

Instability induced in an interface of liquid metals by electron wind

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It is shown that when current flows perpendicular to an interface between two immiscible liquid metals, the interface becomes unstable under the action of the electromechanical forces that are concentrated near the surface and are due to disequilibrium between the electrostatic forces acting on the ions and the force of the electrons that drag the ions. The instability manifests itself in generation of capillary waves and in an effective reversal of the sign of the surface-tension coefficient. For a given metal pair, the instability sets in at a certain polarity of the current and when the current density exceeds a threshold value. The feasibility of inducing instability during thermal breakdown of semiconductors that are metallized when molten is considered.

I. INTRODUCTION AND BASIC RESULTS

The present paper is devoted to the stability of a plane interface of immiscible liquid metals carrying an electric current perpendicular to the interface. A known problem, the stability of a plane liquid-metal surface in an electric field, was dealt with by Frenkel.¹ The surface is stable because the electric field E acting on the surface charges produces in the metal an additional negative pressure $E^2/8\pi$. In principle, the same circumstance can cause instability at the interface of liquid metal. This latter instability, however, in a perpendicular electric field, calls for special investigation, for in this case an electric current flows and produces electromechanical (EM) forces that act alongside the already mentioned electrostatic field and will be shown to exceed the latter in a number of interesting cases.

EM forces^{2,3} in a metal are due the imbalance (electron wind) produced, on the electron mean free path l , between the force exerted on the ions by the external field and the force with which the electrons act on the ions in the course of scattering. To understand the importance of the EM forces to the metal interface it suffices to use their estimate, given in Ref. 3, for a current parallel to the interface between crystallites. Per unit area, these forces are of the order of $j_0 p_F/e$, where j_0 is the current density and p_F the Fermi momentum (this estimate will be confirmed in the main part of the paper also for the interface of liquid metals with different current direction). Obviously, these EM forces, which are linear in the field, prevail over the force $E^2/8\pi$ if the field is not too high. They become comparable when $E \propto \sigma p_F/e$, where σ is the electric conductivity of the metal. For typical metals this field is fantastically strong, $\sim 10^{10}$ W/cm.

The EM force density \mathbf{F} is uniquely expressed, according to Ref. 4, in terms of a nonequilibrium increment $f(\mathbf{p}, \mathbf{r})$ to the electron Fermi distribution function f_0 of the electrons:

$$F_\alpha = \frac{\partial \Psi_{\alpha\beta}}{\partial x_\beta} = \frac{\partial}{\partial x_\beta} \frac{2}{(2\pi\hbar)^3} \int \Lambda_{\alpha\beta}(\mathbf{p}) f(\mathbf{p}, \mathbf{r}) d^3p, \quad (1)$$

where $\Lambda_{\alpha\beta}$ is a tensor determined by the electron energy and \mathbf{p} is the electron quasimomentum. The problem of the inter-

face stability calls therefore for simultaneous solution of the hydrodynamic equations with the kinetic equation for the electrons and the electrodynamics equations. The situation is simpler, however, because the electronic processes can be regarded independently of the relatively slow hydrodynamic motions by assuming that the electron subsystem follows the latter adiabatically. Supporting this assumption are the short characteristic electron-relaxation times (of the Maxwell time and the relaxation time in the kinetic equation) compared with the period of the waves on the interface. In addition, the action of the magnetic field H of the current on the electron distribution function can be regarded small enough and neglected, $(e\tau/mc)H \ll 1$ (c is the speed of light and m is the electron effective mass). This limits in turn the current density and the system size dimension transverse to the current. To avoid complications due to the boundary conditions on the side surfaces, we shall nevertheless assume the system to be long compared with the other characteristic lengths, including the hydrodynamic ones.

One more simplification is neglect of the displacement currents, since the metals have high electric conductivity. As a result of these simplifications $f(\mathbf{p}, \mathbf{r})$ is determined by the kinetic equation, by the conduction-current continuity equation, which is equivalent to the local electroneutrality equation⁵

$$\int f(\mathbf{p}, \mathbf{r}) d^3p = 0, \quad (2)$$

and by the equation $\nabla \times \mathbf{E} = 0$.¹ Even in this case, however, the problem is quite complicated. Further simplification and a qualitative interpretation of the instability require a detailed analysis of the EM forces for different current directions relative to the interface. This analysis is presented in Secs. 2 and 3. Here we report briefly the main results for specular reflection of the electrons from the surface.

If the current is perpendicular to the plane surface, the EM forces are also perpendicular to the surface, but are oppositely directed away from its two sides, so that the total force acting on both metals is zero. For each metal, the EM force per unit surface area is estimated at $k_n f_0 p_F/e$, where

k_n is a numerical factor that depends on the conditions of the passage of the electrons through the interface. The mechanical effect of the EM forces constitutes in this case deformation (compression or tension) of the metals in the subsurface layers.

If the current is parallel to the interface, the EM forces act along the surface, likewise in opposite directions on its two sides. The total force acting on the two metals is zero. In this geometry the EM forces lead to relative motion of the metals along the common boundary. The force per unit surface of each metal is estimated at $k_t f_0 p_F / e$, where k_t is a numerical coefficient that depends on the conditions of the passage of the electrons through the interface and is not equal to k_n . Moreover, cases are possible when one of these coefficients is noticeably smaller than the other. For example, if the coefficient T of electron passage through the interface is close to unity, then $k_n \ll k_t$, and the normal component of the EM forces is small. This is the situation for a pair of metals whose electron Fermi energies differ little, and the probabilities of electron scattering by the ions differ substantially, so that the electric conductivities of the metals are unequal. On the contrary, if $T \ll 1$, then $k_n \gg k_t$, and the tangential forces are considerably weaker than the normal ones.

Consider now the waves on the interface. The EM forces for this case are calculated in Sec. 4, and the waves themselves are investigated in Sec. 5. We present below a qualitative description of the main results. In the calculation the EM forces one cannot, strictly speaking, separate the normal and tangential current components, since the current changes direction in a subsurface layer of thickness l . Here, however, in the qualitative description, we shall assume this separation and justify the assumption by the fact that the wavelength is large compared with l in both metals. In the absence of EM forces on the interface, internal capillary waves are present and attenuate weakly if the viscosities of the liquids are small. The EM forces produced by both the normal and tangential components of the current alter the wave dispersion equation on the surface, but the main effect is due to the tangential component. This is shown by the following arguments.

Assume that a low-amplitude plane wave is present on the surface. The action of the EM forces on the surface of the liquid is determined by a layer of thickness l near the surface (in the estimates that follow we assume for simplicity that the parameters of both metals, e.g., the electron mean free paths, the viscosities, densities, etc., are of the same order). We consider first the effect of the tangential component of the current. Along the surface, from the point of its maximum to its minimum, the tangential current changes by an amount $\sim \xi_0 q j_0$ (where ξ_0 is the wave amplitude, q the wave vector, and j_0 the unperturbed current density), so that in a layer of order l this current produces on the two sides of the surface volume forces

$$F^{(t)} \sim \xi_0 q k_t j_0 p_F / e l, \quad (3)$$

that act in a tangential direction and make the liquids move in opposite directions. The normal component of the current along the surface also changes by an amount $\sim \xi_0 q j_0$, as a

result of which a pressure drop $\sim \xi_0 q k_n j_0 p_F / e$ and a corresponding pressure gradient $\nabla P^{(n)} \sim \xi_0 q^2 k_n j_0 p_F / e$ are produced along the surface. When comparing the action of the normal component of the current with the tangential one, $\nabla P^{(n)}$ must be matched to $F^{(t)}$. It can be seen that compared with the tangential current, the effect of the normal component of the current is ql times weaker: $|\nabla P^{(n)} / F^{(t)}| \sim ql k_n / k_t$. We shall therefore focus our attention hereafter on the tangential component, and to get around the question of the normal component, we shall consider a metal pair for which $k_n \ll k_t$.

The action of the EM forces on the surface is obvious—they launch capillary waves at one polarity of the current and contribute to their damping at the other. Indeed, the EM forces are linear in the current and therefore reverse sign when the current polarity is reversed. We choose the polarity such that the tangential components are directed from the minima on the interface to the maxima (see Fig. 1). This will be the case if the unperturbed current directed along the z axis into the lower metal, labeled by the subscript 1, has a higher electric conductivity than the upper (2). Let the electron mobility in metal 1 be higher than in metal 2. The EM forces in metal 1 are directed from the minima to the maxima, and conversely in metal 2. The direction of the EM forces can be easily established from the following considerations. The tangential electric field is the same on both sides of the interface, so that the electrons in metal 1 acquire a higher velocity than those in metal 2. Far from the interface, the forces exerted on the ions by the electric field and by the electrons are in balance. Near the interface, at a distance on the order of l , this balance is upset in such a way that the electrons in metal 1 have a lower directional momentum than in the volume, and therefore the dynamic equilibrium is upset in favor of the electric-field force. In metal 2, on the contrary, the equilibrium shifts towards the electron drag of the ions.

In this situation the EM forces drag each liquid, at any phase of the wave, towards the crest of its surface and increase the amplitudes of the capillary waves with time. At low viscosity, when the liquid vortical-motion penetration length $(\nu/\omega)^{1/2}$ (ν is the kinematic viscosity and ω the frequency) is small compared with the wave lengths, the liquids are set in motion in a layer of thickness $\sim (\nu/\omega)^{1/2}$ at the

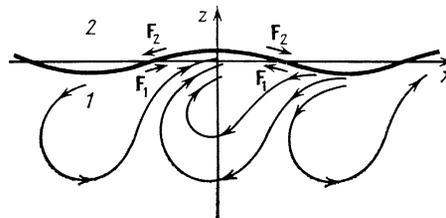


FIG. 1. Distribution of tangential components of the EM forces and of the fluxes in the case when the wave is unstable on the interface of liquid metals 1 and 2. The surface is marked by the heavy line, the forces F_1 and F_2 by short arrows, and the liquid flows are marked by the solid lines for metal 1 only.

surface with an additional velocity $\Delta V_t \sim F^{(t)} l / \eta$ (η is the dynamic viscosity), so that an additional volume of liquid $\sim \Delta V_t (v/\omega)^{1/2}$ is injected into the crests of the waves in each liquid per unit time, meaning an increase $\dot{\zeta}_0 \sim \Delta V_t (v/\omega)^{1/2} q$ in the amplitude per unit time. The instability growth rate can be estimated as the ratio of the increase W of the wave energy per unit time to the wave energy. The wave energy increase is equal to the work performed against the surface-tension forces: $\dot{W} \sim \alpha \dot{\zeta}_0 \dot{\zeta}_0$, where α is the surface-tension coefficient and the wave energy is $\sim \alpha \zeta_0^2 / 2$, so that the growth rate is

$$\omega'' \sim \frac{2\alpha \dot{\zeta}_0 \dot{\zeta}_0}{\alpha \zeta_0^2} \sim \frac{F^{(t)} l}{\zeta_0 \eta} \left(\frac{v}{\omega} \right)^{1/2} q.$$

If account is taken of expression (3) in the capillary-wave dispersion equation $\omega \approx (\alpha/\rho)^{1/2} q^{3/2}$, where ρ is the sum of the densities of the liquids, we obtain

$$\omega'' \sim k_t (j_0 p_F / e) q^{3/2} \rho^{-1/2} \alpha^{-1/2} v^{-1/2}. \quad (4)$$

This estimate differs from the calculated result given in Sec. 5 by only an inessential numerical factor. The EM forces cause self-excitation of the waves when ω'' exceeds the damping decrement as a result of the viscosity. The threshold condition is Eq. (34).

The action of the EM forces on the surface is not limited to the buildup of capillary waves. They also alter effectively the surface-tension coefficient, which is given by Eq. (29). The surface-tension coefficient decreases at the same current polarity at which the capillary waves are built up, and can in principle become negative.

We investigate below the EM forces and the interface waves on the basis of the nearly-free-electron approximation in the theory of liquid metals.⁶ In this approximation the metals have different Fermi energies and different electron-ion scattering probabilities. For the tensor $\Lambda_{\alpha\beta}$ we have the expression

$$\Lambda_{\alpha\beta} = -m v_\alpha v_\beta, \quad (5)$$

where m is approximately equal to the mass of the free electron, and v_α is the electron velocity. We consider first the EM forces for a plane surface.

2. CURRENT PERPENDICULAR TO SURFACE

If the current is perpendicular to the plane interface of the metals, electron scattering by the surface produces an additional electric field (on top of the volume field) needed to preserve continuity of the current. The distribution functions in such a situation were calculated for an intercrystallite interface to determine its electric resistance.⁷ We assume first that the electrons are diffusely scattered from the surface. The problem of interest to us does not differ in principle from that in Ref. 7, and we present the result directly:

$$f(\mathbf{p}, \mathbf{r}) = e \frac{\partial f_0}{\partial \mathcal{E}} \psi(\mathbf{p}, z). \quad (6)$$

Here \mathcal{E} is the electron energy, z is the electric normal to the interface and directed away from the first metal, the quantities pertaining to this metal will be labeled by the sub-

script 1, and those for the second metal by 2. For metal 1 we have

$$\begin{aligned} \psi_1^+ &= E_1 l_1 \left[y - \chi_1(\xi_1) + \frac{1}{y} \int_{-\infty}^{\xi_1} d\xi_1' \chi_1(\xi_1') \exp \frac{\xi_1' - \xi_1}{y} \right], \\ \psi_1^- &= -E_1 l_1 \left[y \left(1 - \exp \frac{\xi_1}{y} \right) + \chi_1(\xi_1) \right. \\ &\quad \left. - \frac{1}{y} \int_{\xi_1}^0 d\xi_1' \chi_1(\xi_1') \exp \frac{\xi_1 - \xi_1'}{y} + A_1 \exp \frac{\xi_1}{y} \right], \quad (7) \end{aligned}$$

E_1 is the field far from the surface, $y = |\cos \theta|$, θ is the polar angle, $\xi_1 = z/l_1$ is the dimensionless coordinate, the superscripts “+” and “-” denote the signs of the corresponding normal components of the electron velocity,

$$A_1 = \frac{2}{3} - 2 \int_{-\infty}^0 d\xi_1' \chi_1(\xi_1') \mathcal{E}_2(-\xi_1'),$$

$\mathcal{E}_n(\xi)$ are integral exponential functions,⁸ $\chi(\xi)$ is a dimensionless potential whose gradient determines the additional electric field ΔE produced by electron scattering from the surface:

$$\Delta E_1 = -E_1 \partial \chi_1(\xi_1) / \partial \xi_1,$$

and $\chi_1(\xi_1)$ is defined by the following integral equation:

$$\begin{aligned} \chi_1(\xi_1) &= \frac{1}{2} \int_{-\infty}^0 d\xi_1' \chi_1(\xi_1') \mathcal{E}_1(|\xi_1 - \xi_1'|) \\ &\quad - \mathcal{E}_2(-\xi_1) \int_{-\infty}^0 d\xi_1' \chi_1(\xi_1') \mathcal{E}_2(-\xi_1') \\ &= \frac{1}{2} \mathcal{E}_3(-\xi_1) - \frac{1}{3} \mathcal{E}_2(-\xi_1). \end{aligned}$$

The solution of this equation was investigated in Ref. 7.

Substituting (6), (7), and (5) in (1) we calculate $\Psi_{\alpha\beta}$. The only nonzero component is Ψ_{zz} :

$$\begin{aligned} \Psi_{zz} &= \frac{j_0 p_F}{e} \left\{ \frac{3}{2} \mathcal{E}_5(-\xi_1) - \mathcal{E}_4(-\xi_1) - \chi_1(\xi_1) \right. \\ &\quad \left. + 3 \mathcal{E}_4(-\xi_1) \int_{-\infty}^0 d\xi_1' \chi_1(\xi_1') \mathcal{E}_2(-\xi_1') \right. \\ &\quad \left. + \frac{3}{2} \int_{-\infty}^0 d\xi_1' \chi_1(\xi_1') \mathcal{E}_3(|\xi_1 - \xi_1'|) \right\}. \quad (8) \end{aligned}$$

Similar expressions are obtained also for metal 2. The volume-force distribution $F_z = \partial \Psi_{zz} / \partial z$ that follows from them is shown schematically in Fig. 2. It can be seen that the forces to the left and right of the interface are directed along the electric field. Direct calculation verifies, however, that the volume forces are exactly offset by the surface force F_S due to the electron scattering from the interface.

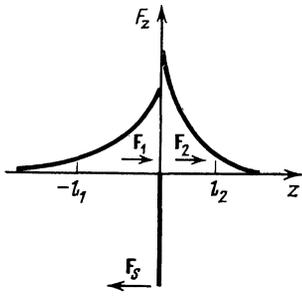


FIG. 2. Distribution of the forces near metal interface perpendicular to the current flow. Diffuse scattering.

The physical meaning of this result is the following. Far from the interface the volume density of the forces is zero because the electric force acting on the ion is balanced by the electron drag force. The electric field near the interface is increased by the additional electron scattering from the surface, so that the electric force prevails over the ion drag by the electrons, and the electron momentum acquired in the additional field is transferred to the surface in the course of the scattering.

An approximate quantitative measure of the forces is the total volume force acting on one of the metals:

$$F_{\text{tot1}} = \int_{-\infty}^0 F_{1z}(z) dz = \Psi_{1zz}(0).$$

According to (8), $F_{\text{tot1}} = k_n j_0 p_F / e$, where k_n is a numerical factor that depends on the variation of the potential $\chi(\xi)$. In order of magnitude, $k_n \sim 10^{-1}$.

It must be noted, however, that more realistic for the interface of liquid metals is specular scattering of the electrons, since the small-scale roughnesses of the surface, which are primarily responsible for the diffuse scattering, are smoothed-out by the surface-tension forces.²⁾ Still, solution of the problem with diffuse scattering is expedient, since it can be carried through to conclusion. For specular reflection, the problem is much more complicated and can be solved only in special cases. We shall not dwell on it, since the conclusions obtained for diffuse scattering remain in force here, too. Namely, the total EM force acting on both metals is zero—this is essentially the consequence of electroneutrality. The surface force and the volume forces of the metals in contact balance one another. Depending on the current direction and on the conditions of passing through the boundary they either compress or distend the metals in the subsurface layer. The order of magnitude of the force is $k_n j_0 p_F / l$, where k_n is a factor that depends on the scattering conditions and is usually small.

An important particular case in what follows is specular scattering from the interface of metals that have only different probabilities of electron scattering by the ions, while their Fermi surfaces are identical. This is a convenient model system for the analysis of waves on an interface. In such a system both metals have the same electron distribution function, equal to its value far from the interface:

$\psi_1 = E_1 l_1 \cos \theta = E_2 l_2 \cos \theta = \psi_2$. Near the interface, over distances l_1 and l_2 , the waves are not distorted—the field changes from E_1 to E_2 over the screening length. There are no EM forces.

3. CURRENT PARALLEL TO SURFACE

When the current is parallel to the plane interface the electric field is independent of the coordinates and to find the distribution function it suffices to solve the kinetic equation. We direct, as before the z axis normal to the interface from metal 1 to metal 2 (Fig. 3a). A similar problem was solved by Kagaov and Fiks³ for crystallite interfaces. In our case the electron dispersion laws $\mathcal{E}(\mathbf{k})$ are different (Fig. 3b):

$$\mathcal{E}_1(\mathbf{k}_1) = \hbar^2 k_1^2 / 2m, \quad \mathcal{E}_2(\mathbf{k}_2) = \Delta + \hbar^2 k_2^2 / 2m,$$

where $\Delta = \mathcal{E}_{F1} - \mathcal{E}_{F2}$ is the difference between the Fermi levels of metals 1 and 2.

We assume the scattering from the interface to be specular. Since the system is homogeneous along the surface, the refraction of the electron wave by the surface is determined by the equations for the conservation of the tangential components of the momentum and of the energy, which lead to the following connection between the polar angles θ_1 and θ_2

$$\sin \theta_2 = \sin \theta_1 / \sin \theta_m,$$

θ_m is the maximum incidence angle at which passage of a wave into metal 1 is possible (Fig. 3c). For the subsequent significant transitions on the Fermi level we have $\sin \theta_m = \mathcal{E}_{F2} / \mathcal{E}_{F1}$. The coefficient of transmission through the interface is

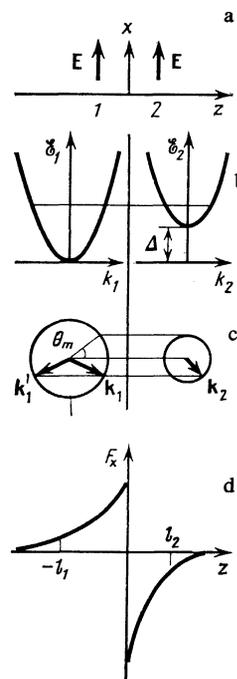


FIG. 3. EM forces in the case of a current parallel to the interface: a—arrangement of metals; b—dispersion laws; c—equal-energy surfaces (k_1 , k_1' , and k_2 are the wave vectors of the incident, reflected, and transmitted waves); d—distribution of the forces.

$$T = \begin{cases} \frac{4 \cos \theta_1 (\sin^2 \theta_m - \sin^2 \theta_1)^{1/2}}{[\cos \theta_1 + (\sin^2 \theta_m - \sin^2 \theta_1)^{1/2}]^2}, & \theta_1 \leq \theta_m, \\ 0, & \theta_1 \geq \theta_m. \end{cases}$$

Solving the kinetic equation with the following boundary conditions for ψ^+ and ψ^- :

$$\psi_1^- = (1-T)\psi_1^+ + T\psi_2^-, \quad \psi_2^+ = T\psi_1^+ + (1-T)\psi_2^- \quad (9)$$

and substituting the result in (1), we find the density of the volume forces:

$$F_{1z} = 3eEn \left(1 - \frac{\tau_2}{\tau_1}\right) \int_0^{\sin \theta_m} dx \frac{x^3 (\sin^2 \theta_m - x^2)^{1/2}}{[(1-x^2)^{1/2} + (\sin^2 \theta_m - x^2)^{1/2}]^2} \times \exp \frac{\xi_1}{(1-x^2)^{1/2}}$$

for $z < 0$ and

$$F_{2z} = 3eEn \left(1 - \frac{\tau_1}{\tau_2}\right) \int_0^{\sin \theta_m} dx \frac{x^3 (1-x^2)^{1/2}}{[(1-x^2)^{1/2} + (\sin^2 \theta_m - x^2)^{1/2}]^2} \times \exp \left[- \frac{\xi_2}{(\sin^2 \theta_m - x^2)^{1/2}} \right]$$

for $z > 0$, where τ_1 and τ_2 are the relaxation times, and n is the electron density in metal 1, where \mathcal{E}_F is larger.

It can be seen that the sign of the forces is determined by the ratio τ_1/τ_2 , i.e., by the ratio of the electron mobilities. The metal with the lower mobility is acted upon by a force directed along the electron motion—counter to the current, and conversely for the metal with the higher mobility. A plot of the distribution of the forces is shown in Fig. 3d. The total forces

$$F_{\text{tot}1} = \int_{-\infty}^0 F_1 dz, \quad F_{\text{tot}2} = \int_0^{\infty} F_2 dz,$$

acting on each of the metals are equal and opposite:

$$F_{\text{tot}1} = -F_{\text{tot}2} = eEnl_1 (1 - \tau_2/\tau_1) g(\sin \theta_m), \quad (10)$$

where

$$g(\sin \theta_m) = 3(1 - \sin^2 \theta_m) \int_0^{\sin \theta_m} dx \frac{(\sin^2 \theta_m - x^2) x^2}{(1-x^2)^3 (1+x)}.$$

A plot of $g(\sin \theta_m)$ is shown in Fig. 4. The function reaches its maximum value $3/16$ at $\sin \theta_m = 1$, i.e., when the Fermi energies of the metals in contact are equal and the interface does not scatter the electron waves. With increasing difference $\mathcal{E}_{F1} - \mathcal{E}_{F2}$ the electron drag weakens rapidly. The physical reason is that T decreases when the difference $\mathcal{E}_{F1} - \mathcal{E}_{F2}$ is increased, i.e., transitions of electrons between the metals become more difficult, and it is just these transitions that produce the electron-drag forces.

According to (10), an estimate of the total tangential force can be represented in the same form as the normal force: $F_{\text{tot}} = k_t j_0 p_F / e$, where k_t is a factor equal to $(1 - \tau_2/\tau_1) g(\sin \theta_m)$ for metal 1 and to $(\tau_1/\tau_2 - 1) (\mathcal{E}_{F1}/\mathcal{E}_{F2})^2 \times g(\sin \theta_m)$ for metal 2.

4. EM FORCES FOR WAVES ON THE INTERFACE

The functions $\psi_1(\mathbf{p}, \mathbf{r})$ and $\psi_2(\mathbf{p}, \mathbf{r})$ and the electric field next to a nonplanar interface are determined by the kinetic equation

$$\mathbf{v} \partial \psi_{1,2} / \partial \mathbf{r} + \psi_{1,2} / \tau_{1,2} = \mathbf{E}(\mathbf{r}) \mathbf{v} \quad (11)$$

and by the equation that follows from (2) and (6)

$$\int \psi_{1,2}(\mathbf{p}, \mathbf{r}) \sin \theta \, d\theta \, d\beta = 0, \quad (12)$$

where β is the azimuthal angle. The solution of these equations must satisfy the boundary condition (9) on the surface $z = \zeta(x, y)$ and must be bounded as $|z| \rightarrow \infty$.

We solve this problem below for plane linear waves $z = \zeta(x) = \zeta_0 \cos qx$ on the interface, assuming the waves to be long compared with the mean free path ($ql_1, ql_2 \ll 1$) and the amplitude to be small compared with l_1 and l_2 . To simplify the problem further, the calculation is for a model pair of metals having different relaxation times but equal Fermi energies. The boundary condition (9) is then simplified because now $T = 1$ and we have consequently on the interface

$$\psi_1^+ = \psi_2^+, \quad \psi_1^- = \psi_2^-. \quad (13)$$

The external field is assumed parallel to the z axis that is perpendicular to the unperturbed surface. Linearizing (11), we separate in the sought quantities ψ_1 , ψ_2 and \mathbf{E} the increments that are due to the wave and depend on x like $\exp(iqx)$:

$$\psi = \psi_0 + \begin{cases} \Delta \psi_1 \\ \Delta \psi_2 \end{cases}, \quad \mathbf{E} = \begin{cases} \mathbf{E}_1 - \partial \Delta \varphi_1 / \partial \mathbf{r} \\ \mathbf{E}_2 - \partial \Delta \varphi_2 / \partial \mathbf{r} \end{cases}.$$

The subscripts "1" and "2" pertain here to metals that occupy the half-spaces $z < \zeta$ and $z > \zeta$; E_1 and E_2 are the electric field strengths in metals 1 and 2 in the absence of a wave; they are connected with one another and with the unperturbed current density by the relation

$$E_1 l_1 = E_2 l_2 = j_0 p_F / e^2 n, \quad \psi_0 = (j_0 p_F / e^2 n) \cos \theta,$$

$\Delta \varphi_1$ and $\Delta \varphi_2$ are the perturbations of the potentials in metals 1 and 2. For the amplitudes $\Delta \psi_1$, $\Delta \psi_2$, $\Delta \varphi_1$ and $\Delta \varphi_2$ that depend only on z we have from (11) and (13)

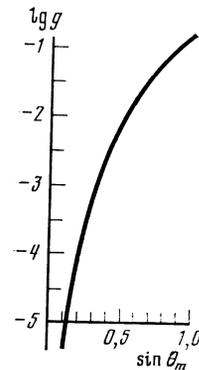


FIG. 4.

$$\Delta\psi_1^+(\xi_1) = -\Delta\varphi_1(\xi_1) + \frac{1}{y} \int_{-\infty}^{\xi_1} d\xi_1' \Delta\varphi_1(\xi_1') \exp[\alpha_1(\xi_1' - \xi_1)], \quad (14)$$

$$\Delta\psi_1^-(\xi_1) = -\Delta\varphi_1(\xi_1) + \frac{1}{y} \int_{\xi_1}^0 d\xi_1' \Delta\varphi_1(\xi_1') \exp[-\alpha_1(\xi_1' - \xi_1)]$$

$$+ \left[\Delta\varphi_1(0) - \Delta\varphi_2(0) + \frac{1}{y} \int_0^{\infty} d\xi_2' \Delta\varphi_2(\xi_2') \exp(-\alpha_2\xi_2') \right] \exp(\alpha_1\xi_1),$$

where we have introduced the dimensionless coordinates $\xi_1 = z/l_1$ and $\xi_2 = z/l_2$ for $z < 0$ and $z > 0$, respectively, and designated

$$\alpha_{1,2} = [1 + ie_{1,2}(1-y^2)^{1/2} \cos \beta] / y,$$

$\varepsilon_{1,2} = ql_{1,2}$, ε_1 and ε_2 are small parameters of the problem. The expressions that connect $\Delta\psi_2$ with $\Delta\varphi_1$ and $\Delta\varphi_2$ differ from (14) by the replacement $1 \rightleftharpoons 2$ and by the reversal of the signs of ξ .

Substituting $\Delta\psi_1$ and $\Delta\psi_2$ in (12) and taking the continuity of the potential on the interface into account, we obtain a system of two equations for the potentials $\Delta\varphi_1$ and $\Delta\varphi_2$. One of them is of the form

$$\Delta\varphi_1(\xi_1) - \frac{1}{2} \int_{-\infty}^0 d\xi_1' \Delta\varphi_1(\xi_1') \int_0^1 \frac{dy}{y} \exp\left(-\frac{|\xi_1' - \xi_1|}{y}\right) \times J_0 \left[\varepsilon_1 |\xi_1' - \xi_1| \frac{(1-y^2)^{1/2}}{y} \right] - \frac{1}{2} \int_0^{\infty} d\xi_2' \Delta\varphi_2(\xi_2') \int_0^1 \frac{dy}{y} \exp\left(\frac{\xi_1 - \xi_2'}{y}\right) \times J_0 \left[(\varepsilon_2 \xi_2' - \varepsilon_1 \xi_1) \frac{(1-y^2)^{1/2}}{y} \right] = -\frac{1}{2} (E_2 - E_1) \zeta_0 \int_0^1 dy \exp\left(\frac{\xi_1}{y}\right) \times J_0 \left[-\varepsilon_1 \xi_1 \frac{(1-y^2)^{1/2}}{y} \right] \quad (15)$$

(J_0 is a Bessel function), and in the second the above-mentioned subscripts and signs are interchanged.

We seek for (15) a solution that vanishes far from the interface. This can be conveniently done by writing out explicitly the asymptotic forms of $\Delta\varphi_{1,2}$ as $|\xi_{1,2}| \rightarrow \infty$. These forms can be easily seen to be described by a Laplace equation and are given by $\Delta\varphi_{1,2}(\xi_{1,2}) \propto \exp(\pm \varepsilon_{1,2} \xi_{1,2})$. We put

$$\Delta\varphi_1(\xi_1) = A_1 \exp(\varepsilon_1 \xi_1) + \Delta\bar{\varphi}_1(\xi_1), \quad (16)$$

$$\Delta\varphi_2(\xi_2) = A_2 \exp(-\varepsilon_2 \xi_2) + \Delta\bar{\varphi}_2(\xi_2).$$

In place of the coordinates ξ_1 and ξ_2 we introduce a single coordinate ξ that coincides with ξ_1 at $z < 0$ and with ξ_2 at $z > 0$. We introduce also a single function $\Delta\bar{\varphi}(\xi) = \Delta\bar{\varphi}_1(\xi_1)\Theta(-\xi) + \Delta\bar{\varphi}_2(\xi_2)\Theta(\xi)$, where $\Theta(\xi)$ is the unit function.

The equation for $\Delta\bar{\varphi}(\xi)$ follows from (15) and (16). Taking it into account in the calculation of the integrals that $\varepsilon_{1,2} \ll 1$, we obtain, accurate to $\varepsilon_{1,2}^2$,

$$\Delta\bar{\varphi}(\xi) - \frac{1}{2} \int_{-\infty}^{\infty} d\xi' \Delta\bar{\varphi}(\xi') [K_{11}(|\xi - \xi'|)\Theta(-\xi)\Theta(-\xi') + K_{22}(|\xi - \xi'|)\Theta(\xi)\Theta(\xi') + K_{12}(\xi, \xi')\Theta(-\xi)\Theta(\xi') + K_{21}(\xi, \xi')\Theta(\xi)\Theta(-\xi')] = \frac{1}{2} (E_2 - E_1) \zeta_0 \{ -[\mathcal{E}_2(-\xi) - \frac{1}{4}\varepsilon_1^2 \xi^2 (\mathcal{E}_0(-\xi) - \mathcal{E}_2(-\xi))]\Theta(-\xi) + [\mathcal{E}_2(\xi) - \frac{1}{4}\varepsilon_2^2 \xi^2 (\mathcal{E}_0(\xi) - \mathcal{E}_2(\xi))]\Theta(\xi) \} + \frac{1}{2} (A_2 - A_1) [\mathcal{E}_2(-\xi)\Theta(-\xi) - \mathcal{E}_2(\xi)\Theta(\xi)] - \frac{1}{2} (\varepsilon_1 A_1 + \varepsilon_2 A_2) \mathcal{E}_3(|\xi|) + \frac{1}{8} (A_2 - A_1) \{ -\varepsilon_1^2 \xi^2 [\mathcal{E}_0(-\xi) - \mathcal{E}_2(-\xi)]\Theta(-\xi) + \varepsilon_2^2 \xi^2 [\mathcal{E}_0(\xi) - \mathcal{E}_2(\xi)]\Theta(\xi) \} + \frac{1}{4} (\varepsilon_2 A_2 - \varepsilon_1 A_1) \{ \varepsilon_1 \xi [\mathcal{E}_1(-\xi) - \mathcal{E}_3(-\xi)]\Theta(-\xi) + \varepsilon_2 \xi [\mathcal{E}_1(\xi) - \mathcal{E}_3(\xi)]\Theta(\xi) \} + \frac{1}{4} (\varepsilon_2^2 A_2 - \varepsilon_1^2 A_1) \times \{ -[\mathcal{E}_2(-\xi) - 3\mathcal{E}_4(-\xi)]\Theta(-\xi) + [\mathcal{E}_2(\xi) - 3\mathcal{E}_4(\xi)]\Theta(\xi) \}, \quad (17)$$

where

$$K_{11,22} = \int_0^1 \frac{dy}{y} \exp\left[-\frac{|\xi - \xi'|}{y}\right] J_0 \left[\varepsilon_{1,2} |\xi - \xi'| \frac{(1-y^2)^{1/2}}{y} \right],$$

$$K_{12,21} = \int_0^1 \frac{dy}{y} \exp\left[\pm \frac{\xi - \xi'}{y}\right] J_0 \left[(\varepsilon_{2,1} \xi' - \varepsilon_{1,2} \xi) \frac{(1-y^2)^{1/2}}{y} \right].$$

The function $\Delta\bar{\varphi}(\xi)$ describes the deviations of the potential from the asymptotic behavior over a distance scale on the order of $l_{1,2}$, where $|\xi| \sim 1$, i.e., the deviations due to the finite value of the mean free path. This circumstance allows us to simplify Eq. (17). First, it is likewise convenient to separate in the coefficients A_1 and A_2 of (16) the parts caused by the fact that the mean free path is finite. The order of magnitude of these parts is determined by the parameters ε_1 and ε_2 . We put

$$A_1 = -A\zeta_0/l_1 + A_1^{(4)}, \quad A_2 = A\zeta_0/l_2 + A_2^{(4)},$$

$$A = \frac{j_0 p_F}{e^2 n} \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2},$$

where σ_1 and σ_2 are the electric conductivities of the metals. The quantity A determines the variation of the potential in the limit $ql_{1,2} = 0$. Second, the kernel of the integral in (17) can be simplified:

$$K(\xi, \xi') \approx \mathcal{E}_1(|\xi - \xi'|).$$

Changing next to the dimensionless quantities

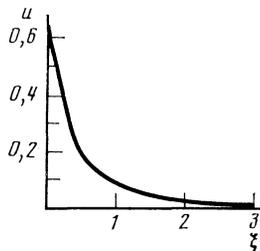


FIG. 5.

$u(\xi) = 4\Delta\varphi(\xi) / (\varepsilon_1 - \varepsilon_2) q \zeta_0 A$, $\bar{A}_{1,2} = 2A_{1,2}^{(1)} / (\varepsilon_1 - \varepsilon_2) q \zeta_0 A$ and leaving out terms of order higher than $\varepsilon_{1,2}^2$, we obtain an equation with a difference kernel

$$u(\xi) - \frac{1}{2} \int_{-\infty}^{\infty} d\xi' u(\xi') \mathcal{E}_1(|\xi - \xi'|) = \mathcal{E}_2(|\xi|) - 3\mathcal{E}_1(|\xi|) + (\bar{A}_2 - \bar{A}_1) [\mathcal{E}_2(-\xi) \Theta(-\xi) - \mathcal{E}_2(\xi) \Theta(\xi)] + (\varepsilon_1 \bar{A}_1 + \varepsilon_2 \bar{A}_2) \mathcal{E}_3(|\xi|),$$

which can be easily solved by taking the Fourier transform. The solution of this equation tends to zero at infinity only if

$$\bar{A}_1 = \bar{A}_2 \approx -3/4(\varepsilon_1 + \varepsilon_2).$$

In this case

$$u(\xi) = \frac{1}{\pi} \int_0^{\infty} dt \frac{(3+t^2) \ln(1+t^2) - 3/2 t(t + \arctg t)}{t^3(t - \arctg t)} \cos(\xi t).$$

A plot of the function $u(\xi)$ is shown in Fig. 5.

The final expression for the potential is

$$\Delta\varphi_1 = -\frac{\zeta_0 A}{l_1} \left[\left(1 + \frac{3}{8} \varepsilon_1 \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right) \exp(\varepsilon_1 \xi_1) - \frac{1}{4} \varepsilon_1 (\varepsilon_1 - \varepsilon_2) u(\xi_1) \right]. \quad (18)$$

The expression for $\Delta\varphi_2$ differs from (18) in that the subscripts 1 and 2 are interchanged and the signs of ξ and A are reversed.

This expression must be used next in the calculation of the function ψ from (14), and the density of the forces from (1), (5), and (6). The calculation yields the following for metal 1:

$$F_{1z} = \frac{3}{2} i \zeta_0 q \frac{enA}{l_1} \left\{ \left[1 + \frac{3}{16} \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon_1 + \varepsilon_2} \right] \times [\mathcal{E}_2(-\xi_1) - \mathcal{E}_1(-\xi_1)] + (\varepsilon_1 - \varepsilon_2) [\mathcal{E}_3(-\xi_1) - \mathcal{E}_5(-\xi_1)] \right\} e^{i q x}, \quad (19)$$

$$F_{1z} = \frac{3}{2} \zeta_0 q \frac{enA}{l_1} \left\{ \varepsilon_1 \xi_1 [\mathcal{E}_2(-\xi_1) - \mathcal{E}_4(-\xi_1)] + \frac{\varepsilon_1 + \varepsilon_2}{2} [3\mathcal{E}_3(-\xi_1) - \mathcal{E}_3(-\xi_1)] + \frac{3}{8} (\varepsilon_1 - \varepsilon_2) \mathcal{E}_4(-\xi_1) \right\}$$

$$+ \frac{\varepsilon_1 - \varepsilon_2}{4} \left[-\frac{2}{3} u'(\xi_1) + \int_0^{\infty} d\xi_2' u(\xi_2') \mathcal{E}_2(\xi_2' - \xi_1) + \int_{-\infty}^0 d\xi_1' u(\xi_1') \mathcal{E}_2(|\xi_1' - \xi_1|) \right] e^{i q x}.$$

The equations for the forces in metal 2 are obtained from (19) by using the same substitutions as in the equations for the potential.

5. SPECTRUM OF WAVES ON THE INTERFACE

When describing the motion of metals acted upon by EM forces it is necessary, generally speaking, to take into account the magnetic field of the currents, i.e., use a system of magnetohydrodynamics equations in which one adds in the Navier-Stokes equation the density \mathbf{F} of the EM forces, which depends on the shape of the interface. Assuming the liquid to be incompressible, we have

$$\text{div } \mathbf{H}_i = 0, \quad (20)$$

$$\text{div } \mathbf{V}_i = 0, \quad (21)$$

$$\frac{\partial \mathbf{H}_i}{\partial t} - \frac{c^2}{4\pi\sigma_i} \nabla^2 \mathbf{H}_i = (\mathbf{H}_i \nabla) \mathbf{V}_i - (\mathbf{V}_i \nabla) \mathbf{H}_i, \quad (22)$$

$$\frac{\partial \mathbf{V}_i}{\partial t} + (\mathbf{V}_i \nabla) \mathbf{V}_i = -\frac{1}{\rho_i} \nabla \left(P_i + \frac{H_i^2}{8\pi} \right) + \frac{1}{\rho_i} \mathbf{F}_i + \frac{1}{4\pi\rho_i} (\mathbf{H}_i \nabla) \mathbf{H}_i + \nu_i \nabla^2 \mathbf{V}_i, \quad (23)$$

where $i = 1$ or 2 , V_i is the velocity of the liquid, ρ_i is the density, and P_i is the pressure. We neglect the gravitational force, assuming the capillary constant to be large compared with the other characteristic lengths.

We consider linear plane waves of the form $\exp[i(qx - \omega t)]$, for which F_i is given by (19). When linearized, Eqs. (22) and (23) take the form

$$-i\omega \Delta \mathbf{H}_i - \frac{c^2}{4\pi\sigma_i} \nabla^2 \Delta \mathbf{H}_i = (\mathbf{H}_0 \nabla) \mathbf{V}_i - (\mathbf{V}_i \nabla) \mathbf{H}_0, \quad (24)$$

$$-i\omega \mathbf{V}_i = -\frac{1}{\rho_i} \nabla \left[P_i + \frac{(\mathbf{H}_0 \Delta \mathbf{H}_i)}{4\pi} \right] + \frac{1}{\rho_i} \mathbf{F}_i + \frac{1}{4\pi\rho_i} [(\mathbf{H}_0 \nabla) \Delta \mathbf{H}_i + (\Delta \mathbf{H}_i \nabla) \mathbf{H}_0] + \nu_i \nabla^2 \mathbf{V}_i. \quad (25)$$

Here \mathbf{H}_0 is the stationary magnetic field and $\Delta \mathbf{H}_i$ is the magnetic field of the wave. Consider the case when \mathbf{H}_0 is perpendicular to the wave-propagation direction. This case is most natural as the limiting case of waves of cylindrical geometry at large distances from the center. We then have in the right-hand side of (25) $(\mathbf{H}_0 \nabla) \Delta \mathbf{H}_i + (\Delta \mathbf{H}_i \nabla) \mathbf{H}_0 = 0$ and in (24) we have $(\mathbf{H}_0 \nabla) \mathbf{V}_i = 0$. The terms $(\mathbf{V}_i \nabla) \mathbf{H}_0$ and $i\omega \Delta \mathbf{H}_i$ in (24) can be regarded as small in view of the low frequency of the hydrodynamic processes; an appropriate criterion will be given below. It turns out as a result that the magnetic field is described by the equation $\nabla^2 \Delta \mathbf{H}_i = 0$, which does not depend on the equations of motion of the liquid.

The boundary conditions for this equation are likewise

independent of V . They follow from the fact that $\Delta \mathbf{H}$ is bounded as $|z| \rightarrow \infty$ and that the tangential components of $\Delta \mathbf{H}$ and of the electric field are continuous. The latter contain the velocity of the liquid:

$$\mathbf{E} = \mathbf{j}/\sigma - (1/c) [\mathbf{V} \times \mathbf{H}],$$

but here, too, the term $(1/c) \mathbf{v} \times \mathbf{H}$ can be omitted. The magnetic field calculated in this manner is parallel to the y axis and is equal to

$$\Delta H_{1,2} = \frac{4\pi i}{c} \frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1} j_0 \xi_0 \mathbf{e}_x \mp qz. \quad (26)$$

The current-density distribution obtained from (2.6) by using the equation $\Delta \mathbf{j} = (c/4\pi) \text{curl } \Delta \mathbf{H}$, is valid far from the surface ($|z| \gg l_1, l_2$) and agrees with the asymptotic distribution obtained for $\Delta \mathbf{j}$ from the conduction-current continuity condition used in Sec. 4. The conditions that allow us to neglect the motion of the liquid when the currents are calculated are the inequalities

$$\omega \ll c^2 q^2 / 4\pi \sigma_{1,2}, \quad c^2 q^2 / 2\pi \sigma_{1,2} R, \quad (27)$$

where R is the conductor radius transverse to the current. With the wave spectrum that will be obtained below [Eq. (33)], the inequalities (27) impose on q the restriction

$$\frac{4\pi \sigma_{1,2}}{c^2} \left(\frac{\alpha}{2\rho} \right)^{1/2} \ll q^{1/2} \ll \frac{c^2}{2\pi \sigma_{1,2} R} \left(\frac{2\rho}{\alpha} \right)^{1/2},$$

which is well satisfied for typical metals. Thus, at $\sigma = 3 \cdot 10^4 \Omega^{-1} \cdot \text{cm}^{-1}$, $\alpha = 10^3 \text{ dyn/cm}$, $\rho = 10 \text{ g/cm}^3$, and $R = 0.1$ this means that $10^{-5} \text{ cm}^{-1} \ll q \ll 10^8 \text{ cm}^{-1}$. We recall that, besides condition (27), neglect of the motion of the liquid imposes a condition on the electron velocity, viz., it must exceed considerably the velocity of the liquid. This condition reduces to the inequality

$$\omega \ll q j_0 / en.$$

Once the magnetic field is determined, the waves on the interface are described by Eqs. (21) and (25), which with allowance for the stipulated simplification and with change of notation $P_i^* = P_i + H^2/8\pi$ reduce to the hydrodynamic equations. The latter are subject to the following boundary conditions on the interface:

$$V_{1x} = V_{2x}, \quad V_{1z} = V_{2z} = \partial \xi / \partial t,$$

$$\eta_1 (\partial V_{1x} / \partial z + \partial V_{1z} / \partial x) - \eta_2 (\partial V_{2x} / \partial z + \partial V_{2z} / \partial x) = 0,$$

$$P_1^* - 2\eta_1 \partial V_{1z} / \partial z - P_2^* + 2\eta_2 \partial V_{2z} / \partial z + \alpha \partial^2 \xi / \partial x^2 = 0.$$

The problem has three characteristic lengths that determine the changes of the pressure and velocity in each of the liquids: the electron mean free path l_i , the wavelength q^{-1} , and the length $(\gamma_i q)^{-1}$, where $\gamma_i = [1 - i\omega/\nu q^2]^{1/2}$, which determines the depth of penetration of the vortical motion. When we determine solutions that do not vanish at infinity, we assume that $q > 0$ and $\text{Re } \gamma_i > 0$. The dispersion equation for the waves on the interface is

$$i\omega [(\gamma_1 + b\gamma_2)(\gamma_1^2 + b\gamma_2^2) + (\gamma_1 + b\gamma_2)^2 + b(\gamma_1 + \gamma_2)^2 + 3(\gamma_1 + b^2\gamma_2) - b(\gamma_1 + \gamma_2) - (b-1)^2] + \frac{2enAl_1}{5\eta_1} q \left(b + \frac{\sigma_2}{\sigma_1} \right) (\gamma_1 \gamma_2 - 1)$$

$$-\alpha^* \eta_1 q (\gamma_1 + b\gamma_2 + 1 + b)$$

$$- \frac{enAl_1^2}{4\eta_1} q^2 \left[\gamma_1 (\gamma_1 + 1) (b\gamma_2 + 1) + \frac{\sigma_2^2}{\sigma_1^2} \gamma_2 (\gamma_2 + 1) (\gamma_1 + b) \right] = 0, \quad (28)$$

where

$$b = \eta_2 / \eta_1, \quad \alpha^* = \alpha^{-2} / 5enA(l_1 + l_2) = \alpha - 0.4(j_0/e)p_F(l_1 - l_2). \quad (29)$$

In the derivation of (28) we used essentially the fact that ε_i is small, not only compared with unity but also compared with $|\gamma_i|^{-1}$.

We assume next for simplicity that the metals are mechanically equivalent and put $\rho_1 = \rho_2 = \rho$ and $\eta_1 = \eta_2 = \eta$. The dispersion equation is then simpler. It is convenient to formulate it for the quantity $\gamma(q)$,

$$\gamma(q) = [1 - i\omega/\nu q^2]^{1/2}, \quad \text{Re } \gamma > 0, \quad (30)$$

rather than for $\omega(q)$. Eliminating ω from (28) with the aid of (30), we have

$$\gamma^4 + \gamma^3 - \gamma^2(1 + \delta) - \gamma(1 + \delta + Q/q) + (a - Q)/q = 0, \quad (31)$$

where

$$Q = j_0 p_F (l_1 - l_2) / 10e\eta\nu, \quad a = \alpha / 2\eta\nu,$$

$$\delta = 5/8 Q (l_1^2 + l_2^2) / (l_1 + l_2).$$

Case of low viscosity

We consider first the case of low viscosity, when $|\gamma| \gg 1$ and the penetration length of the vortical motion is small compared with the wavelength. In addition, we confine ourselves to external fields that are not too strong, such that $Q \ll a|\gamma|^{-1}$. Under these conditions Eq. (31) is solved by iteration. In the zeroth approximation it leads to the dispersion equation for the inner capillary waves:

$$\omega(q) = \nu a^{1/2} q^{3/2}. \quad (32)$$

In the next approximation in $|\gamma|^{-1}$ and $Q|\gamma|/a$ we have

$$\omega(q) = \nu a^{1/2} q^{3/2} \left[1 - \frac{(aq)^{1/2} + Q}{2^{1/2} a^{3/2}} q^{-1/2} \right] - i\nu q^{5/4} \frac{(aq)^{1/2} - Q}{2^{1/2} a^{3/2}}. \quad (33)$$

It can be seen that the imaginary part of the frequency ω'' is always negative at $Q = 0$. If $Q < 0$, then ω'' is positive in the interval $0 < q < q_c = Q^2/a$ and reaches a maximum at $q = q_m = 25/49 q_c$ (Fig. 6). Thus, at $Q > 0$ the surface is unstable to waves with wave vector $q \sim Q^2/a$. The largest growth rate is $\omega_m' = 5^{5/2} \cdot 7^{-7/2} \cdot 2^{-1/2} \nu Q^{7/2} a^{-3/2}$ for waves with $q_m = 25/49 Q^2/a$; their frequency is $\omega_m'' = (5/7)^3 \nu Q^3/a$.

We present now the results in dimensional units. The instability sets in when the external electric field is directed away from the metal with the larger electric conductivity to the one with the lower:

$$q_m = \frac{1}{98} \frac{\rho}{\alpha} \left[\frac{j_0 p_F (l_1 - l_2)}{e\eta} \right]^2,$$

$$\omega_m' = 2 \cdot \left(\frac{5}{7} \right)^3 \cdot 10^{-3} \frac{\rho}{\alpha} \left[\frac{j_0 p_F (l_1 - l_2)}{e\eta} \right]^3,$$

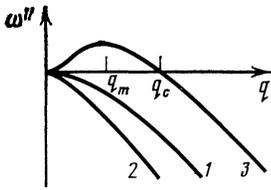


FIG. 6. Growth rate of inner capillary waves $\omega''(q)$: 1) in the absence of current, 2) at $Q < 0$, 3) at $Q > 0$.

$$\omega_m'' = \frac{1}{6860 \cdot 14^{1/2} \eta^{3/2} \alpha^{3/2}} \left[\frac{j_0 p_F (l_1 - l_2)}{e} \right]^{1/2}.$$

The wave with wave vector q turns out to be unstable at

$$j_0 > j_{0c} = 5e(2\alpha\eta\nu q)^{1/2} / p_F(l_1 - l_2). \quad (34)$$

The presence of a limiting value q_c to which $\omega''(q) > 0$ indicates that instability is excited in a bounded surface if the external field exceeds the critical, namely, $Q > Q_c \sim (a/R)^{1/2}$, where R is the transverse dimension.

The conditions for the applicability of Eq. (33) are determined by the assumptions used in its derivation: $1 \ll |\gamma| \ll \epsilon_{1,2}^{-1}$ and $Q/a \ll (q/a)^{1/2} \ll 1$. These equations contain ω and q explicitly. Eliminating ω with the aid of (32), we obtain the inequalities

$$Q/a \ll (q/a)^{1/2} \ll \max[1, (al_1)^{-1/2}, (al_2)^{-1/2}], \quad (35)$$

which determines the validity of (33) in terms of q . As $q \rightarrow 0$ the left-hand inequality of (35) is violated. In this limit, or more accurately at $q \ll Q$, $Q^4/(a-Q)^3$, Eq. (31) also has an unstable solution

$$\omega(q) \approx i\nu q^2 \{ [(a-Q)/Q]^2 - 1 \}.$$

It is however, of little interest because of the extremely small value of the growth rate.

The cause of the instability is that the EM forces excite capillary waves. It can be seen from (32) and (33), within the framework of the approximations employed the function $\omega'(q)$ differs little from the spectrum of the capillary waves, but the EM forces alter substantially the damping of the capillary waves, which become self-excited at $Q > (aq)^{1/2}$, with the growth rate considerably smaller than the frequency.

The EM forces produce vortical streams in liquids. The spatial velocity distribution obtained from Eqs. (21) and (25) takes for metal 1 the form

$$V_{1x} = \frac{i l_1}{q \eta} \int_{-\infty}^{z/l_1} d\xi' \operatorname{ch}[\gamma(qz - \epsilon_1 \xi')] \times \left\{ \int_{-\infty}^{\xi'} d\xi'' \left[i \epsilon_1 F_{1x}(\xi'') + \frac{\partial F_{1z}}{\partial \xi''} \right] \right\} \times \operatorname{ch}[\epsilon_1(\xi' - \xi'')] - F_{1z}(\xi') \Big\} e^{-i\omega t} - i [C_1 e^{qz} + \gamma D_1 e^{\gamma z}] e^{i(qz - \omega t)}, \quad (36)$$

where

$$C_1 = \xi_0 (\nu Q q + i\omega \gamma) / (\gamma - 1), \quad D_1 = -\xi_0 (\nu Q q + i\omega) / (\gamma - 1).$$

The first term of (36) decreases exponentially with increasing $|z|/l_1$, so that at a distance $|z| \gg l_1$ from the surface we have

$$V_{1x} \approx -i [C_1 e^{qz} + \gamma D_1 e^{\gamma z}] e^{i(qz - \omega t)}.$$

In the same region,

$$V_{1z} \approx - [C_1 e^{qz} + D_1 e^{\gamma z}] e^{i(qz - \omega t)}.$$

The fluxes corresponding to this velocity distribution are shown schematically in Fig. 1.

We cite now a condition that allows us to neglect the electrostatic force. It is obtained by comparing the alternating pressure on the surface, obtained from Eqs. (21) and (25), by an alternating pressure of electrostatic origin. For unstable modes with $q \sim q_m$ this condition is

$$\frac{50}{\pi} \frac{|\sigma_1 - \sigma_2|}{\sigma_1 + \sigma_2} \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 \sigma_2^2} \frac{e^2 \rho \nu}{p_F^2 (l_1 - l_2)^2} \ll 1.$$

For $\sigma_1 \sim \sigma_2 \sim 10^4 \Omega^{-1} \cdot \text{cm}^{-1}$, $\rho = 10 \text{ g/cm}^3$, $\nu = 10^{-3} \text{ cm}^2/\text{s}$, $p_F = 10^{-19} \text{ g} \cdot \text{cm/s}$, and $l_1 - l_2 = 3 \cdot 10^{-7} \text{ cm}$ the left-hand side of the inequality is of the order of 10^{-4} .

High viscosity

At extremely high viscosity Eq. (31) has a simple solution because a and Q are small. One of its roots

$$\omega(q) = -i\nu(a/2 - Q)q, \quad (37)$$

becomes unstable at $Q > a/2$.

The instability can be attributed in this case to the reversal of the sign of the effective surface-tension coefficient α^* by the action of the current, in accordance with (29). It is precisely this surface-tension coefficient which determines the dispersion law (37). In the case of low viscosity the surface-tension coefficient is also renormalized, but this effect manifests itself at currents noticeably stronger than those at which the capillary waves become unstable; we neglected it by virtue of the condition $Q \ll a$.

6. CONCLUSION

Let us consider the feasibility of realizing the instability in experiment. A contact between two liquid metals that carries an electric current of high density is realized in natural fashion in semiconductor breakdown. It is known that the electronic stage of semiconductor breakdown, due for example to impact ionization, is followed by a stage of thermal breakdown, in which the current contracts to a thin filament and molten regions can be produced in the contacts.⁹⁻¹¹ The molten region that covers the interface between the metal of the electrode and the semiconductor is a system in which the described instability might be realized. Indeed, when melted, typical semiconductors such as Ge, Si, and GaAs turn into metals.¹² The transverse dimension of the molten region is of the order of several microns, and the current density can reach 10^8 A/cm^2 . If the instability develops rapidly enough, the mixing of the metals during that time can be neglected. The model pair of metals used in Secs. 4 and 5 of the present paper is close to realization in the Al-Si system, which is extensively used. For this pair, in the model of almost free

electrons, the ratio of the Fermi energies is $\mathcal{E}_{\text{FSi}}/\mathcal{E}_{\text{FA1}} \approx 1.2$, and the ratio of the electron mean free paths is $l_{\text{A1}}/l_{\text{Si}} \approx 4$.

Indirect evidence of the instability of the interface is provided also by experiment. It follows from recent data that the semiconductor breakdown is accompanied by transport of microscopic masses of the electrode metal into the semiconductor in the form of a "filament" that penetrates to considerable depths.¹³⁻¹⁷ The propagation direction depends on the polarity of the current and is reversible in a number of cases, viz., at one polarity the metal is drawn into the semiconductor where it forms a conducting filament, and at the other polarity the metal returns to the electrode. This phenomenon has not yet been satisfactorily explained, but the idea that liquid metals can be moved by EM forces, as expounded in this paper, does explain it qualitatively.

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¹⁾The alternating electromagnetic field can be neglected in this equation because it is produced by electrodynamic forces of low frequency. The corresponding conditions are given by Eq. (27).

²⁾This statement is confirmed by the calculation result in Sec. 5, Eq. (33),

according to which the small-scale surface roughnesses are rapidly damped out. Numerically, the decrement for $q \sim p_F/\hbar$ is of the order of 10^{11} s^{-1} and is considerably higher than the frequencies of interest to us.

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