z/a$) for different values of D: 1) $D=0.15$, 2) $D=0.30$, 3) $D=1$. Only the physical branches of the curves are shown for $D>D_0\approx 0.24$.

$z<z_0$ Eq. (5.24) has only one real (negative) root, and $\mu(z_0-0)=\mu_-=-\sqrt{2}$, i.e., $\mu(z)$ has a jump discontinuity at the point z_0 (Fig. 3).

The jump is due to neglect of the effects of the finite Debye and Larmor radii. Certain estimates show that in the vicinity of the point z_0, where the field and the state of the plasma vary so abruptly, the plasma is in general unstable. A detailed investigation of the effects that are produced in the vicinity of z_0 is a problem outside the scope of the present paper.

In sum, we can state that the nonlinear effects described by Eq. (5.24) have thresholds. At $D<D_0$ the field is continuous at $0<z<\infty$, with $\mu_{\max}=\mu(0)\leqslant\mu_+\approx 0.71$ and $N(0)\approx 0.61N_0(N(0)<N(z)<N_0)$. The variations of the field and of the density differ only quantitatively from those that follow from the linear theory. At $D>D_0$ the near-zone structure is qualitatively altered. A singular point $z_0>0$ appears, for which $dE/dz=\infty$ and $E(z_0+0)\neq E(z_0-0)$ (within the framework of the hydrodynamic description of the medium). Generally speaking, kinetic effects become important in the vicinity of z_0; a large field gradient should heat the plasma in this case.

The antenna current corresponding to $D>D_0$ is determined according to (2.10) by the condition ($(\omega/\omega_{ce})^2\ll 1$)

$$I\geqslant I_0\approx 0.047(\omega_{ce}/\omega)T^{1/2}, \tag{5.28}$$

where I_0 is the critical value of the current (in amperes) and T is given in degrees. For the upper ionosphere at heights $H=(1\text{–}3)\times 10^3$ km and at $\omega=0.03\,\omega_{ce}$ we have $I_0\approx 50$ A to 80 A, close to the current used in the experiments of Ref. 4. In the magnetosphere at a height of 3–4 earth radii the experiments are usually performed at $\omega/\omega_{ce}\sim 0.3\text{–}0.5$ (the condition $(\omega/\omega_{ce})^2\ll 1$ can be regarded as satisfied to first order; in this case $T\sim 10^4$ deg. This yields the rather low threshold value $I_0=10$ A. The same order of I_0 follows from (5.28) for laboratory experiments, in which case the condition $\lambda>2\pi a$ is satisfied (λ is the wavelength). The foregoing estimates show that the nonlinear effects described above can occur (and possibly do occur) under real conditions, and must therefore be taken into account when the experimental results are interpreted.

In conclusion, I take the opportunity to express sincere gratitude to L. P. Pitaevskiĭ for helpful discussions of a number of results and to N. A. Ryabov for numerically solving Eq. (5.4).

Note added in press (30 May 1985). The analysis of the transition to $z \to 0$ in Eq. (5.8) contains an error. A recently developed consistent approach leads to the expression $C = -\alpha/\gamma^2(1+\gamma^2)$, which yields for a magnetic antenna

$$\psi = -\alpha(1+\gamma^2)^{-1}R^{-1} - [\alpha/\gamma(1+\gamma^2)](\gamma^2 z^2 - r^2)^{-1/2}.$$

Expressions (5.10) and (5.11) pertain in this case to a source consisting of a magnetic dipole and of a point charge located at $R=0$ and having an amplitude $\alpha|\eta_0|(1+\gamma^2)^{-1}$. Equations (5.12)–(5.15) determine accordingly the field of a magneto-electric sheet (MES) consisting of a circular current I enclosing a distributed surface charge with a density of amplitude $\sigma = (\alpha/\pi a^2)|\eta_0|(1+\gamma^2)^{-1}$. The analysis presented in Sec. 5.3 pertains precisely to the MES, which is a convenient model for the analytic investigation, since its field has no singularity as $\gamma^2 \to 0$. It is probable that the conclusion that the MES is threshold-dependent is valid also for antennas of the magnetic type and other types. The threshold at $\gamma^2 \ll 1$ can in this case be even lower than (5.28), since the effective amplitude of a magnetic antenna increases with decreasing γ. This question is now being investigated numerically.

1)Various aspects of nonlinear penetration of an alternating electromagnetic field into a magnetoactive plasma have been studied in many papers. As to the problems dealt with here, we point out the paper by Gurevich and Pitaevskiĭ[1] and the literature cited in it, as well as a number of papers referred to below.

2)If ω is considerably higher than the lower hybrid frequency, Eq. (2.20) goes over into Eq. (20) of Ref. 1, although the expression given in Ref. 1 for **f** differs from (2.10). This difference is significant only for the r component of Eq. (2.14).

3)It is important here that, for the solution that goes over in the linear approximation into (5.13), the terms of the expansion in powers of r are not singular as $\gamma \to 0$. This is not the case for an electric antenna, where the presence of the resonance cone $r^2 = \gamma^2 z^2$ produces in the solution a singularity as $\gamma \to 0$ even in the linear approximation, as we have seen in Sec. 3.

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