Pair correlation function of a one-dimensional Fermi gas with strong interactions

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We find the "density-density" correlation function of a one-dimensional spinless Fermi gas with strong repulsions between the fermions at zero temperature. The correlator decreases at large distances as a power law and we show that in all orders of large-interaction perturbation theory, the exponent α can be simply expressed in terms of the sound velocity c in the system: $\alpha \propto c^{-1}$.

§1. INTRODUCTION

The problem of finding various correlation functions is of great interest in the theory of one-dimensional quantum systems. In many cases the asymptotic behavior of the correlators at large distances turns out to be particularly important. Moreover, relatively few theoretical results have been obtained so far in this field.

Efetov and Larkin[1] have suggested that the behavior of the correlation functions of one-dimensional Fermi systems at large distances is determined by the long-wavelength gapless excitations. Taking only those excitations into account and neglecting all others one can find the explicit forms of the correlation functions.

Another approach to the problem of evaluating the correlators is connected with the linearization of the quadratic spectrum of the fermions and the introduction of two kinds of particles. It is possible to perform the calculations exactly in the framework of that model.[2–4]

It has been shown[1] that the correlation functions found using these two approaches agree qualitatively. Their characteristic property is a power-law decrease at large distances at zero temperature, and a continuous dependence of the exponents on the interaction constant.

However, both approaches are based on assumptions which are difficult to prove, though they are plausible. Moreover, in Ref. 5 it was noted that the asymptotic behavior of the pair correlation function can be determined by the singularities of the structure factor $S(k)$ for $k = 2k_F$, $4k_F$, and so on (k_F is the Fermi momentum) and hence cannot be connected with sound-like excitations. Nonetheless, Krivnov and Ovchinnikov[5–7] have shown that, notwithstanding this fact, the qualitative behavior of the correlators is unchanged, and have suggested that the corresponding exponents are simply connected with the sound speed in the system.

It is the aim of the present paper to find the asymptotic behavior of the "density-density" correlation function at large distances in a one-dimensional system of spinless fermions with strong binary interactions (repulsions), taking into account all orders of perturbation theory in the large interaction constant.

The Hamiltonian of this system has the form[5]

$$\hat{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g\sum_{i<j}^{N} V(x_i - x_j), \quad \hbar = 2m = 1. \tag{1}$$

Here $V(x)$ is the pair interaction potential, assumed to be long-range, and g is the coupling constant. In what follows we consider everywhere the repulsion case: $g > 0$. It is convenient to specify the boundary conditions by assuming that the particles move on a circle of great length L. One must then assume that $V(x)$ is periodic with period L. Instead of the original V we can introduce a periodic potential

$$\tilde{V}(x) = \sum_{n=-\infty}^{\infty} V(x + nL);$$

this does not affect the results in the thermodynamic limit $L \to \infty, N \to \infty, L/N = a$.

The equal-time "density-density" correlator at zero temperature is defined as follows:

$$G(R) = \rho^2 \int |\Psi(0, R, x_3, \ldots, x_N)|^2 dx_3 \ldots dx_N, \tag{2}$$

where $\rho = a^{-1}$ is the equilibrium density of the system, and Ψ the wave function of the ground state.

It turns out that as $R \to \infty$ the correlator $G(R)$ tends to ρ^2 and the deviation $h(R) = G(R) - \rho^2$ from this value at zero temperature decreases as a power law with increasing distance:

$$h(R) \propto R^{-\alpha}, \quad R \to \infty. \tag{3}$$

The exponent α is sometimes called, by analogy with the theory of phase transitions, the critical exponent. Its determination is one of the problems of the theory.

Krivnov and Ovchinnikov[5] found the exponent α in the limit of strong repulsion: $g \to \infty$. In that case the system with Hamiltonian (1) can be treated rigorously in view of the formation of a Wigner crystal with lattice constant $a = L/N$. This enables us to regard the system as a gas of interacting phonons and the interaction can be neglected in the first approximation. We shall call such an approximation a harmonic one; it is better the larger g. The value of α in the harmonic approximation was found in Ref. 5: $\alpha = 4k_F c_0^{-1}$, where $k_F = \pi a^{-1}$ is the Fermi momentum and c_0 the sound speed in the same approximation. In the same paper the hypothesis was put forward that the exact value of α is equal to $4k_F c^{-1}$ where c is the true sound speed in the system. This result is independent of the actual form of the interaction potential and is valid for sufficiently large c—such that the Wigner crystal picture is valid.

Our further aim is a proof of this assumption; for this it is necessary to go beyond the framework of the harmonic

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approximation and to take into account the contribution of the anharmonic terms. This will be done in § 2 by perturbation theory in terms of a large interaction constant. We obtain in § 3 for the system a convenient representation of the ground state wave function, which is used in $\oint$ 4 to calculate the "density-density" correlator and to find its asymptotic form. In that case we establish to all orders of perturbation theory the connection of α with the sound speed c. We give the proof of the basic relations in the Appendix.

§2. PERTURBATION THEORY AND DIAGRAM TECHNIQUE

To reformulate the problem in phonon language we must change to collective variables. It turns out that the Fourier coefficients of the density operator ρ_p, which have normally been used as collective variables since the work of Bogolyubov and Zubarev,[8] are inconvenient for our goal. Krivnov and Ovchinnikov[5] introduced "lattice" variables φ_n which take the presence of a Wigner lattice into account explicitly even in zeroth approximation:

$$\varphi_n = x_n - na, \quad n=1,\ldots,N. \tag{4}$$

In what follows we shall work both with the spatial Fourier components of (4)

$$\varphi_p(t) = N^{-1/2}\sum_{n=1}^{\infty}\varphi_n(t)\exp(-ipn), \quad \varphi_p^{+}=\varphi_{-p}$$

and with the space-time Fourier components

$$\varphi_{p,\omega} = T^{-1/2}\int_{-\infty}^{\infty} dt\,\varphi_p(t)\exp(-i\omega t).$$

Here p is a dimensionless "momentum" which takes on quasi-discrete values $2\pi ma/L$, where m is an integer, while T is a large time interval introduced in (10) below.

The transformation to the variables φ_p, is canonical whereas the transformation to the ρ_p is not. It is also important that the φ_p, in contrast to the ρ_p, are periodic in p with period 2π. As already noted, the leading term in the asymptotic behavior of $G(R)$ is determined by the singularity of the structure factor at $k=2\pi$, but ρ_p as $p\to 2\pi$ is, in no way simply connected with ρ_p as $p\to 0$. On the other hand, in terms of the variables φ_p we can restrict ourseleves to small p, because of the periodicity, and this simplifies the problem considerably.

In the new variables the Lagrangian corresponding to the Hamiltonian (1) is obtained in the usual way after expanding the potential in a Taylor series in φ_j, and is written in the following form:

$$L(\dot\varphi_p,\varphi_p) = \frac{1}{4g}\sum_{-\pi}^{\pi}\dot\varphi_p\dot\varphi_{-p} - \frac{1}{4g}\sum_{-\pi}^{\pi}\omega_0^2(p)\varphi_p\varphi_{-p}$$

$$-\sum_{n=3}^{\infty}\frac{1}{n!}(gN)^{1-n/2}\sum_{p_l}\Gamma_0^{(n)}(p_1\ldots p_n)\tilde\delta\Big(\sum_{j=1}^{n}p_j\Big)\varphi_{p_1}\ldots\varphi_{p_n}. \tag{5}$$

We have here made the stretching $\varphi_p\to g^{-1/2}\varphi_p$. There is no linear term in φ_p if the potential is periodic. $\omega_0(p)$ describes the free phonon spectrum:[5]

$$\omega_0(p) = \Big\{4g\sum_{n=1}^{\infty}V''(na)(1-\cos np)\Big\}^{1/2}. \tag{6}$$

As $g\to\infty$ the energy of the elementary excitations equals $\omega_0(p)$. These excitations have an acoustic character: as $p\to 0$, $\omega_0 = c_0|p|$ where the sound speed in the harmonic approximation is

$$c_0 = a\Big\{2g\sum_{n=1}^{\infty}V''(na)n^2\Big\}^{1/2}. \tag{7}$$

The remaining terms in the Lagrangian correspond to the terms following the quadratic in the Taylor series expansion of the potential:

$$\Gamma_0^{(n)}(p_1\ldots p_n) = \sum_{k=1}^{\infty}V^{(n)}(ka)\prod_{m=1}^{n}[\exp(ip_m k)-1]. \tag{8}$$

It is important that for small momenta we have

$$\Gamma_0^{(n)}(p_1\ldots p_n)\propto\prod_{j=1}^{n}p_j, \quad p_j\to 0. \tag{9}$$

Finally, $\tilde\delta(p)$ guarantees conservation of quasimomentum accurate to the reciprocal lattice vector, i.e., up to integer multiples of 2π.

Changing in the normal way to an imaginary time we construct for the system (5) a perturbation theory in large g. We choose some large time interval T and write down the action in it as a functional of $\varphi_{p,\omega}$. After transforming the sums over p into integrals we have in the imaginary time

$$S(\varphi_{p,\omega}) = \frac{1}{4g}NT(2\pi)^{-2}\int_{-\infty}^{\infty}d\omega\int_{-\pi}^{\pi}dp[\omega^2+\omega_0^2(p)]\varphi_{p,\omega}\varphi_{-p,-\omega}$$

$$+\sum_{n=3}^{\infty}\frac{1}{n!}g^{1-n/2}(NT)^{n/2}(2\pi)^{2-2n}\int_{-\infty}^{\infty}d\omega_1\ldots d\omega_n\int_{-\pi}^{\pi}dp_1\ldots dp_n$$

$$\times\tilde\delta\Big(\sum_{j=1}^{n}p_j\Big)\delta\Big(\sum_{j=1}^{n}\omega_j\Big)\Gamma_0^{(n)}(p_1\ldots p_n)\varphi_{p_1,\omega_1}\ldots\varphi_{p_n,\omega_n}. \tag{10}$$

To simplify the notation we agree, where this does not lead to confusion, to omit all the integration signs and the δ-functions, as well as not to indicate explicitly the arguments of $\Gamma_0^{(n)}$ and φ. The action can then be written as

$$S(\varphi) = \frac{1}{4g}[\omega^2+\omega_0^2(p)]\varphi^2 + \sum_{n=3}^{\infty}\frac{1}{n!}(gNT)^{1-n/2}\Gamma_0^{(n)}\varphi^n. \tag{11}$$

The integration

$$(2\pi)^{-2}NT\int_{-\pi}^{\pi}dp\int_{-\infty}^{\infty}d\omega \tag{12}$$

is implied.

We get directly from (11) the elements of the diagram technique: the free phonon propagator is

$$G_0^{-1}(p,\omega) = (2g)^{-1}[\omega^2+\omega_0^2(p)] \tag{13}$$

and the n-point vertices are $(gNT)^{1-n/2}\Gamma_0^{(n)}$. In the vertices the momentum conservation law is satisfied modulo 2π and energy is conserved. The integration (12) is performed over the internal independent momenta. Each diagram is multiplied by its own symmetry coefficient which is found by standard rules.[9]

The ground state energy can be written in the form

$$E_0=-\lim_{T\to\infty}\frac{1}{T}\ln\int D\varphi\exp(-S(\varphi)). \tag{14}$$

Here $D\varphi=\Pi_{p,\omega}d\varphi_{p,\omega}$ is the functional-integration symbol. The zero-point oscillation energies in the harmonic approximation as well as the anharmonicity corrections to it are included in E_0. These corrections are a sum of connected vacuum diagrams.

§ 3. THE WAVE FUNCTION

We now obtain a representation of the ground-state wave function in the form of a path integral which is needed in what follows.

It is well known[10] that functional integration with a weight $\exp(-S(\varphi))$ and a fixed value $\varphi_p(t=0)=\varphi_p$ gives as $T\to\infty$ the square of the modulus of the ground-state wave function:

$$|\Psi(\varphi)|^2=\exp(E_0T)\int D\varphi\prod_p\delta(\varphi_p(0)-\varphi_p)\exp(-S(\varphi)).$$

We shall assume Ψ to be real. Introducing an auxiliary integration $DJ=\Pi_p dJ_p$ and changing to Fourier components we have

$$\Psi^2(\varphi)=\exp(E_0T)\int DJ\exp\Big(-i\sum_p J_p\varphi_p\Big)$$

$$\times\int D\varphi\exp\Big\{-S(\varphi)+iT^{-1/2}\sum_{p,\omega}J_p\varphi_{p,\omega}\Big\}. \tag{15}$$

The internal integral over φ has the meaning of the partition function $Z(J)$ of the original system in the presence of a "source" J_p and its coefficient functions are the Green functions of the system[11]

$$G^{(n)}(p_1,\omega_1;\ldots;p_n,\omega_n),\quad n=2,3,\ldots,$$

integrated over the external frequencies with account taken of the conservation laws:

$$Z(J)=\exp\Big\{-\frac{1}{2}\sum_p J_pD^{(2)}(p)J_{-p}$$

$$+\sum_{n=3}^{\infty}\sum_{p_i}D^{(n)}(p_1\ldots p_n)J_{p_1}\ldots J_{p_n}\Big\}, \tag{16}$$

$$D^{(n)}(p_1\,..\,p_n)$$

$$=(2\pi)^{-n+1}\int_{-\infty}^{\infty}\prod_{i=1}^{n}d\omega_i\delta\Big(\sum_{j=1}^{n}\omega_j\Big)G^{(n)}(p_1,\omega_1;\ldots;p_n,\omega_n). \tag{17}$$

In what follows we shall write the sum in (16) in abbreviated form, e.g., as $JD^{(2)}J$, bearing in mind, however, that we are summing here only over p and not over ω.

Substituting $Z(J)$ in the form (16) into (15) we get a representation of the ground state wave function as a path integral:

$$\Psi^2(\varphi)=\int DJ\exp\Big\{-\frac{1}{2}JD^{(2)}J+\sum_{n=3}^{\infty}D^{(n)}J^n-iJ\varphi\Big\}. \tag{18}$$

§ 4. ASYMPTOTIC FORM OF THE "DENSITY-DENSITY" CORRELATOR; CONNECTION WITH SOUND SPEED

We proceed now to find directly the correlator (2). After some simple transformations we can write (2) in the form

$$G(R)\propto\sum_{k=-\infty}^{\infty}\int D'\varphi\Psi^2(\varphi)$$

$$\times\int_{-\infty}^{\infty}d\mu\exp\Big\{i\sum_{p\neq0}E_p(k)\varphi_p\mu+i(ka-R)\mu\Big\}, \tag{19}$$

where $D'\varphi$ indicates integration over all φ_p but φ_0 and

$$E_p(k)=(gN)^{-1/2}(1-e^{ipk}).$$

We substitute (18) into (19) and integrate first over φ and then over J. As a result we get

$$G(R)\propto\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}d\mu\exp\Big\{-\frac{\mu^2}{2}E(k)D^{(2)}E(k)$$

$$+\sum_{n=3}^{\infty}\mu^nD^{(n)}E^n+i\mu(ka-R)\Big\}. \tag{20}$$

One can show that the asymptotic form of this expression in terms of R is determined by the behavior of $D^{(2)}(p)$ at small momenta. In fact, in the case where $D^{(2)}(p)$ has at the origin a first-order pole, $h(R)$ decreases as a power law and the exponent is proportional to the residue at that pole, while the coefficient of proportionality equals $4\pi(ga^2)^{-1}$.

This result is obtained after transforming (20) by using the Poisson summation formula and using the properties of the integrals $D^{(n)}E^n(k)$ for large k.

We show in Appendix I that $D^{(2)}(p)$ has indeed a simple pole at the origin and

$$D^{(2)}(p)\to{}^1/_2gG^{1/2}(p,0),\quad p\to0. \tag{21}$$

Using the relation $G^{-1}=G_0^{-1}+\Pi$ (Π is the polarization operator) and the result from Appendix II

$$\frac{\partial\Pi(p,0)}{\partial p^2}\Big|=(2ga^2)^{-1}(c^2-c_0^2), \tag{22}$$

as well as the explicit form (13) of G_0, we find that the residue of $D^{(2)}(p)$ at the origin equals agc^{-1}, and hence that the critical exponent is

$$\alpha=4\pi\rho c^{-1}, \tag{23}$$

which we wanted to prove.

§ 5. CONCLUSION

We have thus established that the asymptotic form of the binary "density-density" correlation function at large distances, at a rather arbitrary form of the potential $V(x)$, is simply related to the sound speed in the system considered. Knowledge of this relation simplifies greatly the finding of the asymptotic behavior of the correlators, since the problem of evaluating c is appreciably simpler and there exist for its solution methods that give a good approximation.

We note that the connection between the critical exponents of various correlation functions and the sound speed was well known in the literature for some systems of a special form. For instance, in Ref. 11 the long-wavelength asymptotic form was found for the many-particle Green functions in an exactly soluble model of a one-dimensional Bose gas with point interactions. It was shown that it has a power law character and the exponent was expressed in terms of the sound speed. Very recently papers were published by Izergin and Korepin[12,13] where, in the framework of the quantal inverse scattering method, a general method was developed for finding various correlators in this model. In particular, a representation was obtained in Ref. 13 for the "density-density" correlator in the form of a series in inverse powers of the large interaction constant. However, one should note that this model cannot be treated by our method, since the potential is δ-shaped and does not have a long range. This manifests itself in the fact that in contrast to our case the sound speed remains finite at $g=\infty$. Nonetheless, the results of Ref. 13 as $g\to\infty$ agree with (23).

All this enables us to suggest that the postulated connection between the critical exponents of the correlators and the sound speed is quite general. There is undoubted interest in finding similar relations for other correlation functions and also in establishing analogous regular relations in other models, in particular, in the lattice model of a Fermi gas.

In conclusion we thank V. Ya. Krivnov for useful discussions.

APPENDIX I

In this Appendix we show the validity of Eq. (21) to all orders of perturbation theory. We recall the definition of $D^{(2)}(p)$:

$$2\pi D^{(2)}(p)=\int_{-\infty}^{\infty} G(p,\omega)\,d\omega.$$

Introducing the polarization operator $\Pi(p,\omega)$, we write

$$D^{(2)}(p)=\sum_{n=1}^{\infty}(2\pi)^{-1}\int_{-\infty}^{\infty} d\omega G_0{}^n(p,\omega)\,\Pi^{n-1}(p,\omega); \qquad (\mathrm{A}.1)$$

Π does not contain poles in ω, while G_0^n has an nth order pole at the points $\pm i_{\omega}{}_0(p)$ (see (13)). We show in Appendix II that, thanks to the property (9) of the bare vertices, $\Pi(p),\omega)\propto p^2$ as $p\to 0$ independently of ω. The derivatives of Π with respect to ω at small p therefore also possess the same property. Hence it follows that if we are interested in (A.1) only in the leading terms we can retain in the integral, after evaluating the residue, only the terms which do not contain derivatives of Π with respect to ω. Moreover, we can replace $i_{\omega}{}_0(p)$ in the argument of Π by zero. After that (A.1) takes the form

$$D^{(2)}(p)\to g\omega_0{}^{-1}(p)\sum_{n=0}^{\infty}2^{-2n}(2n)!\,(n!)^{-2}[2g\omega_0{}^{-2}\Pi(p,0)]^n,$$

$$p\to 0. \qquad (\mathrm{A}.2)$$

Writing the analogous series for $G(p,0)$:

$$G(p,0)=2g\omega_0{}^{-2}(p)\sum_{n=0}^{\infty}[2g\omega_0{}^{-2}(p)\,\Pi(p,0)]^n \qquad (\mathrm{A}.3)$$

and comparing (A.2) and (A.3) we can verify the validity of (21).

APPENDIX II

Here we prove Eq. (22) in which c is the exact sound speed, c_0 the sound speed in the harmonic approximation, and $\Pi(p,0)$ the polarization operator at zero frequency. It is well known (see, e.g., Ref. 14) that

$$c^2=2\rho\partial^2\varepsilon_0/\partial\rho^2,$$

where ε_0 is the ground-state energy density. It includes a classical part governed by the spacing a between the particles and also the zero-point energy and the anharmonic corrections to it:c_0^2 is obtained by differentiating the classical part. Using (14) we write (22) in the form

$$-\frac{\partial\Pi(p,0)}{\partial p^2}\bigg|_{p=0}=(gNT)^{-1}\left(\frac{\partial^2\ln Z}{\partial a^2}\right)_{N=\mathrm{const}}, \qquad (\mathrm{A}.4)$$

where Z is the partition function.

Our plan for the proof is the following. We first of all elucidate the behavior of the vertices of the effective action for small momenta and obtain for them Ward-type identities. using these identities we can transform (A.4) into a Schwinger-Dyson equation for the polarization operator and thereby complete the proof.

The vertices of the effective action $\Gamma^{(n)}(p_1,\omega_1;\ldots p_n,\omega_n)$ are written in the form of a sum of single-particle irreducible diagrams. One verifies easily that the property (9) which is valid for "bare" vertices $\Gamma_0^{(n)}$ remains valid also for the "dressed" ones (the vertices at zero frequencies are those without the frequency label)

$$\Gamma^{(n)}(p_1,\ldots,p_n)\propto\prod_{j=1}^{n}p_j,\quad p_j\to 0. \qquad (\mathrm{A}.5)$$

We introduce the following notation for the derivatives of the $\Gamma_0^{(n)}$ and of the $\Gamma^{(n)}$:

$${}^{(1)}\Gamma_0^{(n)}(p_1,\ldots,p_{n-1})=\frac{\partial}{\partial p}\bigg|_{p=0}\Gamma_0^{(n)}(p,p_1-p,p_2,\ldots,p_{n-1}), \qquad (\mathrm{A}.6)$$

${}^{(1)}\Gamma^{(n)}$ is defined in exactly the same way. We designate similarly the second derivatives: ${}^{(2)}\Gamma_0^{(n)}(p_1,\ldots p_{n-2})$ (the derivatives are with respect to different momenta). The arguments on the right-hand side of (A.6) take into account the momentum conservation. The sequence of the arguments is unimportant, since $\Gamma^{(n)}$ is a symmetric function.

From (8) and (A.6) we get the derivatives of the bare vertices with respect to a:

$$\frac{\partial G_0}{\partial a}=iG_0\,{}^{(1)}\Gamma_0^{(3)}G_0,\quad \frac{\partial\Gamma_0}{\partial a}=-i\,{}^{(1)}\Gamma_0^{(n+1)}. \qquad (\mathrm{A}.7)$$

These equations give a recipe for differentiating directly in diagram language. Using (A.7) one shows easily that the following Ward identities are valid for the single-particle irreducible vertexes:

$$\partial G^{-1}/\partial a=-i^{(1)}\Gamma^{(3)},\quad \partial\Gamma^{(n)}/\partial a=-i^{(1)}\Gamma^{(n+1)}. \qquad (\mathrm{A}.8)$$

Indeed, attaching in all possible ways an additional line with zero momentum to the diagrams $\Gamma^{(n)}$ we obtain all the diagrams ${}^{(1)}\Gamma^{(n+1)}$, and furthermore with the correct symmetry coefficients.

We now find the right-hand side of (A.4):

$$\frac{\partial^2 \ln Z}{\partial a^2}=\frac{i}{2}\frac{\partial G}{\partial a}{}^{(1)}\Gamma_0^{(3)}+\frac{1}{2}G\,{}^{(2)}\Gamma_0^{(4)}$$
$$+\sum_{n=3}^{\infty}\frac{i}{n!}(gNT)^{1-n/2}\,{}^{(1)}\Gamma_0^{(n+1)}\frac{\partial}{\partial a}\langle\varphi^n\rangle$$
$$+\sum_{n=3}^{\infty}\frac{1}{n!}(gNT)^{1-n/2}\langle\varphi^n\rangle\,{}^{(2)}\Gamma_0^{(n+2)}. \qquad \text{(A.9)}$$

Here the unconnected correlators are indicated by angle brackets. We wish to compare this expression with the Schwinger-Dyson equation for the polarization operator which is obtained after adding to (10) a term with a "source" J and differentiating the Schwinger equation $\langle\delta S/\delta\varphi\rangle=J$ with respect to J at $J=0$. (Here $\delta/\delta\varphi$ is a functional derivative.) As a result we have

$$\Pi(p)=\sum_{n=2}^{\infty}\frac{1}{n!}(gNT)^{(1-n)/2}\Gamma_0^{(n)}(p)G^{-1}(p)\frac{\delta}{\delta J}\langle\varphi^n\rangle|_{J=0}. \qquad \text{(A.10)}$$

For the sake of convenience we have indicated here only the arguments over which we do not integrate. We can express the right-hand side of (A.10) in terms of irreducible vertices and obtain a diagram series for Π.

The mean value $\langle\varphi^n\rangle$ in (A.9) is an uncoupled correlator which is equal to the sum of products of all possible connected ones. We denote by $\langle\varphi^n\rangle_1$ that part of the sum in which $\langle\varphi\rangle$ does not occur at all and by $\langle\varphi^n\rangle_2$ the part of the sum in which $\langle\varphi\rangle$ occurs but not to a power higher than unity. One sees easily that

$$\langle\varphi^n\rangle_2=n\langle\varphi\rangle\langle\varphi^{n-1}\rangle_1+\langle\varphi^n\rangle_1.$$

It is also clear that if $J=0$ we have $\langle\varphi\rangle=0$. We can thus replace $\langle\varphi^n\rangle$ by $\langle\varphi^n\rangle_1$ in (A.9) and $\langle\varphi^n\rangle$ by $\langle\varphi^n\rangle_2$. We can then write (A.10) in the form

$$gNT\Pi(p)=\frac{1}{2}(gNT)^{1/2}\Gamma_0^{(3)}(p)G^{-1}(p)\frac{\delta G}{\delta J}$$
$$+\sum_{n=3}^{\infty}\frac{1}{n!}(gNT)^{1-n/2}(gNT)^{1/2}\Gamma_0^{(n+1)}(p)G^{-1}(p)\frac{\delta}{\delta J}\langle\varphi^n\rangle_1$$
$$+\frac{1}{2}\Gamma_0^{(4)}(p,p)G+\sum_{n=3}^{\infty}\frac{1}{n!}(gNT)^{1-n/2}\Gamma_0^{(n+2)}(p,p)\langle\varphi^n\rangle_1.$$

It now remains to note that after differentiating Π with respect to the momentum the vertices $\Gamma_0^{(n)}(p)$ turn into ${}^{(1)}\Gamma_0^{(n)}$ and the operation $(gNT)^{1/2}G^{-1}\delta/\delta J$ on $\langle\varphi^n\rangle_1$ turns out to be exactly $-i\partial/\partial a$, thanks to the Ward identity (A.8). Indeed, $\langle\varphi^n\rangle_1$ is in the form of a sum of contractions of single-particle-irreducible vertices and exact propagators; $\delta/\delta J$ joins in turn to each single-particle irreducible vertex or propagator a new end, while G^{-1} amputates it. When the momentum tends to zero and at zero frequency this is, according to (A.8), equivalent to differentiation with respect to a. The left-hand and right-hand sides of (A.4) are thus the same.

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