Quantum chaos in stationary coherent states

K. N. Alekseev and G. P. Berman

Institute of Physics, Siberian Branch, Academy of Sciences of the USSR
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Analysis of the dynamics of a nonlinear quantum system in a stationary coherent state yields the conditions for a transition from regular motion to chaotic motion.

1. INTRODUCTION

Under certain conditions, the motion of a classical dynamic system will become stochastic.[1,2] A quantum-mechanical analysis of such systems runs into serious difficulties, both analytic and numerical, and only the simplest of systems have been studied so far. The dynamic behavior of quantum systems which are stochastic in the classical limit is found to be more stable than a classical stochastic motion, at least for systems with one and a half degrees of freedom, (one degree of freedom interacting with an external field which has a regular time dependence[3–6]).

In this paper we show that even in the simplest case of one and a half degrees of freedom it is possible to find a class of quantum systems for which the motion goes stochastic. These quantum systems are related to so-called stationary coherent states.[7–11] Although such systems are not typical from the physical standpoint, their analysis may be of interest in connection with the problem of the transition to chaos in quantum systems.

A coherent state is defined as an eigenstate of an annihilation operator,

$$a_s|\alpha\rangle=\alpha|\alpha\rangle, \tag{1.1}$$

where $a_s = a(t=0)$ (the subscript s specifies the Schrödinger picture), and $\alpha=\alpha(t=0)$. In the Heisenberg picture (denoted by the subscript g), the operator $a_g(t)$ depends on the time, and the coherent state $|\alpha\rangle$ in (1.1) is "stationary" if it is an eigenstate of the operator $a_g(t)$ for all t:

$$a_g(t)|\alpha\rangle=\alpha(t)|\alpha\rangle. \tag{1.2}$$

Equation (1.2) imposes some severe restrictions on the possible Hamiltonians. It turns out, however, that these conditions are met by some systems which have the property of exhibiting a stochastic behavior in a certain region of parameters. The stochastic behavior here is analogous to that in quantum systems.

2. BASIC EQUATIONS

Following Refs. 7–11, we list some properties of stationary coherent states which we will make use of below. In the Heisenberg picture we have

$$i\hbar\dot{a}_g(t)=[a_g(t),\ H_g(t,\ a_g(t),\ a_g^+(t))]=\partial H_g/\partial a_g^+(t). \tag{2.1}$$

We consider the class of Hamiltonians which have stationary coherent states, i.e., those for which condition (1.2) holds. According to Ref. 11, a necessary and sufficient condition for a coherent state to be stationary is

$$[a_g(t),\ \dot{a}_g(t)]|\alpha\rangle=0, \tag{2.2}$$

where the stationary coherent state $|\alpha\rangle$ does not depend on t in the Heisenberg picture and is determined by Eqs. (1.1) and (1.2). The state $|\alpha\rangle$ can of course be written as an expansion in the energy states $|n\rangle$:

$$|\alpha\rangle=\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle=\hat{D}(\alpha,\alpha^*)|0\rangle, \tag{2.3}$$

where

$$\hat{D}(\alpha,\ \alpha^*)\equiv\exp(\alpha a_s^+-\alpha^* a_s).$$

Condition (2.2) can be written

$$[a_g(t),[a_g(t),H_g(t)]]|\alpha\rangle=\left\{\frac{\partial^2}{\partial a_g^{+2}(t)}H_g(t,a_g(t),a_g^+(t))\right\}|\alpha\rangle=0. \tag{2.4}$$

We write the Hamiltonian in the form

$$H_g(t,a_g(t),a_g^+(t))=H_g^{(0)}(t,a_g(t))+a_g^+(t)H_g^{(1)}(t,a_g(t))+a_g^{+2}(t)H_g^{(2)}(t,a_g(t),a_g^+(t)), \tag{2.5}$$

where

$$H_g^{(2)}=\sum_{m=0}^{\infty}a_g^{+m}(t)h_m(t,a_g(t)). \tag{2.6}$$

In general, H can be chosen to be non-Hermitian. The necessary and sufficient condition for the existence of stationary coherent states can also be expressed as the requirement that there exist a function $\alpha(t)$ which satisfies the equations[11]

$$h_m(t,\ \alpha(t))=0;\quad m=0,\ 1,\dots \tag{2.7}$$

$$i\hbar\dot{\alpha}(t)=H_g^{(1)}(a_g(t)=d(t),t). \tag{2.8}$$

That conditions (2.7) and (2.8) are necessary conditions follows from the equation

$$\langle\alpha|i\hbar\dot{a}_g(t)-[a_g(t),\ H_g(t)]|\alpha\rangle=0 \tag{2.9}$$

under conditions (1.2), (2.4), and (2.6). The proof of sufficiency uses the state

$$|\psi,\ t\rangle=F(t)U^+(t)|\alpha(t)\rangle, \tag{2.10}$$

where $|\alpha(t)\rangle$ is the eigenfunction of the Schrödinger operator a_s in (1.1) with the eigenvalue $\alpha(t)$,

$$a_s|\alpha(t)\rangle=\alpha(t)|\alpha(t)\rangle, \tag{2.11}$$

and the function $\alpha(t)$ satisfies Eqs. (2.7) and (2.8). We write

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the state $|\alpha(t)\rangle$ in the form

$$|\alpha(t)\rangle=\bar{D}(\alpha(t),\ \alpha^*(t))|0\rangle=\exp\{\alpha(t)a_s^+-\alpha^*(t)a_s\}|0\rangle. \tag{2.12}$$

In (2.10), $U(t)$ is an evolution operator which satisfies the equation

$$i\hbar\dot{U}(t)=U(t)H_g(t),\quad H_g(t)=U^+(t)H_sU(t). \tag{2.13}$$

Function (2.10) is an eigenfunction of the operator $a_g(t) = U^+(t)a_s U(t)$, with the eigenvalue $\alpha(t)$. From (2.10) and (2.11) we have

$$\begin{aligned} a_g(t)|\psi,\ t\rangle&=U^+(t)a_sU(t)|\psi,\ t\rangle\\ &=U^+(t)a_sU(t)F(t)U^+(t)|\alpha(t)\rangle\\ &=F(t)U^+(t)a_s|\alpha(t)\rangle=\alpha(t)|\psi,\ t\rangle. \end{aligned} \tag{2.14}$$

The proof that conditions (2.7) and (2.8) are sufficient to satisfy Eq. (1.2) reduces to the following according to Ref. 11: We require that conditions (2.7) and (2.8) hold, and we write $F(t)$ in (2.10) as

$$F(t)=\exp\varphi(t),$$

$$\varphi(t)=-\frac{i}{\hbar}\int_0^t H_g^{(0)}(t',\alpha(t'))dt'+\int_0^t\dot{\alpha}^*(t')\alpha(t')dt'. \tag{2.15}$$

In this case the state $|\psi,t\rangle$ is stationary, $(\partial/\partial t)|\psi,t\rangle=0$, so that

$$|\psi,\ t\rangle=|\psi,\ 0\rangle=|\alpha(0)\rangle=|\alpha\rangle. \tag{2.16}$$

Using (2.16), we find from (2.14)

$$a_g(t)|\psi,\ 0\rangle=\alpha(t)|\psi,\ 0\rangle. \tag{2.17}$$

The state $|\psi,0\rangle=|\alpha\rangle$ is thus a stationary coherent state. The evolution of the wave function of a stationary coherent state is described in the Schrödinger picture by

$$\begin{aligned} |\alpha,\ t\rangle&=U(t)|\alpha\rangle=U(t)|\psi,\ t\rangle\\ &=U(t)F(t)U^+(t)|\alpha(t)\rangle\\ &=F(t)|\alpha(t)\rangle=F(t)\bar{D}(\alpha(t),\ \alpha^*(t))|0\rangle, \end{aligned} \tag{2.18}$$

where $\alpha(t)$ satisfies Eqs. (2.7) and (2.8), and the function $F(t)$ is of the form in (2.15) or, when (2.8) is used,

$$F(t)=\exp\left\{\frac{i}{\hbar}\int_0^t[H_g^{(0)}(t',\alpha(t'))+\alpha(t')H_g^{(1)*}(t',\alpha(t'))]dt'\right\}. \tag{2.19}$$

3. DYNAMIC CHAOS IN STATIONARY COHERENT STATES

We now wish to show how the evolution of a quantum system in a stationary coherent state can lead to chaos. As an example we consider the Hermitian Hamiltonian

$$H=\hbar\omega a^+a+\hbar^2\mu[a^{+2}\alpha(t)(a-\alpha(t))+(a^+-\alpha^*(t))\alpha^*(t)a^2]+\lambda\hbar^{1/2}(a^++a)f(t). \tag{3.1}$$

In (3.1), $f(t)$ is a given function of time, $\alpha(t)$ is a function to be determined, and ω,μ, and λ are parameters. We write (3.1) in the form in (2.5), (2.6), where

$$\begin{aligned} H_g^{(0)}(t)&=\hbar^{1/2}\lambda a_g(t)f(t)-\hbar^2\mu\alpha^{*2}(t)a_g^2(t),\\ H_g^{(1)}(t)&=\hbar\omega a_g(t)+\hbar^2\mu\alpha^*(t)a_g^2(t)+\hbar^{1/2}\lambda f(t),\\ H_g^{(2)}(t)&=\hbar^2\mu\alpha(t)(a_g(t)-\alpha(t)). \end{aligned} \tag{3.2}$$

Equation (2.7) for $h_0(t,\alpha(t))$ holds identically for Hamiltonian (3.1), (3.2), and Eq. (2.8) becomes

$$i\dot{\alpha}(t)=\omega\alpha(t)+\mu\hbar|\alpha(t)|^2\alpha(t)+\hbar^{-1/2}\lambda f(t). \tag{3.3}$$

To simplify the analysis, we choose $f(t)$ as a sequence of δ-function pulses which is periodic in time:

$$f(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT). \tag{3.4}$$

In this case, Eq. (3.3) becomes

$$i\frac{dz(\tau)}{d\tau}=z(\tau)+|z(\tau)|^2z(\tau)+\varepsilon\sum_{n=-\infty}^{\infty}\delta(\tau-n\tilde{T}), \tag{3.5}$$

where

$$z(\tau)=(\mu\hbar/\omega)^{1/2}\alpha(t),\quad \varepsilon=\lambda(\mu/\omega)^{1/2},\quad \tau=\omega t,\quad \tilde{T}=\omega T. \tag{3.6}$$

Equation (3.5) can be written in Hamiltonian form:

$$\frac{dz}{d\tau}=-i\frac{\partial}{\partial z^*}H(t,z,z^*),\quad \frac{dz^*}{d\tau}=i\frac{\partial}{\partial z}H(t,z,z^*) \tag{3.7}$$

$$H(t,z,z^*)=zz^*+\frac{1}{2}z^2z^{*2}+\varepsilon(z+z^*)\sum_{n=-\infty}^{\infty}\delta(\tau-n\tilde{T}). \tag{3.8}$$

The criterion for stochastic behavior for system (3.5) was calculated in Ref. 6 in connection with a study of the classical limit ($\hbar=0$) for the dynamics of a nonlinear quantum oscillator excited by a periodic sequence of δ-function pulses. We introduce

$$z_n\equiv z(t_n-0),\quad \bar{z}\equiv z(t_n+0),\quad t_n=n\tilde{T}. \tag{3.9}$$

From (3.5) we have

$$z_n=\exp\{-i(1+|\bar{z}_{n-1}|^2)\tilde{T}\}\bar{z}_{n-1},\quad \bar{z}_{n-1}=z_{n-1}-i\varepsilon. \tag{3.10}$$

From (3.10) we find the recurrence relation

$$z_n=\exp\{-i(1+|z_{n-1}-i\varepsilon|^2)\tilde{T}\}(z_{n-1}-i\varepsilon). \tag{3.11}$$

We replace the canonical variables z_n, z_n^* by the canonically conjugate action I_n and the canonically conjugate phase θ_n in accordance with

$$z_n=I_n^{1/2}\exp(-i\theta_n). \tag{3.12}$$

In terms of the variables I_n,θ_n, transformations (3.11) become

$$\begin{aligned} I_n&=I_{n-1}+2\varepsilon I_{n-1}^{1/2}\sin\theta_{n-1}+\varepsilon^2,\\ \theta_n&=\operatorname{arctg}(\operatorname{tg}\theta_{n-1}+\varepsilon/I_{n-1}^{1/2}\cos\theta_{n-1})+(1+I_n)\tilde{T}. \end{aligned} \tag{3.13}$$

The condition for stochastic behavior for transformations (3.13) can be written in the form[1,2,6]

$$\frac{\partial\theta_{n+1}}{\partial\theta_n}=\left(1+\frac{\varepsilon}{I_n^{1/2}}\sin\theta_n\right)\left(1+\frac{\varepsilon^2}{I_n}+\frac{2\varepsilon}{I_n^{1/2}}\sin\theta_n\right)^{-1}+2\varepsilon\tilde{T}I_n^{1/2}\cos\theta_n>1. \tag{3.14}$$

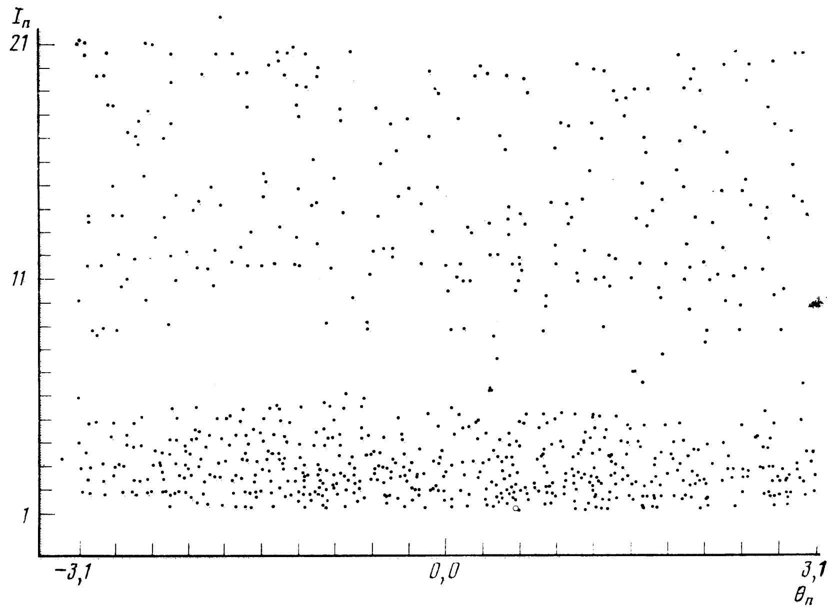

FIG. 1. Stochastic path in the θ_n, I_n phase plane ($\theta_0 = 0, I_0 = 1, \varepsilon = 0.1, \overline{T} = 10$).

For a relatively small perturbation ($\varepsilon / I_n^{1/2} \ll 1$) the condition for stochastic behavior for transformations (3.11) and (3.12) takes the following form, according to (3.14):

$$K_n = 2\varepsilon \tilde{T} I_n^{1/2} > 1. \quad (3.15)$$

Accordingly, if we require that the function $\alpha(t) = (\omega/\mu\hbar)^{1/2} z(\tau)$ in Hamiltonian (3.1) satisfy Eq. (3.5), the coherent state $|\alpha\rangle$ for a system with Hamiltonian (3.1) will be stationary. The expectation value of the operator $a_g(\tau)$ in this state is described as a function of $z(\tau)$ by

$$\langle \alpha | a_g(\tau) | \alpha \rangle = (\omega/\mu\hbar)^{1/2} z(\tau). \quad (3.16)$$

The behavior of the function $z(\tau)$ depends strongly on the "stochastic parameter" K_n in (3.15). If $K_n < 1$, the function $z(\tau)$ varies regularly over time. If $K_n > 1$, the behavior of the function $z(\tau)$ is stochastic. The evolution of a wave function in the Schrödinger representation is described by the following expression for the Hamiltonian given by (2.18), (2.19), (3.2):

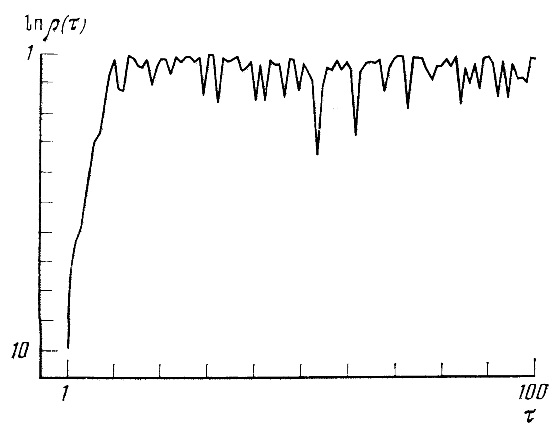

FIG. 2. Local instability of the path shown in Fig. 1. $\rho(0) = 5 \cdot 10^{-5} (\tau = n)$.

$$|z, \tau\rangle = U(\tau) |\psi, 0\rangle = F(\tau) |z(\tau)\rangle$$
$$= \exp\left\{ \frac{i\omega}{\mu\hbar} \int_0^\tau (1 + 2|z(\tau')|^2) |z(\tau')|^2 d\tau' \right\} |z(\tau)\rangle,$$
$$|z(\tau)\rangle = \hat{D}(z(\tau), z^*(\tau)) |0\rangle. \quad (3.17)$$

Numerical analysis of transformations (3.11) has been carried out in an effort to determine the conditions for a transition from regular motion to chaotic motion. Figure 1 shows a typical stochastic path in the (θ_n, I_n) phase plane. We see from Fig. 1 that the chaos boundary agrees well with the estimate $I_b \approx (2\varepsilon\tilde{T})^{-2} = 1/4$ which follows from the condition $K_b \approx 1$. There are some differences from the value of I_b, but they can be attributed to the first term in (3.14) ($\varepsilon / I_b^{1/2} = 0.2$) and the finite path length. Figure 2 shows the time (τ) dependence of the logarithm of the distance $\rho(\tau) = |z(\tau) - z'(\tau)|$ between the $z(\tau)$ path shown in Fig. 1 and the path $z'(\tau)$, which is close to the former path at $\tau = 0$. We see from Fig. 2 that the path $z(\tau)$ is locally unstable; this is a characteristic property of a stocastic regime of motion.

4. CONCLUSION

We have been discussing the special (limiting) case of quantum dynamics in which the state of a nonlinear quantum system which is coherent at $t = 0$ remains coherent, uniformly over time, even for random motion. In this sense we can speak in terms of a maximum possible correspondence between the stationary coherent state and the classical limit. This example of a stationary coherent state shows that, even in the simplest case of one and a half degrees of freedom, quantum systems can exhibit stochastic dynamics. The class of Hamiltonians which allow stationary coherent states, however, is a rather special one. The primary reason for this circumstance is the incorporation of certain functions of the type $\alpha(t)$ in the original Hamiltonian [see (3.1)]. These functions are determined from the corresponding dynamic equations. The regular behavior of the functions $\alpha(t)$ thus gives rise to a corresponding regular time dependence in the original Hamiltonian. Under certain conditions on the parameters of the original Hamiltonian ($K > 1$), the requirements for the existence of stationary coherent states lead to stochastic behavior of the functional $\alpha(t)$, in terms of which various observable quantities are calculated. This means that a random process will appear in the original Hamiltonian. We can also expect stationary coherent states to prove unstable with respect to an additional perturbation of the Hamiltonians [of the type in (3.1)] which allow these states.

It nevertheless seems useful to study this class of Hamiltonians in connection with the problem of stochastic behavior in quantum dynamic systems and to extend this study to the extent possible to systems of physical interest. We also note that there would be no difficulty in extending these results to systems with more degrees of freedom.

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