# Shallow nuclear levels and radiative transitions in hadronic atoms

V. S. Popov, A. E. Kudryavtsev, V. I. Lisin, and V. D. Mur

*Institute of Theoretical and Experimental Physics*
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The position of a nuclear level perturbing the Coulomb spectrum can be calculated in the analytic theory of nuclear level shifts[1-3] from the level shift of a hadronic atom. This is illustrated by the example of the $K^{-4}$He atom. Experimental $2p$-level shifts[4,5] suggest that this system may have a weakly-bound $p$-state with binding energy and width $\varepsilon \sim \gamma \sim 0.5$ MeV. The probabilities of radiative transitions to this level and the cross section for its creation in nuclear reactions on $^6$Li are calculated. The possible existence of states in which $K^-$ and $\bar{p}$ are weakly bound to other light nuclei is examined. An exact solution is obtained for a model Coulomb problem with short-range interaction, and is used to determine the range of validity of the initial approximations.

**1.** There has been increasing experimental and theoretical interest in recent years in systems bound by a Coulomb potential that is distorted at short[1)] distances. Analyses of the spectra of $\bar{p}p$ and $\Sigma^- p$ hadronic atoms have been based on the equation[1-3]

$$\left\{\lambda+2\zeta\left[\psi\left(1-\frac{\zeta}{\lambda}\right)+\ln\frac{\lambda}{|\zeta|}\right]\right\}$$

$$\times\prod_{j=1}^{l}\left(\frac{\zeta^2}{j^2}-\lambda^2\right)=\frac{1}{a_l^{(cs)}}+\frac{1}{2}r_l^{(cs)}\lambda^2, \tag{1}$$

which relates atomic $l$-level shifts and widths with the low energy-scattering parameters. In this equation, $\lambda=(-2E/E_c)^{1/2}$, $E=E_0-i\Gamma/2$ is the level energy, $l$ is the angular momentum, $\zeta=-Z_1Z_2$, $\psi(z)=\Gamma'(z)/\Gamma(z)$, $a_l^{(cs)}$ and $r_l^{(cs)}$ are the Coulomb-nuclear scattering length and effective radius. Equation (1) is independent of the specific model of the strong potential $V_s(r)$ and is valid for

$$\lambda r_0 \ll 1. \tag{2}$$

Its properties were examined in detail in Ref. 2.

Three independent experimental groups[4,5] have recently reported evidence for the existence of an anomalously large $2p$-level shift in the $K^{-4}$He atom (i.e., the state in which the $K^-$meson is bound to the $\alpha$-particle). This shift is $\Delta E_{2p}=\mathrm{Re}\,E_{2p}-E_{2p}^{(0)}>0$, i.e., the $2p$-level is pushed upward. The experimental shift and width of the $2p$-level[5] are listed in Table I. Calculations of these parameters using the optical potential[2)]

$$V_{opt}(r)=\frac{2\pi\hbar^2}{m}\left(1+\frac{m}{m_N}\right)\bar{a}\rho(r)$$

yield $\Delta E_{2p}=0.2$ eV and $\Gamma_{2p}=2$ eV, which are lower by more than an order of magnitude as compared with the corresponding experimental values.

Anomalously large atomic level shifts are usually due to the presence of a near-zero level (real, virtual, or quasistationary) in the strong potential $V_s$. It is interesting to consider the situation in the $K^{-4}$He atom from this point of view (a brief summary of our results was published previously in Ref. 6).

The plan of our paper is as follows. Section 2 gives a calculation of the position and width of a shallow nuclear $p$-level perturbing the atomic spectrum of kaonic helium. Section 3 discusses radiative transitions in hadronic atoms, and Section 4 gives the estimated cross section for the formation of a bound $K^{-4}$He state in nuclear reactions on $^6$Li. Section 5 gives a criterion for the existence of a shallow nuclear level with arbitrary $l$ in the system, and notes some hadronic atoms that could be interesting from this point of view. Section 6 examines a model Coulomb problem with short-range interaction, which has an exact solution. Analysis of exact solutions yields the range of validity of (1) and the other equations used in the theory of hadronic atoms.

**2.** Kaonic helium. For the $K^{-4}$He system, the reduced mass is $m=436.0$ MeV, $\zeta=2$, $E_C=23.2$ keV, and the Bohr radius is $a_B=L/2=31.0$ F.

The effective radius $r_1^{(cs)}$ ($l=1$) in (1) is calculated as follows. We know that the nucleon density in the $\alpha$-particle is satisfactorily described by the Gaussian distribution $\rho(r)=\mathrm{const}\cdot\exp(-r^2/r_0^2)$. In the optical model, the interaction potential $V_s$ between the $K$-meson and the $^4$He nucleus has the same form. If we know $V_s$, we can relate the effective radius $r_1^{(s)}$ of the strong interaction with the Coulomb interaction turned off ($\zeta=0$) to the root-mean-square charge radius $\langle r_{\mathrm{ch}}^2\rangle^{1/2}$:

$$r_1^{(s)}=-\alpha_1/\langle r_{ch}^2\rangle^{1/2}. \tag{3}$$

The dimensionless coefficient $\alpha_1$ is not very dependent on the specific strong-interaction model (for example, $\alpha_1=2.32$, $2.52$, and $2.89$ for the rectangular well, the Gaussian potential, and the exponential potential, respectively). If we introduce the Coulomb potential into $r_1^{(s)}$ in accordance with Ref. 7, i.e.,

$$r_1^{(cs)}=r_1^{(s)}-4\zeta(\ln|r_1^{(s)}/\zeta|-\beta_1), \tag{4}$$

we obtain the Coulomb-nuclear radius $r_1^{(cs)}$ ($\beta_1$ is the constant to be calculated: $\beta_1=0.70$ for the rectangular well and $\beta_1=0.74$ for the separable Yamaguchi potential). If we take $\langle r_{\mathrm{ch}}^2\rangle^{1/2}=1.67F$ for the $\alpha$-particle,[8] we obtain $r_1^{(s)}=-94.8\,L^{-1}$ and $r_1^{(cs)}=-120L^{-1}$ for the Gaussian potential, $r_1^{(s)}=-87.3\,L^{-1}$ and $r_1^{(cs)}=-112L^{-1}$ for the rectangular well, and so on.

Coulomb renormalization of the effective radius is thus seen to be quite substantial. This is a specific feature of the $p$-

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TABLE I. Results for kaonic helium.

| № | $\Delta E_{2p}$, eV | $\Gamma_{2p}$, eV | $\varepsilon$, MeV | $\gamma$, MeV | $a_1^{(cs)}$, Fm$^3$ |
|---|---|---|---|---|---|
| 1 | 35±12 | 30±30 | 0,76 | 0,38 | 60−25$i$ |
| 2 | 50±12 | 100±40 | 0,27 | 0,31 | 85−82$i$ |
| 3 | 43±8 | 55±34 | 0,48 | 0,36 | 73−45$i$ |

Note. The values of the $2p$-level shift and width are taken from the papers cited in Ref. 5. The values of $\varepsilon$, $\gamma$, etc. were calculated from (1) for the average values of the $2p$-level shift and width without using the experimental uncertainties. We took $r_1^{(s)} = -95/L = -1.53\ \mathrm{F}^{-1}$, which corresponds to the Gaussian potential $V_s(r) \propto \exp(-r^2/r_0^2)$.

wave and is due to the presence of the "large logarithm" $\ln|r_1^{(s)}/\zeta| \sim \ln(a_B/r_0) \gg 1$ in (4). It has long been known[9,10] that the Coulomb logarithm appears in the correction to the scattering length $a_0^{(cs)} - a_0^{(s)}$ for the $s$-wave. For an arbitrary $l$, the logarithm $\ln(a_B/r_0)$ appears only in the Coulomb correction to the coefficient of $k^{2l}$ in the expansion for the effective radius[7] while in the remaining terms of this expansion the Coulomb renormalization is of the order of $r_0/a_B \ll 1$.

Using the values of $r_1^{(s)}$ given above, we have varied this parameter between $-85L^{-1}$ and $-110L^{-1}$ ($L = 62.0$ F). For given $r_1^{(cs)}$, $\Delta E_{2p}$ and $\Gamma_{2p}$, Eq. (1) gives us the scattering length $a_1^{(cs)}$ in the $p$-wave, and also the position and width of the remaining $p$-levels.

It turns out that, in addition to the up-shifted atomic $p$-levels, the $K^{-4}$He system has a deeper nuclear level that distorts the Coulomb spectrum. Its positions is sensitive to the shift $\Delta E_{2p}$ and, to a lesser extent, to the effective radius $r_1^{(s)}$ (see Ref. 11 for further details). It is clear from Table I that the measured position and, especially, the width of the $2p$-level of the $K^{-4}$He atom cannot as yet be regarded as established. This is responsible for the appreciable discrepancy between the values of $\varepsilon$ and $\gamma$ calculated from variants 1–3 in Table I. We have varied $\Gamma_{2p}$ in the range 0–100 eV allowed by experiment. Calculations of $\varepsilon$ and $\gamma$ by the method described above show that the $K^{-4}$He system can have a stationary state with $l = 1$. Its binding energy and width are again of the order of a few hundred keV (Fig. 1), i.e., they are very small on the nuclear scale.

The mean radius of the $p$-state with a low ($\lambda r_0 \ll 1$) binding energy is given by

$$\langle r\rangle = \frac{2}{|r_1^{(s)}|}\left\{\ln\left|\frac{r_1^{(s)}}{\lambda}\right| - \beta + O(\lambda r_0)\right\}, \qquad (5)$$

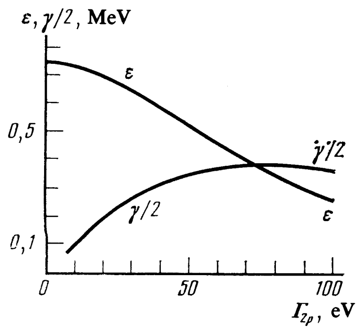


FIG. 1. Position and width of the nuclear $p$-level as functions of the width of the atomic $2p$ state ($\Delta E_{2p} = 43$ eV and $r_1^{(s)} = -95L^{-1}$).

where $\beta \simeq 0.7$–$0.8$ is a constant that is not very sensitive to the shape of the potential $V_s$ (Ref. 11). For $\varepsilon \sim 0.5$ MeV, we obtain $\langle r\rangle \simeq 2.5$ F, which exceeds the range of nuclear forces by a factor of about two.

The possible existence of a weakly-bound nuclear level is probably the most interesting consequence of the observed[4,5] shift of the atomic $2p$-level. For weak absorption, the nuclear level appears as a result of the rearrangement of the Coulomb spectrum.[12,13,1] Increased absorption is accompanied by a transition from the rearrangement regime to the oscillator regime in the motion of the atomic levels.[14] It may be shown that all the atomic $nl$-levels correspond to the rearrangement regime if absorption in the system is not too large, i.e.,

$$\xi_l \equiv \frac{(l!)^2}{2\pi}\,\mathrm{Im}\left(\frac{a_B^{2l+1}}{a_l^{(cs)}}\right) < \xi_l^{(cr)}, \qquad (6)$$

where $\xi_0^{(cr)} = 0.991$ for $s$-states[2] and $\xi_1^{(cr)} = 1$ for $l \geqslant 1$ (see Ref. 11).

**3.** Radiative transitions in hadronic atoms. A possible way of detecting the presence of the nuclear $p$-level in the $K^{-4}$He system is to observe radiative transitions to this state from atomic levels. Transitions from the $s$- and $d$-levels $\Delta l = l' - l = \pm 1$ are possible in the dipole approximation, but the $s$- and $d$-levels are not the first to be populated during transitions from high orbits in the $K$-meson.

The general case of an $E1$ tranition from an unperturbed $nl'$-level to a nuclear $\nu l$-level is examined in Appendix A ($\nu = \zeta/\lambda$ is the analog of the principal quantum number, where $\nu = 1, 2, 3, \ldots$ for the unperturbed Coulomb spectrum). We shall now give formulas for the $nd \to \nu p$ transition probabilities that are important for kaonic helium. In the case of a deep level ($\varepsilon \gg E_C$ or $\nu \ll 1$) the influence of the Coulomb interaction on the wave function of the nuclear state $\nu l$ can be neglected, and we have

$$w_0(nd \to \nu p) = \omega_0 \frac{8\nu^2}{15n^3}\left(1 - \frac{1}{n^2}\right)\left(1 - \frac{4}{n^2}\right)|r_1^{(s)} a_B|^{-1},$$
$$\omega_0 = \zeta^4\left(\frac{Z_1 m_2 - Z_2 m_1}{m_1 + m_2}\right)^2 \alpha^3 \frac{E_C}{\hbar}, \qquad (7)$$

where $\omega_0 = 0.179$ for $K^{-4}$He and $\alpha = e^2/\hbar c \approx 1/137$. Since $|r_1^{(s)} a_B|^{-1} \sim r_0/a_B$ the probabilities of radiative transitions vanish in the limit of zero-range nuclear forces (this is in contrast to the $np \to \nu s$ transition; cf. Ref. 2). Inclusion of the Coulomb interaction for $r > r_0$ leads to the additional factor $C_l^{\pm}(\nu)$ i.e.,

TABLE II. Effect of the Coulomb interaction on the probability of transitions to the $p$-level.

| $\nu$ | $3d\to\nu p$ | $1s\to\nu p$ | $\nu$ | $3d\to\nu p$ | $1s\to\nu p$ |
|---|---|---|---|---|---|
| 0,05 | 1,028 | 0,873 | 0,55 | 1,465 | 0,121 |
| 0,10 | 1,058 | 0,756 | 0,60 | 1,534 | 8,82(−2) |
| 0,15 | 1,090 | 0,651 | 0,65 | 1,609 | 6,14(−2) |
| 0,20 | 1,124 | 0,555 | 0,70 | 1,693 | 4,02(−2) |
| 0,25 | 1,162 | 0,468 | 0,75 | 1,785 | 2,43(−2) |
| 0,30 | 1,202 | 0,391 | 0,80 | 1,888 | 1,30(−2) |
| 0,35 | 1,246 | 0,322 | 0,85 | 2,004 | 5,8(−3) |
| 0,40 | 1,294 | 0,260 | 0,90 | 2,133 | 1,8(−3) |
| 0,45 | 1,346 | 0,207 | 0,95 | 2,280 | 2,4(−4) |
| 0,50 | 1,403 | 0,161 | 1,00 | 2,448 | 0 |

Note. The functions $C_l^{\pm}(\nu)$ are tabulated [see (8)]. As usual, 8.82 ( − 2) means 0.0882, etc.

$$w(n,\ l\pm1\to\nu l)=w_0C_l^{\pm}(\nu), \tag{8}$$

where $w_0$ corresponds to $\zeta = 0$ and is given by (7). The general formulas for $C_l^{\pm}$ are somewhat unwieldy and are reproduced in Appendix A. for the $3d\to\nu p$ transition, we have

$$w(3d\to\nu p)=\frac{64}{6561}\frac{\omega_0}{|r_1^{(s)}a_B|}\frac{\nu^2(9-\nu^2)}{[(2-\nu)(4-\nu)(5-\nu)]^2}$$

$$\times\left\{\left(\frac{2}{1+\nu/3}\right)^{\nu+4}{}_2F_1\left(2-\nu,-(1+\nu);6-\nu;\frac{1}{2}\left(1-\frac{\nu}{3}\right)\right)\right\}^2. \tag{9}$$

The factors $C_l^{\pm}(\nu)$ become equal to unity [see (A4)] as $\nu\to0$. The numerical value of $C_l^{\pm}(\nu)$ for the $3d\to\nu p$ and $np\to\nu s$ transitions are listed in Tables II and III (the latter case refers, for example, to the $\bar{p}p$-atom[2]).

To calculate the $\gamma$-ray spectrum emitted as a result of $3d\to\nu p$ radiative transitions, and the absolute probability of these transitions, we must average (9) over the Breit-Wigner distribution. For a narrow resonance, this yields

$$R=\frac{w(3d\to\nu p)}{w(3d\to2p)}=0{,}97\frac{\nu^2C_1^{+}(\nu)}{|r_1^{(s)}a_B|}. \tag{9'}$$

For variants 1–3 in Table I, Re $\nu$ ranges from 0.23 to 0.30, and hence $R = 10^{-3}$–$2\times10^{-3}$ for $r_1^{(s)} = -95/L$ (the Coulomb correction will increase the transition probability by roughly 20%). Since $w(3d\to\nu p)\propto\nu^2\propto\varepsilon^{-1}$ for $\nu\ll1$, the $\gamma$-ray spectrum has a peak near $\omega_0 = (\zeta^2/2)(1/\nu^2 - 1/9)$ provided $\gamma < 1.2\omega_0$ (see Appendix A). It is not clear at present whether this condition is satisfied (for this to be so, we must have $\Gamma_{2p} < 30$ eV). We must therefore consider other ways of detecting the bound state of $K^-{}^4$He.

**4.** The simplest way of generating this state is the reaction

$$K^-+{}^6\mathrm{Li}\to(K^-\,{}^4\mathrm{He})+d$$

in which one records deuterons emitted into the forward cone. Because of the cluster nature of the nucleus ${}^6\mathrm{Li}\rightleftarrows d+\alpha$, the principal contribution to the cross section is provided by the triangular diagram of Fig. 2, which can be evaluated in a standard way. The function $\psi_{d\alpha}(r)$ for the relative motion of the $d$ and $\alpha$ was taken in the approximation of zero-range nuclear forces, and the wave function for the system $K^-{}^4$He with $l = 1$ was taken for $V_s(r) = -V_0\delta(r-r_0)$. The corresponding $Kd$-scattering amplitude was taken outside the integral sign at the point corresponding to backward scattering:

TABLE III. Coulomb corrections for $np\to\nu s$ radiative transitions.

| $\nu$ | $n=2$ | $n=3$ | $n=4$ | $n=\infty$ | $\beta$ |
|---|---|---|---|---|---|
| 0,05 | 9,665 | 9,661 | 9,660 | 9,658 | 0,9020 |
| 0,10 | 9,324 | 9,311 | 9,306 | 9.300 | 0,8078 |
| 0,15 | 8,977 | 8,948 | 8,938 | 8,925 | 0,7170 |
| 0,20 | 8,624 | 8,573 | 8,556 | 8,533 | 0,6311 |
| 0,25 | 8,264 | 8,186 | 8,159 | 8,125 | 0,5503 |
| 0,30 | 7,895 | 7,786 | 7.749 | 7,701 | 0,4741 |
| 0,35 | 7,519 | 7,374 | 7,325 | 7,262 | 0,4030 |
| 0,40 | 7,134 | 6,951 | 6,889 | 6,810 | 0,3375 |
| 0,45 | 6,741 | 6,517 | 6,442 | 6,347 | 0,2776 |
| 0,50 | 6,341 | 6,074 | 5,985 | 5,874 | 0,2239 |
| 0,55 | 5,935 | 5,624 | 5,522 | 5,395 | 0,1761 |
| 0,60 | 5,523 | 5.169 | 5,054 | 4,913 | 0,1344 |
| 0,65 | 5,109 | 4,712 | 4,585 | 4,431 | 9,89(−2) |
| 0,70 | 4,693 | 4,256 | 4,119 | 3,955 | 6,37(−2) |
| 0,75 | 4,278 | 3,805 | 3,661 | 3,487 | 4,58(−2) |
| 0,80 | 3,868 | 3,364 | 3,213 | 3,035 | 2,76(−2) |
| 0,85 | 3,466 | 2,936 | 2,782 | 2,601 | 1,45(−2) |
| 0,90 | 3,075 | 2,527 | 2,371 | 2,192 | 5,98(−3) |
| 0,95 | 2,699 | 2,140 | 1,986 | 1,812 | 1,37(−3) |
| 1,00 | 2,341 | 1,780 | 1,631 | 1,465 | 0 |
| 1,05 | 2,005 | 1,450 | 1,309 | 1,155 | 1,12(−3) |
| 1,10 | 1,694 | 1,153 | 1,022 | 0.882 | 4,00(−3) |
| 1.15 | 1,410 | 0,892 | 0,773 | 0,650 | 7,91(−3) |
| 1,20 | 1,155 | 0,668 | 0,563 | 0.458 | 1,22(−2) |
| 1,25 | 0,929 | 0,480 | 0.390 | 0,304 | 1,63(−2) |

Note. The values of the functions $C_0^{+}(\nu)$ are multiplied by 10. The coefficient $\beta$ represents the correction for the effective range [see (A5)].

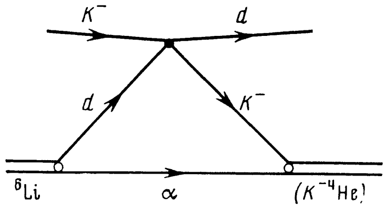

FIG. 2. Feynman diagram for the $K^- + {}^6\text{Li} \to (K^-\alpha) + d$ reaction.

$$f_{Kd} = 2f_{KN}F_d(p_d),$$

where $F_d$ is the deuteron formfactor and $p_d$ is the momentum transferred to the deuteron. We now reproduce the estimated cross section at $K$-meson energy corresponding to the excitation of the $\Lambda(1520)$ resonance in the $K^-N$ system, i.e., $p_{\text{lab}} = 389$ MeV/$c$;

$$\frac{d\sigma}{d\Omega}(\theta_{lab}=180°) = 2(2l+1)^2\lambda\!\!\!^{-2}\left(\frac{\Gamma_{el}}{\Gamma_{tot}}\right)^2\left(\frac{m_{K\,\text{He}}}{m_{Kd}}\right)^2 F_d^2(p_d)F_{kd}^2(\Delta'), \quad (10)$$

where $l = 2$ corresponds to the fact that $\Lambda(1520)$ is a $d$-wave resonance, $\Gamma_{el}/\Gamma_{tot} = 0.45$ for $\Lambda(1520)$, and the transition form factor is given by

$$F_{Kd}(\Delta') = \int \psi_{d\alpha}(\mathbf{r})\psi_{Kd}(\mathbf{r})e^{i\Delta'\mathbf{r}}\,dr^3,$$

$$\Delta' = \frac{m_\alpha}{m_\alpha + m_K}\Delta,$$

where $\Delta' \approx 200$ MeV/$c$ and $\Delta$ is the momentum transferred to the $K^-{}^4$He system. Under the above assumptions about the wave functions, the numerical calculation yields $F_{Kd} \approx 0.28$ and $d\sigma/d\Omega$ ($\theta_{\text{lab}} = 180°$)$\approx 60$ microbarn/radian.

**5.** At first sight, the existence of a level with binding energy $\varepsilon \sim 0.5$ MeV seems surprising, since the $2p$-level shift in the $K^-{}^4$He atom is relatively small: $\delta_{2p} \sim 7\cdot 10^{-3}$ where $\delta_{nl} = \Delta E_{nl}/(E^{(0)}_{n+1,l} - E^{(0)}_{nl})$. We note, in this connection, the general criterion for the existence of a weakly-bound nuclear state. If we use perturbation theory in the scattering length[3] to determine the level shift, and assume that $a_l^{(s)}$ can be taken equal to the scattering length for a rigid sphere of radius $r_0$, we obtain the following "critical" value for the parameter $\delta_{nl}$:

$$\delta_{nl}^{(cr)} \approx \frac{(n+l)!}{(2l)!(2l+1)!(n-l-1)!}\left(\frac{2r_0}{na_B}\right)^{2l+1}. \quad (11)$$

This parameter falls rapidly with increasing $l$. When $\delta \gg \delta^{(cr)}$, the perturbation of the atomic spectrum must be regarded as strong, and the system may have a weakly-bound state whose position is determined by (1).

The condition $\delta \gtrsim \delta^{(cr)}$ provides us with a rapid way of estimating the possible existence of weakly-bound nuclear levels in hadronic atoms. For example, for $K^-{}^4$He, $l = 1$, $r_0/a_B \approx 1/20$ and $\delta_{2p}^{(cr)} \sim 10^{-4}$. For $K^-$Li, we have[16] $l = 1$, $\Delta E_{2p} = 2 \pm 26$ eV, $\Gamma_{2p} = 55 \pm 28$ eV, and $\delta_{2p} = (1.9 \pm 0.9)\cdot 10^{-3}$. For $r_0/a_B \simeq 0.1$, we find from (11) that

$$\delta_{2p}^{(cr)} \sim 0.6\cdot 10^{-3} < \delta_{2p}.$$

For $\bar{p}{}^4$He, we have the estimates

$$r_0/a_B \approx 0.07, \quad \delta_{2p}^{(cr)} \approx 1.7\cdot 10^{-4}.$$

If we calculate $\delta_{2p}$ for the average $2p$-level shift and width given in Ref. 17 ($\Delta E_{2p} = 50 \pm 18$ eV and $\Gamma_{2p} = 105 \pm 65$ eV), we obtain $\delta_{2p} \sim 7\cdot 10^{-3}$. The lower value $\delta_{2p} \sim 10^{-3}$ follows from Ref. 5 $\Delta E_{2p} = 12 \pm 14$ eV, $\Gamma_{2p} = 0^{+30}_{-0}$ eV.

In the above cases, the measured $\delta_{2p}$ is definitely greater than $\delta_{2p}^{(cr)}$ which is an indication that nuclear states with $\varepsilon$ and $\gamma$ which are low (on the nuclear scale) may be present. Moreover, estimates show that $\delta \sim \delta^{(cr)}$ for $K^-{}^7$Li and other systems (we cannot reproduce exact values because the experimental data are still subject to considerable uncertainty). At any rate, all these systems deserve further detailed experimental study.

Only the $s$- and $p$-levels were involved in the examples examined above. This is not surprising since the relative width of the region of rearrangement of the atomic spectrum (for $r_0 \ll a_B$) decreases with increasing $l$:[4]

$$\Delta g/g \propto (r_0/a_B)^{2l-2}, \quad l \geq 1; \quad (12)$$

where

$$g = \frac{2m}{\hbar^2}\int_0^\infty |V_s(r)|\,r\,dr = \bar{V}_s r_0^2,$$

and $\bar{V}_s$ is the characteristic depth of the strong potential.

The rearrangement phenomenon consisting of a strong perturbation of the atomic spectrum by a weakly-bound nuclear level was first examined by Zel'dovich[12] in connection with the question of electron energy levels in primary semiconductors. It was shown[12,13] that atomic $s$-levels are sensitive to a short-range potential if it contains a level (real or virtual) with a low binding energy $\varepsilon \ll \hbar^2/2mr_0^2$, and the width of the region in which the spectrum becomes rearranged was estimated ($\sim r_0/a_B \ll 1$ for $l = 0$). This phenomenon is now referred to as the Zel'dovich effect[18–20] (for states with arbitrary angular momentum $l$). Although the Zel'dovich effect has been investigated for $s$-levels in all its details,[1,2,12,13,18,19] it was only recently that it was shown[20] that it had a number of new qualitative features for $l \neq 0$.

**6.** Exactly solvable model. So far, our approach has been based on the model-independent equation[5] (1), whose range of validity is defined by (2). The limits of the range of validity of this approximation can be established more precisely by comparing it with the exact solutions of the Coulomb problem with a short-range interaction. To do this, consider the model potential

$$V_s(r) = -\frac{g}{2r_0}\delta(r - r_0), \quad (13)$$

where $g$ is the dimensionless coupling constant. The discrete spectrum is then determined from the equation

$$\frac{\Gamma(l+1-\nu)}{(2l+1)!z}M_{\nu,\,l+1/2}(z)W_{\nu,\,l+1/2}(z) = g^{-1}, \quad (14)$$

where $\nu = \zeta/\lambda$, $z = 2\lambda r_0$ (see Appendix C). As $\zeta \to 0$ (Coulomb interaction "turned off"), this equation assumes the form

$$I_{l+1/2}(\lambda r_0)K_{l+1/2}(\lambda r_0) = g^{-1}, \quad (14')$$

from which it follows that there is only one bound state in

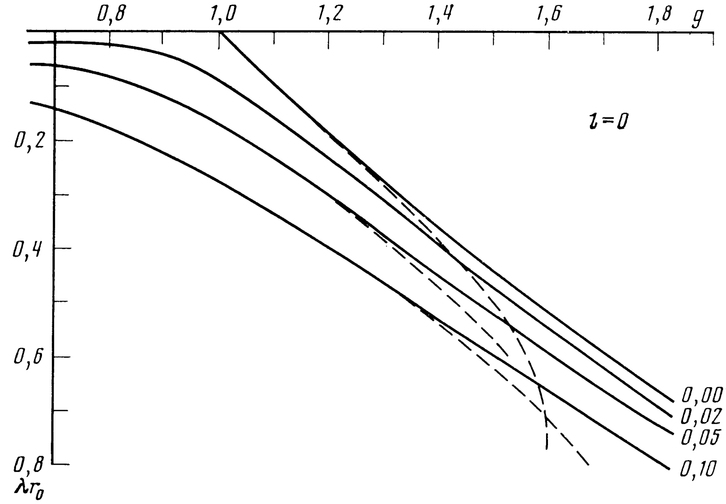


FIG. 3. The product $\lambda r_0$ as a function of the coupling constant $g[\lambda = (2\varepsilon)^{1/2}$; $\varepsilon$ is the binding energy). The solid curves are based on (14) and the broken curves on the approximate equation (1). The values of the parameter $\zeta r_0 = r_0/a_B$ are indicated against the curve.

each partial wave which appears for $g > g_l = 2l + 1$. By considering the scattering problem, we can readily find the scattering length and the effective range for the potential (13):

$$a_l^{(s)} = \tilde{a}_l\left(1 - \frac{g_l}{g}\right)^{-1}, \quad r_l^{(s)} = \tilde{r}_l\left(1 - \frac{2l-1}{g}\right), \tag{15}$$

where $\tilde{a}_l$ and $\tilde{r}_l$ represent the corresponding quantities for a perfectly rigid (impenetrable) sphere of radius $r_0$:

$$\tilde{a}_l = \alpha_l r_0^{2l+1}, \quad \tilde{r}_l = \rho_l r_0^{1-2l},$$

$$\alpha_l = [(2l+1)!!(2l-1)!!]^{-1}, \quad \rho_l = -\frac{2(2l+1)}{(2l+3)(2l-1)\alpha_l}.$$

When the Coulomb interaction is taken into account (i.e., $\zeta \neq 0$), the formulas become much more complicated. The Coulomb-nuclear scattering length and the effective range are then given by (see also Ref. 19).

$$a_l^{(cs)} = \tilde{a}_l\left(\frac{\xi_l}{\eta_l} - \frac{2l+1}{\eta_l^2 g}\right)^{-1},$$

$$r_l^{(cs)} = \tilde{r}_l\left[\frac{2l-1}{2l+1}\frac{\xi_l\tilde{\eta}_l}{2\eta_l^2} + \frac{2l+3}{2l+1}\frac{\tilde{\xi}_l}{2\eta_l} - \frac{(2l-1)\tilde{\eta}_l}{\eta_l^2 g}\right]. \tag{16}$$

The functions $\xi_l(x)$, $\eta_l(x)$, and so on, where $x = 2\zeta r_0$, were determined earlier in Ref. 25 [see Eqs. (2.4)–(2.8)]. All these functions become equal to unity for $\zeta = 0$ and (16) becomes identical with (15).

We have used (14) to calculate the binding energy in the ground state as a function of $g$ for several values of the ratio $\zeta r_0 = r_0/a_B$ (Fig. 3). We note that, when $\zeta = 0$, the bound state appears only for $g > 1$, whereas for $\zeta > 0$ it arises for all $g > 0$ (we then have $\lambda \to \zeta$ as $g \to 0$. It is clear from Fig. 3 that: (1) the uncertainty in the energy calculated from (1) does not exceed 10% when $\lambda r_0 < 0.3$, i.e., the binding energy is $\varepsilon < 0.1\hbar^2/2mr_0^2$, and (2) the range of validity of (1) expands somewhat as $\zeta r_0$ increases. For example, for the $\bar{p}p$-atom, equation (1) is quite accurate up to $\varepsilon \sim 5$–10 MeV.

We now turn to the determination of the "purely nuclear" scattering length $a_l^{(s)}$ from the Coulomb-nuclear length $a_l^{(cs)}$ found by low-energy phase-shift analysis. In a previous paper,[2] we found that ($l = 0$)

$$\frac{1}{a_{cs}} = (1 + 2b_1\zeta r_s)\frac{1}{a_s} - 2\zeta(\ln|\zeta| r_s + c_0 + c_1\zeta r_s), \tag{17}$$

which is a generalization of the well-known Schwinger formula,[9] obtained from (17) by substituting $b_1 = c_1 = 0$, i.e., by neglecting corrections of the order of $\zeta r_0$. Similarly, for states with $l \neq 0$, we have[25]

$$\frac{1}{a_l^{(cs)}} = \frac{1}{a_l^{(s)}} + \frac{1}{\tilde{a}_l}[d_l\zeta r_0 + O((\zeta r_0)^2)]. \tag{18}$$

The coefficients $b_1$, $c_0$, $c_1$, and $d$ depend on the shape of the strong potential $V_s(r)$, and explicit expressions for them are given in Refs. 2 and 25. For the $\delta$-function potential (13), we have

$$b_1 = 3/4, \quad c_0 = 2C - 1/2 + \ln 3/2 = 1{,}060,$$
$$c_1 = 7/4, \quad d_l = [l(l+1)]^{-1}, \quad r_s = 4/3\, r_0, \tag{19}$$

where $C = -\psi(1) = 0.577...$

Figure 4 compares (16) and (17) for $l = 0$ ($\zeta > 0$ corresponds to Coulomb attraction and $\zeta < 0$ to Coulomb repulsion). It is clear that the Schwinger formula is valid in the relatively narrow range in which $r_0/a_B < 1/20$, but its accuracy deteriorates rapidly as the parameter $r_0/a_B$ increases. On the other hand, inclusion of corrections $\sim r_0$ (as described in Ref. 2) results in a considerable improvement in the situation.

For $l \neq 0$ states, we have from (16)

$$\frac{\tilde{a}_l}{a_l^{(cs)}} = \frac{\zeta r_0}{l(l+1)} + \frac{2(4l+7)}{(l+1)^2(2l-1)(2l+3)}(\zeta r_0)^2 + \dots \tag{20}$$

where $g = g_l$, i.e., $1/a_l^{(s)}$. The first term in this expansion reproduces (18), and the correction $\sim(\zeta r_0)^2$ decreases with increasing $l$. An analogous picture arises for the Coulomb corrections to the effective range (see Ref. 26, in which the $\delta$-function interaction on the $r = r_0$ sphere is examined in greater detail).

Our results show that (1), (17), and (18) have wide range of validity in the parameter $\lambda r_0$ and $r_0/a_B$.

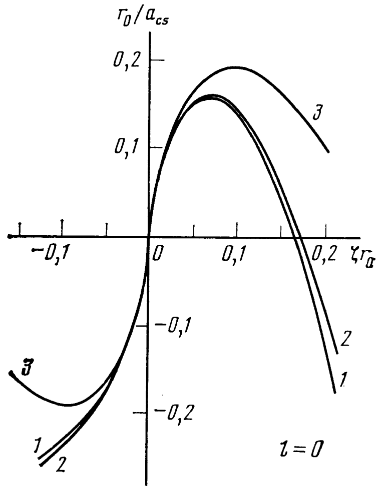


FIG. 4. The ratio $r_0/a_{cs}$ for the model described by (13) with $l=0$ and $g=g_0=1$ ($a_s=\infty$). Curves 1, 2, and 3 correspond to the exact solution (16), formula (17), and the Schwinger formula ($b_1=c_1=0$), respectively.

## APPENDIX A

Let us now consider the probabilities of electric dipole transitions ($nl'\to\nu l$, $l'=l\pm1$) in a hadronic atom. The specific feature of this problem is that the shift of the upper $nl'$-level can be neglected, while the position of the lower (nuclear) $\nu l$-level is arbitrary. We shall not reproduce any derivations and give only the final formulas.

The $E$ 1-transition probability can be factored as shown in (8). The first factor corresponds to $\zeta=0$:

$$w_0(n,l\pm1\to\nu l)=\frac{\omega_0}{n^3}P_l^{\pm}\Pi_{n,l\pm1}\frac{\nu^{\pm2}}{\zeta^{1-2l}|r_l^{(s)}|},\tag{A1}$$

where $l\geqslant1$,

$$P_l^{+}=\frac{16}{3}[(l!)^2(l+1)(2l+3)]^{-1},$$

$$P_l^{-}=\frac{4}{3}\frac{l(2l+1)^2}{(2l-1)[(l-1)!]^2},$$

$$\Pi_{nl}=\left(1-\frac{1}{n^2}\right)\left(1-\frac{2^2}{n^2}\right)\ldots\left(1-\frac{l^2}{n^2}\right),\qquad \Pi_{n0}\equiv1.$$

We note that $(\zeta^{1-2l}|r_l^{(s)}|)^{-1}\sim(r_0/a_B)^{2l-1}\ll1$ and the numerical factors $P^{\pm}$ decrease rapidly with increasing $l$: $P_1^{+}=0.533$, $P_2^{+}=6.35\cdot10^{-2}$, $P_3^{+}=4.12\cdot10^{-3}$. This leads to a corresponding reduction in the transition probability.

The Coulomb factors $C_l^{\pm}$ will now be given for the case $l'=l+1$, which is the most important for hadronic atoms (for the formulas for $l'=l-1$, see Ref. 11). When $n=l'+1$ (transition from the lower atomic level with orbital angular momentum $l'$), we have for $0<\nu<l+2$

$$C_l^{+}(\nu)=\left(1-\frac{\nu}{l+2}\right)\left(1+\frac{\nu}{l+2}\right)^{-(2l+2\nu+5)}$$
$$\times\left\{2^{l+\nu+2}\frac{3(l+1)}{(l+1-\nu)(l+3-\nu)(l+4-\nu)}\right.$$
$$\left.\times{}_2F_1\left(l+1-\nu,-l-\nu;l+5-\nu;\frac{1}{2}\left(1-\frac{\nu}{l+2}\right)\right)\right\}^2.\tag{A2}$$

The expression for $C_l^{+}(\nu)$ in the case of a transition from an arbitrary atomic state ($n=l'+1$) is analogous to (A2) but contains the sum of $n-l'$ hypergeometric functions. The formulas become very unwieldy as $n$ increases, but a simple expression is obtained in the limit $n\to\infty$:

$$C_l^{+}(\nu)=\{2(l+1)(l+2)I_l(\nu)\}^2,$$
$$I_l(\nu)=\int_0^1 e^{-2\nu x}(1+x)^{l+\nu}(1-x)^{l-\nu}\left(1-\frac{\nu x}{l+2}\right)x^3\,dx.\tag{A3}$$

As $\nu\to0$, i.e., for a deep ($\varepsilon\gg E_C$) nuclear level, the Coulomb correction approaches unity:

$$C_l^{\pm}(\nu)=1+c_l^{\pm}\nu+O(\nu^2),\tag{A4}$$

where

$$c_l^{+}=\pi^{1/2}\Gamma(l+1)/\Gamma(l+5/2)\quad\text{for}\quad l\geqslant1,$$
$$c_0^{+}=-2/3\ \text{и}\ c_l^{-}=-2\pi^{1/2}\Gamma(l)/\Gamma(l+3/2).$$

With increasing $\nu=\zeta/\lambda$, the correction increases when $l'=l+1$ and $l\geqslant1$, and decreases when $l'=l-1$ or $l=0$. For $\nu\sim1$ (and small $l$), the correction is already $\sim100\%$ (see Table II). Furthermore, we note that the coefficients $c_l^{\pm}$ are independent of $n$ [the dependence on the principal quantum number $n$ of the initial state appears in the higher-order terms in the expansion (A4), beginning with $\nu^2$].

To calculate the $\gamma$-ray spectrum recorded experimentally, we must multiply the probability $w(\nu)$ by the Breit-Wigner distribution

$$\frac{\gamma}{2\pi}\left[(\omega-\omega_0)^2+\frac{\gamma^2}{4}\right]^{-1},\qquad \omega_0=\frac{\zeta^2}{2}\left(\frac{1}{\nu^2}-\frac{1}{n^2}\right).$$

Since $w(\nu)\propto\nu^{\pm2}\propto\omega^{\mp1}$ for $\nu\ll1$ (the upper and lower signs correspond to $l'=l\pm1$, respectively), we obtain the $\gamma$-ray spectrum for $nd\to\nu p$ transitions, which has a peak at the frequency

$$\omega_r^{+}=\frac{1}{3}\omega_0\left[2+\left(1-\frac{3\gamma^2}{8\omega_0^2}\right)^{1/2}\right].$$

The peak vanishes for $\gamma>(2/\sqrt{3})\omega_0=1.155\omega_0$. The $\gamma$-ray spectrum always contains a peak at the frequency $\omega_r^{-}=(\omega_0^2+\gamma^2/4)^{1/2}$ for transitions of the form $ns\to\nu p$ ($l'=l-1$). When $\gamma\ll\omega_0$, we have $\omega_r^{\pm}=\omega_0(1\mp\gamma^2/8\omega_0^2)$.

The $np\to\nu s$ transition probabilities, corrected for the effective radius $r_s$, are given by

$$w(np\to\nu s)=w_0C_0^{+}[1+\beta\lambda r_s+O((\lambda r_s)^2)],\tag{A5}$$

where (using the notation of Ref. 2)

$$w_0=\omega_0\frac{16}{9n^3}\left(1-\frac{1}{n^2}\right)\nu,$$
$$C_0^{+}(\nu)=\frac{f(n,\nu)}{4J(\nu,0)},\qquad \beta=[\Gamma^2(1-\nu)J(\nu,0)]^{-1}.\tag{A6}$$

The Coulomb corrections (Tables II and III) were calculated from (A2), (A3), and (A6).

## APPENDIX B

It is well-known that the Schroedinger equation with the potential $V=V_s+V_C$, the sum of the short-range $V_s$

and the Coulomb $V_C(r) = -\zeta/r$ potentials, can be solved analytically only in very rare cases. The fact that exact solutions can be obtained for the model defined by (13) is due to the fact that this potential is equivalent to the boundary condition

$$\varphi_l'(r_0+0)-\varphi_l'(r_0-0)+\frac{g}{r_0}\varphi_l(r_0)=0. \tag{B1}$$

The wave function $\varphi_l(r) = rR_l(r)$ can be expressed in terms of the Whittaker function in the case of the discrete spectrum $(E = -\lambda^2/2 < 0)$:

$$\varphi_l(r)=\begin{cases} c_1 M_{\nu,\, l+1/2}(2\lambda r), & 0<r<r_0 \\ c_2 W_{\nu,\, l+1/2}(2\lambda r), & r>r_0 \end{cases}. \tag{B2}$$

Substituting (B2) into the mating condition (B1), and using the Wronskian

$$M'_{\nu,\, l+1/2}(z)\,W_{\nu,\, l+1/2}(z)-M_{\nu,\, l+1/2}(z)\,W'_{\nu,\, l+1/2}(z)=\frac{(2l+1)!}{\Gamma(l+1-\nu)},$$

we obtain (14).

The curves of Fig. 3 were calculated with the aid of the tables given in Ref. 27 and the relation

$$W_{\nu,l+1/2}(z)=e^{-z/2}z^{l+1}G(l+1-\nu,\ 2l+2;\ z),$$

where $G(\alpha,\gamma;z)\equiv\Psi(\alpha,\gamma;z)$ (see Ref. 28). The dashed curves were obtained from the approximate equation (1) with $l = 0$ and

$$r_{cs}=\frac{4}{3}r_0\left(1+\frac{7}{4}\zeta r_0\right),$$

which corresponds to the inclusion of the first Coulomb correction.[25] Finally, the $\zeta = 0$ curve was determined from (14') and the corresponding dotted curve from

$$\lambda=\frac{2}{a_s}\left[1+\left(1-\frac{2r_s}{a_s}\right)^{1/2}\right]^{-1}, \tag{B3}$$

which gives the solution of (1) with $\zeta = 0$. Here, $a_s = r_0 g/(g-1) > 0$, $r_s = (4/3)r_0$.

Figure 4 was calculated from (16). We note that the functions $\xi_l$, $\eta_l$, and so on, have different analytic form, depending on the sign of $\zeta$. For example, for $s$-scattering,

$$\frac{r_0}{a_{cs}}=-\begin{cases}\left[\dfrac{\rho}{2J_1(\rho)}\right]^2 g^{-1}+\dfrac{\pi}{4}\dfrac{\rho^2 N_1(\rho)}{J_1(\rho)}, & \zeta>0 \\ \left[\dfrac{\rho}{2I_1(\rho)}\right]^2 g^{-1}-\dfrac{\rho^2 K_1(\rho)}{2I_1(\rho)}, & \zeta<0\end{cases}, \tag{B4}$$

where $l = 0, \rho = (8|\zeta|r_0)^{1/2}$, and $J_1$, $N_1$, $I_1$, and $K_1$ are Bessel functions. The formulas for $r_1^{(cs)}$ in the $p$-wave have an analogous form but are more unwieldy[26] (the case $\zeta > 0$ then corresponds to Coulomb attraction and $\zeta < 0$ to Coulomb repulsion).

[1)]I.e., for $r \sim r_0 \ll a_B$, where $r_0$ is the range of nuclear forces and $a_B = |\zeta|^{-1}$ is the Bohr radius. We are using the atomic system of units in which $\hbar = m = e = 1$; the unit of binding energy is $E_c = me^4/\hbar^2$ and the unit of length is $L = \hbar^2/me^2 = |\zeta| a_B$.

[2)]See Ref. 5. Here $\rho(r)$ is the density of nuclear matter and $\bar{a}$ is the effective $\bar{K}N$-scattering length, extrapolated to the heavier $K$-mesonic atom in accordance with the level shift and width.

[3)]See Eq. (5) in Ref. 15:

[4)]We note that Refs. 1 and 2 give an incorrect esimate for the width of the region in which the spectrum becomes rearranged in the case of states with nonzero orbital angular momentum. For $l \geqslant 1$, the nuclear level successively crosses each of the atomic $nl$-levels as the coupling constant $g$ increases. The term-crossing region is very narrow $(\Delta g \propto (r_0/a_B)^{l+3/2}$, and the entire range of values of $g$ in which the nuclear level crosses the atomic spectrum is much wider and is given by (12). The reader is referred to Ref. 20 for further details.

[5)]This equation is widely used in the theory of hadronic atoms (see, for example, Refs. 2, 3, 6, and 11). It can be deduced by analytic continuation of the effective-range expansion (in the presence of the Coulomb interaction) to the discrete spectrum.[2,21,22] Another way of deriving it is based on the method of evolution with respect to the coupling constant,[23] described in Ref. 24.


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