Bloch oscillations in small Josephson junctions

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A quantum-statistics theory is derived for the processes which occur in small Josephson junctions at low temperatures. When the current, the temperature, and the quasiparticle admittance are all below certain limits, the dynamics of the system is similar to that of a quantum-mechanical particle in a one-dimensional periodic potential. The possible occurrence of "Bloch oscillations" in the junction is analyzed on the basis of an adiabatic Hamiltonian. The current-voltage characteristic of the system is derived. The factors most important in suppressing or masking the Bloch oscillations are discussed. The possibility of observing these effects experimentally is analyzed.

1. INTRODUCTION

"Secondary" (or "real") quantum microscopic effects in weak superconductivity have recently attracted considerable interest. These effects, which were discussed as early as 1963 by Anderson,[1] occur because the Josephson phase difference φ cannot in general be treated as a classical variable and must be instead treated as an operator which does not commute with the electric Josephson-junction charge operator Q:

$$[\varphi,\ Q]=2ei. \tag{1}$$

Until recently, attention has focused primarily on the case in which the elementary value

$$E_Q=e^2/2C \tag{2}$$

of the electrical energy of the junction $Q_2/2C$ is not too large:

$$E_Q\ll\max\,[E_J,\ T], \tag{3}$$

where E_J is the amplitude of the Josephson coupling energy,

$$U_J=-E_J\cos\varphi, \tag{4}$$

C is the capacitance of the junction, and T is the temperature (which will be expressed below in energy units). Under condition (3) a nonvanishing value of the commutator in (1) leads to (first) small quantum fluctuations of φ and Q and (second) a nonvanishing probability for a macroscopic quantum tunneling; see the review by Likharev.[2] Neither of these effects, however, caused qualitative changes in the behavior of the junctions, and each can be described phenomenologically by simply introducing an effective temperature $T^*>T$ in the "classical" dynamic equations of the Josephson effect.[3]

The consequences of second quantization may provide significantly more information in the case $E_Q\gtrsim E_J\gtrsim T$. Although several attempts have been undertaken to analyze this case (e.g., Refs. 4–9), the most specific processes (more on this below) have escaped the attention of these authors, apparently because interest was focused on relatively complex disordered structures consisting of many junctions. Only recently[10,11] has the simplest case, in which a current $I(t)$ fixed by an external system flows across a single junction of small area, been examined in detail. However, Rogovin and Nagel[10] and Widom *et al.*[11] reached fundamentally different conclusions regarding the properties of such a junction, since they (implicitly) made opposite assumptions regarding the translational properties of the variable φ.

Our purpose in the present paper is to analyze the processes which occur at an isolated junction for sufficiently low levels of the perturbing factors: the current $I(t)$, the temperature T, and the quasiparticle admittance $Y(\omega)$. Since the translational properties of the phase difference φ are crucial to the analysis below, we begin with this topic.

2. TRANSLATIONAL PROPERTIES OF THE PHASE DIFFERENCE φ

If the voltage $V=Q/C$ across the Josephson junction is not too high, specifically, if

$$e|V|\ll\Delta_{1,2}(T), \tag{5}$$

where $\Delta_{1,2}(T)$ are the energy gaps in the superconductors which form the junction, we know (Ref. 12, for example) that all the properties of the junction can be described successfully by the "adiabatic" Hamiltonian

$$\hat{H}=\frac{Q^2}{2C}+U_J(\varphi)+\frac{\hbar}{2e}[I_q\{x\}-I(t)]\varphi+\hat{H}_q\{x\}. \tag{6}$$

Here $\hat{H}_q$ and $\{x\}$ are respectively the Hamiltonian and the set of coordinates of the ensemble of quasiparticles, which serves as a heat reservoir in this case. This ensemble is related to the superfluid subsystem in which we are interested through the quasiparticle current operator I_q, which appears in (6) in the same way as the external current $I(t)$. We assume that this external current is a classical function of the time, as we are justified in doing if the impedance of the source of this current is sufficiently high: $|Z_e Y|\gg 1$.

If the energy dissipation rate, determined by I_q, and the external current $I(t)$ are sufficiently small, the Hamiltonian (6) takes the very simple form

$$\hat{H}_0=\frac{Q^2}{2C}+U_J(\varphi)=4E_Q\frac{\partial^2}{\partial\varphi^2}-E_J\cos\varphi. \tag{7}$$

This Hamiltonian and commutation relation (1) are completely analogous to those which describe the properties of two well-known systems:

1) a planar quantum pendulum with moment of inertia $(\hbar/2e)^2C$, angular momentum $(\hbar/2e)Q$, and deviation angle φ from the equilibrium position $\varphi=0$ in a gravitational force field;

2) a one-dimensional quantum particle with a mass $(\hbar/2e)^2C$, momentum $(\hbar/2e)Q$, and coordiante φ in the field of

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the periodic-potential $U_J(\varphi)$.

At energies which are not too low ($E \gtrsim E_J$), however, the properties of these two systems are fundamentally different, because of the different translational properties of the variable φ. Specifically, after the translation

$$\varphi \to \varphi + 2\pi \tag{8}$$

the state of the quantum pendulum is precisely the same as its original state, so that in the φ "coordinate" representation its weight function is periodic with a period of 2π, leading us immediately to the well-known picture of a discrete energy spectrum (Ref. 13, for example). For the one-dimensional quantum particle, in contrast, the states before and after translation (8) are fundamentally different, so that we must incorporate in the wave function some Bloch components with arbitrary quasimomenta $\hbar k$ (Refs. 14 and 15, for example):

$$\psi(\varphi) = \sum_s \int dk c_k^{(s)} \psi_k^{(s)}, \qquad \psi_k^{(s)} = u_k^{(s)}(\varphi) e^{ik\varphi},$$

$$u_k^{(s)}(\varphi) = u_k^{(s)}(\varphi + 2\pi), \qquad s = 0, 1, 2, \dots . \tag{9}$$

We then immediately find the band structure of the energy spectrum and other effects which are well known in solid state physics. For Josephson junctions, the first of these possibilities was adopted (implicitly) in Ref. 10, while the second was adopted in Ref. 11 (again, without proof).[1)]

Obviously, if we wish to determine the actual translational properties of the Josephson phase difference φ we must go beyond Hamiltonian (7); e.g., we might use the adiabatic Hamiltonian (6). The question can be solved most easily in the case $I(t) \neq 0$. In this case, translation (8) leads to a finite change in the term $-(\hbar/2e)I(t)\varphi$, which describes simply a change in the energy of the current source:

$$\Delta E_e = -2\pi(\hbar/2e)I(t) = -\Phi_0 I(t) \neq 0. \tag{10}$$

In principle, this change can be measured arbitrarily accurately, so that the states of the junction before and after translation (8) are fundamentally different. As for the value $I = 0$, we note that the same result could be obtained for it either from continuity considerations or by the following approach: We cannot assume that the operator I_q, which figures in $\hat{H}$ in precisely the same way as the current $I(t)$, is ever exactly zero, so that the states before and after the 2π phase translation may also be distinguished, in principle, by the change in the energy of the heat reservoir.

We thus see that the states of a Josephson junction which differ in phase by 2π are always distinguishable in principle. For this reason, the properties of junctions with small dimensions are similar to the properties of a particle in a one-dimensional potential (although the properties are not analogous in the two cases because of the different nature of the scattering processes). If $I(t)$ and $I_q\{x\}$ are sufficiently small we can thus describe the processes in the junction by perturbation theory, using as a basis the system of Bloch functions $\psi_k^{(s)}$ (9), which are periodic in k with a period 1. Since the Schrödinger equation for the unperturbed Hamiltonian H_0 is, in the φ representation, the well-known Mathieu equation, the properties of these basis functions are well known. In particular, in the most interesing case, $E_Q \gtrsim E_J$, the energy spectrum consists of a lower band $E^{(0)}(k)$, which is separated at $E \approx E_Q$ from the upper bands (which essentially merge) by an energy gap of magnitude E_J (Fig. 1).

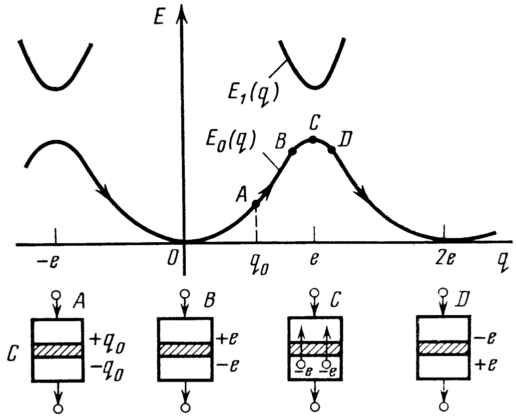

FIG. 1. The function $E^{(s)}(k)$ for a junction with $E_Q > E_J$; diagram of the discrete transfer of a Cooper pair when the "quasicharge" $q = 2ek$ crosses the boundary of the Brillouin zone ($q = e$).

3. LANGEVIN OPERATOR EQUATION FOR THE QUASICHARGE

If the temperature T is quite low, then in the case $I(t)$, $I_q\{x\} \to 0$ the system will be trapped in the lower energy band ($s = 0$), so that the band index s cannot vary over time; only the quantity $q = 2ek$ can vary. This quantity differs from the charge Q to precisely the same extent that the quasimomentum of a quantum particle in a crystal differs from its momentum. We could thus naturally call Q a *quasicharge*. The most convenient technique for describing the changes in Q is to use Langevin operator equations, which are used extensively in quantum radiophysics for systems of weakly coupled harmonic oscillators (Ref. 17, for example).

To derive this equation for our system we break up φ into two components,

$$\varphi = \varphi_q + \varphi_s, \tag{11}$$

in such a way that φ_q is related to q by a commutation relation analogous to (1): $[\varphi_q, q] = 2ei$. The remainder of the phase operator, φ_s, then describes only interband transitions, and it commutes with q (Ref. 15). In the representation in which the time dependence associated with $\hat{H}_0$ (7) is incorporated in the state vector of the system, Hamiltonian (6) corresponds to the following system of operator equations:

$$\dot{q} = I(t) - I_q, \tag{12a}$$

$$\dot{I}_q = -\frac{i}{\hbar}[I_q, H_q], \tag{12b}$$

$$\dot{H}_q = \frac{i}{2e}\varphi[I_q, H_q], \tag{12c}$$

$$\dot{\varphi} = \frac{2e}{\hbar}V, \tag{12d}$$

$$\dot{\varphi}_s = -\frac{i}{2e}[I(t) - I_q][\varphi_q, \varphi_s]. \tag{12e}$$

Here the operator V is determined by its matrix elements in the Schrödinger picture:

$$V_{qq'}^{ss'}=\left[\frac{\partial E^{(s)}(q)}{\partial q}\delta_{ss'}+i\frac{E^{(s)}-E^{(s')}}{2e}(1-\delta_{ss'})\right]\delta(q-q'). \quad (13)$$

It is also a simple matter to derive the matrix elements of the commutator $[\varphi_q, \varphi_s]$:

$$[\varphi_q,\varphi_s]_{qq'}^{ss'}=2ei\frac{\partial\varphi^{ss'}(q)}{\partial q}\delta(q-q'),$$

where the $\varphi^{ss'}(q)$ are the elements of the operator φ_s,

$$\varphi^{ss'}(q)=2e\left\langle q,s'\left|i\frac{\partial}{\partial q}\right|q,s\right\rangle, \quad (14)$$

which are expressed in terms of the solution of the Mathieu basis equation.

Solving Eqs. (12) by perturbation theory, we find, in second order in the interaction,

$$\dot q=I(t)-[\langle I_q\rangle+\tilde I(t)]. \quad (15)$$

The random operator $\tilde I(t)$ satisfies the equation

$$\tilde I(t)=\tilde I(-\infty)+\frac{i}{\hbar}\int_{-\infty}^{t}dt'e^{\varepsilon(t'-t)}[\tilde I(t'),H(t')] \quad (16)$$

(where H is the unperturbed Hamiltonian of the heat reservoir, H_q), and $\langle I_q\rangle$ satisfies the equation

$$\langle I_q(t)\rangle=\frac{1}{2e\hbar}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt''e^{\varepsilon(t''-t)}\varphi(t'')\times\langle[\tilde I(t'),[\tilde I(t''),H(t'')]]\rangle, \quad (17)$$

where $\langle\ldots\rangle$ means an average over the states of the unperturbed heat reservoir.

Making the usual assumptions regarding the continuity of the energy spectrum of the heat reservoir, we can use the well-known[18] statistical properties of the solution of an equation of the type in (16):

$$\langle\tilde I(t)\rangle=0,$$

$$\frac{1}{2}\langle\tilde I(t)\tilde I(t+\tau)+\tilde I(t+\tau)\tilde I(t)\rangle=2\int_0^\infty S_I(\omega)\cos\omega\tau\,d\omega, \quad (18a)$$

$$S_I(\omega)=\pi^{-1}\,\mathrm{Re}\,Y(\omega)\Theta(\omega,T),\qquad \Theta(\omega,T)=\frac{\hbar\omega}{2}\mathrm{cth}\frac{\hbar\omega}{2T}, \quad (18b)$$

where Re Y is the real part of the complex admittance of the junction due to the quasiparticle current I_q, given by

$$\mathrm{Re}\,Y(\omega)=(\hbar\omega)^{-1}\int dE\sigma(E)f(E)\times[\sigma(E+\hbar\omega)|I_+|^2-\sigma(E-\hbar\omega)|I_-|^2]. \quad (19)$$

Here $\sigma(E)$ is the state density of the heat reservoir near the energy E, $f(E)$ is the energy distribution of the reservoir, which we assume to be an equilibrium distribution,

$$f(E)=Z^{-1}\exp\{-E/T\},\qquad Z=\int dE\sigma(E)\exp\{-E/T\}, \quad (20)$$

and the $I_\pm$ are the matrix elements of the operator I_q,

$$I_\pm=\langle E\pm\hbar\omega|I_q|E\rangle. \quad (21)$$

To find the expectation value $\langle I_q\rangle$ we note that in the energy representation we are concerned with only the diagonal matrix elements of the commutator in (17),

$$\langle[\ldots]\rangle=\mathrm{Sp}(\rho[\ldots])=\sum_m f(E_m)[\ldots]_{mm}, \quad (22)$$

which can be found quite easily:

$$[\tilde I(t'),[\tilde I(t''),H(t'')]]_{mm}=\sum_n 2(E_m-E_n)|I_{nm}|^2\cos[(E_m-E_n)(t'-t'')]. \quad (23)$$

Expanding the operator φ in a Fourier integral, and transforming from a summation to an integration in (22), we find

$$\langle I_q(t)\rangle=\int d\omega e^{-i\omega t}(-i\hbar\omega\varphi_\omega/2e)K(\omega), \quad (24)$$

$$K(\omega)=\lim_{\varepsilon\to+0}\int dE\sigma(E)f(E)\times\int dE'\sigma(E')|I_{EE'}|^2\frac{\omega'}{\omega^2}\left[\frac{1}{\omega+\omega'+i\varepsilon}+\frac{1}{\omega-\omega'+i\varepsilon}\right] \quad (25)$$

$$\omega'=(E'-E)/\hbar.$$

However, using expression (19) for Re $Y(\omega)$, and using the Kramers-Kronig dispersion relations, we see that $K(\omega)$ is none other than the total quasiparticle admittance of the junction: $K(\omega)\equiv Y(\omega)$. Consequently, using (12d), we can put Eq. (15) in the form

$$\dot q=I(t)-\tilde I(t)-\int d\omega e^{-i\omega t}Y(\omega)V_\omega. \quad (26)$$

As long as we are concerned with the single-band approximation we can replace V by $dE^{(0)}/dq$, according to (15). Furthermore, our adiabatic approach [see (16)] is valid only if all of the important frequencies for the changes in q and V satisfy the condition $\hbar\omega\ll\Delta_{1,2}(T)$. The frequency dispersion of the quasiparticle admittance $Y(\omega)$, on the other hand, is important only at frequencies on the order of $\Delta_{1,2}/\hbar$, so that we can set $Y(\omega)=Y(0)=G$ in (26). Under these assumptions, Eq. (26) takes the simple form

$$\dot q=I(t)-G\frac{dE^{(0)}(q)}{dq}-\tilde I(t). \quad (27)$$

We proceed now to the solution of this equation.

4. BLOCH OSCILLATIONS AND THE CURRENT-VOLTAGE CHARACTERISTIC

For the time being we ignore the Langevin operator of the "fluctuation current" $\tilde I(t)$. The quasicharge q is then a well-defined classical variable according to Eq. (27). Finding the solutions $q_0(t)$ for various values of the average current $\bar I$ (the average here is over the time), we can also find the current-voltage characteristic (cvc) of the junction, $\bar V(\bar I)$, since an averaging of (27) over the time gives us

$$\bar V=G^{-1}(\bar I-\bar{\dot q}). \quad (28)$$

In particular, if the current remains constant over time, $I(t)=\bar I$, then for all currents below the threshold value

$$I_t=G(dE^{(0)}/dq)_{max} \quad (29)$$

there exists a stable solution $q_0=\mathrm{const}$ (point A in Fig. 1). It can be seen from (28) that such low currents correspond to a linear initial region of the cvc: $\bar V=G^{-1}\bar I$ (Fig. 2). If $\bar I$ is in-

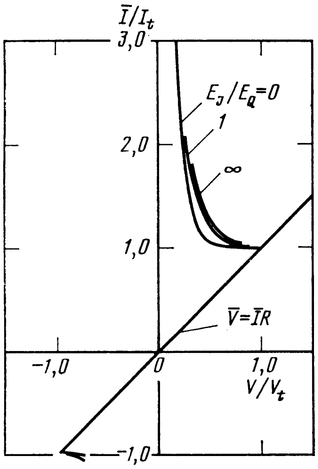

FIG. 2. Voltage-current characteristic of a junction for various values of the ratio E_J/E_Q.

stead greater than I_t, then the solution $q_0(t)$ is periodic in the time with a frequency ω_B which increases monotonically with increasing $\bar{I}$. Because of the strict $2e$ periodicity of the function $E^{(0)}(q)$, this frequency is $2\pi(\bar{\dot{q}}/2e)$, so that the relation

$$\omega_B=(\pi/e)\,(\bar{I}-G\bar{V}) \tag{30}$$

holds in all cases, according to (28). Here the second term in (28) partially cancels the first, so that a region with a negative slope, $R_d \equiv d\bar{V}/d\bar{I}<0$, appears on the cvc of the junction (Fig. 2).

The oscillations in (30) are an exact analog of the "Bloch oscillations" (or "Stark oscillations") which are well-known in solid state theory and can arise in spatially periodic conducting structures to which a static electric field is applied (Refs. 14 and 15, for example). Although we know of no direct experimental detection of such oscillations, the result of their quantization—the so-called Wannier-Stark energy ladder[14,19]—has apparently been detected in several experiments with narrow-gap semiconductors.[20,21] In experiments with semiconductor superlattices with extremely thin ($\sim$1-nm) layers it has been possible to observe[22] a descending cvc region induced by Bloch oscillations.

To the best of our knowledge, Eq. (30) was first written for Josephson junctions, in an incomplete form ($\omega_B = \pi\bar{I}/e$), by Widom *et al.*,[23] who did not prove it (it was written essentially on the basis of dimensionality considerations). This equation was derived by Widom *et al.* in Ref. (11), in the same incomplete form, on the basis of an analogy between our system and a quantum particle, although the validity of this analogy was not proved. Finally, two of the authors of the present paper made a brief report in Ref. (24) of the derivation of Eq. (30), in its general form, from Eq. (27).

Since the oscillations in (30) are of fundamental importance to the dynamics of small Josephson junctions, we will "translate" the familiar description of this process in solid state theory[14,15] into the language of Josephson junctions. We consider the most graphic case: $E_J \ll E_Q$ (Fig. 1). In this case the macroscopic quantum tunneling of the phase through the maxima of the potential U_J is so intense that the state of the junction is described, not by any classical value of φ, but by a broad wave packet, (9) ("broad" along the φ scale). According to the uncertainty relation $\Delta\varphi\Delta q \gtrsim e$, such a packet can have a small width along the scale of the quasicharge q in the limit $T\to 0$, so that q is nearly a classical variable.

As long as the value of q is not close to any of the points ne, the corresponding wave number $k = q/2e$ does not coincide with the half-period of the reciprocal lattice of the potential $U_J(\varphi)$; accordingly, the reflection of Bloch wave (9) from the maxima of the potential U_J is slight. In this case, the packet in (9) is almost a simple plane wave $\exp\{ik\varphi\}$, and the expectation value (over the quantum ensemble) of the supercurrent, $\langle I_s\rangle \propto \langle k|\sin\varphi|k\rangle$, is essentially zero. Accordingly, at $q\neq ne$ there is simply a recharging of a capacitor with the difference between the currents $I(t)$ and $\langle I_q\rangle$ (point A in Fig. 1). This result can also be seen from Eq. (27) when we note that with $q\neq ne$ the expectation value (over the ensemble) of the real charge Q is approximately equal to $(q-ne)$, so that we have[2)]

$$\langle\dot{Q}\rangle\approx I(t)-\langle Q\rangle/\tau,\quad \tau=C/G. \tag{31}$$

When q approaches the boundary of the first Brillouin zone (e.g., the value $q=e$; see points B–D in Fig. 1), however, the coherent above-barrier reflection of the Bloch wave from the maxima of the potential U_J leads to the formation of an intense standing wave with a wave number $k\approx 1/2$. Here the expectation value of I_s becomes nonzero, and as q cuts through the crest of the lower zone (as it goes from B to D in Fig. 1) the supercurrent transfers precisely one Cooper pair, $\Delta\langle Q\rangle = -2e$, from one superconductor to the other. Accordingly, at point D the junction is recharged, $\langle Q\rangle\approx q-2e\approx -e$, so that it will subsequently undergo a reverse recharging by the current (31), to the value $+e$; then the process will repeat at the frequency (30). It is important to note that although this system also contains "ordinary" Josephson oscillations with the frequency $\omega_J = 2e\bar{V}/\hbar$ these oscillations are of negligible importance in the expectation values (over the ensemble) of V and Q.

Returning to the cvc, we note that although its shape depends slightly on the relation between E_Q and E_J, we have

$$v=j-\begin{cases}(j^2-1)^{1/2} & \text{for}\quad E_J\gg E_Q\\ 2\ln^{-1}|(j+1)/(j-1)| & \text{for}\quad E_Q\gg E_J\end{cases}, \tag{32}$$

where $v=\bar{V}/V_t$, $j=\bar{I}/I_t>1$, and $V_t=G^{-1}I_t$, the threshold value I_t itself depends strongly on this relation. If $E_Q \ll E_J$, both I_t and V_t are exponentially small:

$$I_t=\begin{cases}(\pi/2)\,\delta^{(0)}G/e & \text{for}\quad E_J\gg E_Q\\ eG/C & \text{for}\quad E_J\ll E_Q\end{cases}, \tag{33}$$

where $\delta^{(0)}$ is the width of the lower energy band, given in the case $E_J\gg E_Q$ by

$$\delta^{(0)}\approx\hbar\omega_p\exp\{-8E_J/\hbar\omega_p\},\quad \hbar\omega_p=(8E_QE_J)^{1/2}. \tag{34}$$

If now we pass not only a direct current but also an alternating current through the junction, $I(t)=\bar{I}+I_A\cos\omega t$, this current can synchronize the Bloch oscillations in (30) both at the fundamental frequency $\omega_B=\omega$ and at its harmonics and subharmonics $\omega_B=(n/m)\omega$. According to

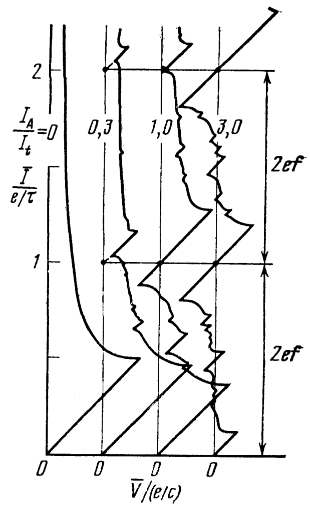

FIG. 3. Effect of an external microwave current of amplitude I_A on the voltage-current characteristic of a junction with $E_J/E_Q = 1$.

the general formula in (30), this effect gives rise to strictly parallel "resistive steps" with an identical slope $R_d = G^{-1}$ on the cvc (Fig. 3). These steps (or their continuations) intersect the current axis at the values

$$\bar{I}=I_{n/m}=(n/m)2ef, \quad f=\omega/2\pi. \tag{35}$$

It is quite obvious that this phenomenon could in principle be exploited to develop a fundamental current standard, which would make it possible to close the "metrological triangle" for the quantities ω, I, V (along with the "ordinary" Josephson effect and the quantum Hall effect).[24]

5. INTRABAND FLUCTUATIONS

Let us examine the effects which disrupt the coherent Bloch oscillations. We begin with fluctuations of q in the lower band, whose sources are described by the Langevin term $\tilde{I}(t)$ in Eq. (27). As long as the fluctuations $\tilde{q} = q - q_0(t)$ are small, we can—in precisely the same manner as in the "classical" theory of the Josephson effect (Chap. 4 in Ref. 3)—linearize Eq. (27) in terms of these fluctuations, transform to the Fourier representation, and immediately calculate the symmetrized spectral density of the quantity $\tilde{q}$. With $I(t) = \bar{I}$ we find by this procedure

$$S_{\dot{q}}(\omega)=\begin{cases}[1+(\omega\tau_d)^{-2}]^{-1}S_I(\omega) & \text{for } |\bar{I}|\leqslant I_t \\ (1+|R_d|/R)^{-2}\sum_n|\alpha_n|^2S_I(\omega-n\omega_B) & \text{for } |\bar{I}|\geqslant I_t\end{cases}, \tag{36}$$

$$\tau_d=RC_d, \quad C_d=d^2E^{(0)}/dq^2|_{q=q_0}, \tag{37}$$

where the moduli of the coefficients $\alpha_n(\omega)$ in the limit $\omega\to 0$ decrease rapidly with increasing n ($\alpha_0 = 1$), especially if $|\tilde{I}|\gg I_t$.

As Eqs. (36) show, we have $S_{\dot{q}}(\omega)\to 0$ as $\omega\to 0$ under the condition $|\tilde{I}|\leqslant I_t$, so that the system simply undergoes small fluctuations near the value of q_0. At $|\tilde{I}|>I_q$, however, the value of $S_{\dot{q}}(0)$ is nonzero, so that there is a nonzero diffusion of the total phase of the Bloch oscillations, $\theta = \omega_B t + \pi\tilde{q}/e$, i.e., these oscillations have a nonzero line width $2\Gamma_B$. If the fluctuations satisfy the condition for a classical nature,

$$\hbar\Gamma_B\ll T, \tag{38}$$

then $S_I(\omega)$ is constant over the interval $0<\omega\lesssim\Gamma_B$, and the line has a Lorentzian shape with a half-width

$$\Gamma_B=\pi(\pi/e)^2S_{\dot{q}}(0). \tag{39}$$

According to (36) and (39), at currents $\bar{I}$ slightly above I_t the line half-width rapidly approaches the value

$$\Gamma_T=(\pi/e)^2GT. \tag{40}$$

If condition (38) does not hold, i.e., if quantum fluctuations are predominant,[3)] we can estimate Γ_B by equating the mean square value of the fluctuations of the instantaneous Bloch frequency, $\tilde{\omega}_B = (\pi/e)\dot{\tilde{q}}$, to the contribution of the Langevin force in the band $0<\omega\lesssim\Gamma_B$. If the current is not too small ($\bar{I} - I_t \gtrsim I_t$), we find by this procedure

$$\Gamma_B\approx\Gamma_T(1-GR_0)^{-1}, \quad R_0\approx R_Q=\pi\hbar/2e^2. \tag{41}$$

The expressions given above for Γ_B are quantitatively correct under the condition $\Gamma_B\ll\omega_B$, but they can also be used to estimate the values of the parameters for which the intraband fluctuations completely "smear" the Bloch oscillations. For this purpose we must equate Γ_B to the characteristic value of ω_B, which is $(\pi/e)I_t$. We then find the following conditions under which the effects of the thermal and quantum fluctuations, respectively, are small:

$$T\ll\begin{cases}\delta^{(0)}, & E_J\gg E_Q \\ E_Q, & E_Q\gg E_J\end{cases}, \tag{42}$$

$$GR_Q\ll 1. \tag{43}$$

We note that the quasiparticle conductance of the junction, G^{-1}, must be substantially larger than the quantum unit of resistance, $R_Q\approx 7\ k\Omega$. On the other hand, narrow-band Josephson oscillations are possible only if $GR_Q\gg 1$. For $E_Q\gtrsim E_J$, therefore, it is the value $GR_Q\sim 1$ which serves as the boundary between the two possible oscillation effects in Josephson junctions.[4)]

6. INTERBAND TRANSITIONS

Bloch oscillations can also be disrupted by transitions of the system to upper bands. These transitions are analyzed more conveniently by switching from Langevin equations (12) to equations for the density matrix $\rho^{ss'}(q,q')$, averaged over the heat reservoir. We will write the conditions (which follow from these equations) under which the probabilities for interband transitions are small, since these conditions lend themselves to a simple physical interpretation.

Thermal excitation to the higher-line bands is negligible as long as T is below the energy gap between the fundamental value ($s = 0$) and the next higher band ($s = 1$). We therefore have the conditions

$$T\ll\begin{cases}(E_JE_Q)^{1/2}, & E_J\gg E_Q \\ E_J, & E_Q\gg E_J\end{cases}. \tag{44}$$

A current may also cause an excitation to higher-line bands through Zener tunneling.[14] This effect will be kept weak if the quantum of frequency of the higher significant harmonic of the function $q_0(t)$ is smaller than the energy gap between the bands:

$$\hbar\omega_B \ll \begin{cases} (E_J E_Q)^{1/2}, & E_J \gg E_Q \\ E_J^2/E_Q, & E_Q \gg E_J \end{cases}. \tag{45}$$

Finally, a nonzero quasiparticle conductance G leads to a loss of coherence of the states with different values of q over a time on the order of $\tau = C/G$. This effect is unimportant as long as the corresponding "smearing" of states along the energy scale, $\delta E \approx \hbar/\tau$, is much smaller than the energy gap between the lower and upper bands:

$$\hbar\omega_t \ll \begin{cases} (E_J E_Q)^{1/2}, & E_J \gg E_Q \\ E_J, & E_Q \gg E_J \end{cases}, \tag{46}$$

where $\omega_t = (\pi/e)I_t$.

7. CONDITIONS FOR EXPERIMENTAL OBSERVATION

The set of relations (42)–(46), the inequality $|\bar{I}| \gtrsim I_t$, and the conditions for the applicability of our analysis $[eV_t \ll \Delta_{1,2}(T)]$ determine the conditions for an observation of the Bloch oscillations in Josephson junctions (the first step might be to simply measure cvc of the type in Figs. 2 and 3). A comparison of these conditions shows that if $E_J \gg E_Q$ the effect will be very difficult to observe because of the exponentially small width of the lower band, $\delta^{(0)}$, in (34). According to (42) and (46), this circumstance will lead to a slight smearing of the oscillations by fluctuations. On the other hand, if E_J falls below $2E_Q$, restrictions (44) and (45) will loom extremely large. Accordingly, for fixed dimensions of the junction, i.e., for fixed values of C and E_Q, we should choose a Josephson coupling force such that E_J is comparable in magnitude to E_Q, more precisely, such that $E_J \sim 2E_Q$. When this choice is made, the conditions written above reduce to

$$\hbar\tau^{-1} \ll \hbar\omega_B \ll E_Q, \quad T \ll E_Q. \tag{47}$$

To estimate the parameter values, we choose the small but attainable capacitance $C \approx 3 \cdot 10^{-15}$ F, i.e., $E_Q = e^2/2C \approx 5 \cdot 10^{-24}$ J, which corresponds (for example) to tunnel junctions made from lead alloys with an area $S \approx 0.1\ \mu\text{m}^2$. The value $E_J \approx 2E_Q$ is reached at $I_c \approx 30$ nA in such a junction; i.e., we have $j_c = I_c/S \approx 30$ A/cm^2. Such a critical current density is a completely typical value for tunnel junctions. The threshold voltage V_t of such a junction would be $V_t \approx 0.5e/C \approx 30\ \mu$V for $2\Delta/e \approx 3$ mV, so that the adiabatic Hamiltonian (6) which we have used should still be completely suitable. Assigning the product $I_c G^{-1}$ the completely realistic value of 10 mV, we find $G^{-1} \approx 3 \cdot 10^5\ \Omega$. This value is much larger than R_Q, so that conditions (46) hold over the frequency range 10^9 Hz $\lesssim \omega_B/2\pi \lesssim 10^{10}$ Hz at temperatures[5)] $T \ll 0.3$ K.

This region of parameter values may not be the best possible choice. Nevertheless, these estimates clearly reveal the main difficulties which must be overcome in the design of an experiment: It is necessary to fabricate junctions of small area, to use low temperatures, and to keep the technical fluctuations of the current at extremely low values (the effective amplitude of the technical fluctuations must be much lower than the threshold current I_t, which is ~ 0.1 nA in our case). However, all of these difficulties can be overcome at the present state of the art in thin-film technology and in low-temperature experimental techniques, so that we might hope to see an early experimental observation of the effects discussed in this paper.

8. CONCLUSION

We have used the simple adiabatic Hamiltonian in (6) to derive a theory for some extremely specific processes which should be observed in Josephson junctions of very small area at extremely low temperatures. The picture of these processes drawn above, including Bloch oscillations at the frequency in (30), seems to us to be quite realistic for junctions with parameters on the order of those described above.

Nevertheless, we must stress the urgent need for the derivation of a microscopic theory for such effects, based, for example, on the familiar tunneling Hamiltonian,[12] to which some terms $Q^2/2C$ and $-(\hbar/2e)I(t)\varphi$ are added. Such a theory would automatically incorporate the quantization of not only the electric charge carried by the supercurrent but also the charge carried by the quasiparticle component of the current (judging from the results of Refs. 27 and 28, allowance for this quantization will simply expand the region of parameters in which narrow-band Bloch oscillations occur). Finally, the microscopic theory may answer the question of how Bloch oscillations are related to other consequences of the quantization of electric charge which are important at $e^2/C \gtrsim \Delta(T)$ (Ref. 29, for example).

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[1)]In neither the "classical" theory of the Josephson effect[3] nor case (3) are the translational properties of φ unimportant. Actually, a problem arises only when the "macroscopic quantum interference"[16] of different quasistationary states of the phase, differing by translation (8), is important.

[2)]For our purposes it is more convenient to use the representation of expanded, rather than associated, energy bands, so that we are assuming that q is defined on the interval $[-\infty, +\infty]$.

[3)]These fluctuations implicitly incorporate the quantization of the Bloch oscillations, i.e., the existence of a Wannier-Stark ladder.[19]

[4)]Condition (43) is the same as the delocalization condition $\eta < 1/2\pi$ which was derived in Refs. 25 and 26 for an analogous system.

[5)]We recall that G^{-1} is the effective low-frequency resistance, which, in tunnel junctions at low temperatures, is much larger than the normal resistance of these junctions, $R_N \approx \Delta(T)/I_c e$.

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