Spin glass with helical short-range order

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We study the helical-structure spin glass that is formed in a system of randomly arrayed vector magnetic moments with oscillating large-radius interaction. The low-temperature phase differs from the paramagnetic one, but does not have the Edwards-Anderson parameter ($q_{\rm EA}=0$) in the case of a fully isotropic interaction. The anisotropy of the interaction leads to $q_{\rm AE}\neq 0$, and the susceptibility $\chi(T)$ either has a smoothed-out cusp at $T\approx T_c$ or increases right down to low temperatures (depending on the anisotropy). In the former case the nonlinear susceptibility $\widetilde{\chi}=\partial^2\chi/\partial h^2$ is very high in absolute value (but finite) near the maximum of $\chi(T)$. Very weak magnetic fields $h\sim h_0$ (where h_0 depends on the anisotropy parameter) smooth out and then destroy completely the maximum of $\chi(T)$.

I. INTRODUCTION

A theoretical description of highly disordered systems such as spin glasses is made complicated[1] by the lack of a translationally invariant order parameter and hence of explicit "slow"[1)] variables that describe the continuous deformations of the ground state. The model proposed by Mattis[2] for spin-glass without frustrations,[3] however, does have latent slow variables that are uniquely connected with the initial spins, and this connection is determined by a set of random numbers that vary with the realizations of the system. This suggests the possible existence of some latent slow variables also in the more realistic spin-glass models. A model of this kind, which describes a random array of spins (at points $\mathbf{x}_i$ and with density c) in three-dimensional space, with an interaction

$$V_0^{\alpha\beta}(r)=\delta_{\alpha\beta}W_0\frac{\varkappa p_0}{2\pi r}\sin p_0 r e^{-\varkappa r}, \tag{1.1}$$

was proposed and investigated by us[4–6] for the case of Ising spins. It has turned out that at sufficiently high densities c (more accurately, at $\gamma=\varkappa p_0^2/4\pi c\ll 1$) the spin configuration is described by a sinusoidal modulation wave $\langle\sigma_i\rangle\propto\cos(\mathbf{Q}\cdot\mathbf{x}_i+\varphi)$, where $\mathbf{Q}$ is a wave vector of arbitrary direction ($Q=p_0$) and φ is the sought slow variable. Both the thermal fluctuations φ and the static strains (due to the disorder of the system) increase over large scales, so that the sinusoidal structure is preserved only locally. Nonetheless, such a "distorted sinusoidal" phase differs in principle from the paramagnetic one, and the two should thus be separated by a thermodynamic-phase-transition point.

Clearly, however, latent slow variables exist in not just any spin glass. In particular, in the Edwards-Anderson model[1] with large (but finite!) number of nearest neighbors, a phase transition of an entirely new type takes place (in a three-dimensional system),[7] not connected with formation of any (even very well hidden) macroscopic mean values, so that the question of slow variables no longer arises.

It makes sense thus to distinguish between two types of spin glass: one with latent sinusoidal or (see below) helical structure and the other having no spatial structure at all. From the viewpoint of spin-glass theory, the "general situation" corresponds more readily to the second type, but the first type is apparently also realized in nature and is therefore worthy of study. In this paper we investigate in detail the properties of the low-temperature phase of a classical spin glass of the first type with a vector (Heisenberg or planar) spin. The question of the phase transition from the paramagnetic to the low-temperature phase remains unclear, and will not be dealt with here, i.e., we confine ourselves to the temperature region $\tau=(T_c-T)/T_c\gg\gamma^{2/3}$ (for a derivation of this criterion see Ref. 5).

We emphasize that we are considering only equilibrium thermodynamics (which may be very difficult to achieve in experiment). The plan of the paper is the following: In Sec. II we discuss certain experimentally investigated forms of spin glass, in which a distorted helical structure can exist. It is shown that the RKKY interaction ($V(r)\sim W_0 r^{-3}\cos 2p_F r$) customarily used to describe the interaction of magnetic moments in alloys such as CuMn near the phase-transition temperature T_c is equivalent to an interaction of type (1.1) with $\varkappa\sim\min(c^{1/3},p_F)$, $p_0=2p_F$. It is shown next in the same section that the interaction (1.1) can lead to formation of a helical structure only at $\gamma=\varkappa p_0^2/4\pi c\lesssim 1$, whereas at $\gamma\gg 1$ (low densities) the problem is totally equivalent to the utterly random Edwards-Anderson model (second spin-glass type). In particular, the alloy $Cu_{1-x}Mn_x$ is expected to exhibit a helical structure at $x\gtrsim 10\%$ (see in this connection the recent experiment of Cable *et al.*[8]).

The third section of the paper is devoted to a derivation of the effective Hamiltonian of a helical structure in the case of Heisenberg spins. We shall show that this structure is characterized (locally) by two vectors: the wave vector $\mathbf{Q}=\nabla\theta$ of the helix and by the vector $\mathbf{n}$ (in spin space) that determines the direction of the normal to the plane of rotation of the spins in the helix. In the absence of spin-orbit interactions the vectors $\mathbf{Q}$ and $\mathbf{n}$ belong to different spaces and are therefore independent. When the direction of $\mathbf{n}$ is slowly varied, an "elastic" energy appears in the form

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$$H_0[\mathbf{n}]=J\int(\nabla\mathbf{n})^2\,d^3x, \qquad (1.2)$$

where the constant J is of pure fluctuational origin ($J=0$ in the mean-field approximation). The field deformations θ are described by the effective Hamiltonian

$$H_0[\theta]=\int\left[{}^1/_8A\left((\nabla\theta)^2-p_0^2\right)^2+{}^1/_2B(\nabla^2\theta)^2\right]d^3x. \qquad (1.3)$$

The fluctuations of the density of the disposition of the magnetic atoms are neglected in Eqs. (1.2) and (1.3). Allowance for these fluctuations adds to (1.2) and (1.3) the terms $\widetilde{H}[n]$ and $\widetilde{H}[\theta]$, respectively:

$$\widetilde{H}[\mathbf{n}]\propto\int g_{\mu\nu}(\mathbf{x})\,\partial_\mu\mathbf{n}\partial_\nu\mathbf{n}\,d^3x, \qquad (1.2')$$

$$\widetilde{H}[\theta]\propto\int f_\mu(\mathbf{x})\,\partial_\mu\theta\,d^3x, \qquad (1.3')$$

where $g_{\mu\nu}(\mathbf{x})$ and $f_\mu(\mathbf{x})$ are random Gaussian fields with small correlation radii. A dimensionality analysis shows that the fluctuations (both thermal and those due to disorder) of the field $\mathbf{n}$ remain finite over large spatial scales, whereas the fluctuations of the field θ diverge strongly. This means that to investigate the stability of our helical phase (neglecting spin-orbit interactions) it suffices to consider the Hamiltonian $H[\theta]=H_0[\theta]+\widetilde{H}[\theta]$, defined by Eqs. (1.3) and (1.3'). The Hamiltonian $H[\theta]$ agrees to within numerical coefficients with the Hamiltonian that describes the helical phase of a planar XY magnet.

In Sec. IV we study the properties of the distorted helical phase of vector magnets. In the absence of anisotropic interactions, a low-temperature phase is produced with strong thermal fluctuations of the field θ: $\langle\!\langle(\theta(0)-\theta(\mathbf{x}))^2\rangle\!\rangle\propto x^{1/2}$, therefore the mean value of the spin is $|\langle\sigma_i\rangle|=\langle\cos\theta(x_i)\rangle=0$ and the magnetic susceptibility χ obeys the Curie law $\chi\propto 1/T$ at all T (here and below the angle brackets denote thermodynamic averaging, the double brackets an irreducible correlator, and a superior bar averaging over the realizations). Nonetheless, this low-temperature phase differs from the paramagnetic one, as can be seen from the behavior of the correlator of the wave vectors of the structure $Q(\mathbf{x})$: $\langle Q(0)Q(\mathbf{x})\rangle\propto x^{-\gamma}$ (a situation reminiscent of a two-dimensional XY ferromagnet).[9]

In Sec. V we study the effects of dipole-dipole interaction and of anisotropy. The presence of a dipole-dipole interaction, no matter how small, alters qualitatively the properties of the system. The point is that the dipole energy is a minimum when $\mathbf{Q}$ and $\mathbf{n}$ are parallel. This means that if $\mathbf{n}$ is fixed (say on account of spin anisotropy of the easy-plane type) lifting of the degeneracy in the directions of $\mathbf{Q}$ and cutoff of the long-wave fluctuations. A similar effect results also, in the absence of any spin anisotropy, from small anisotropy of the initial interaction in coordinate space. The spins acquire therefore local mean values $\langle\sigma_i\rangle\neq 0$ (in which case, of course, the magnetic moment averaged over the system is $\langle\bar{\sigma}\rangle=0$), and the magnetic susceptibility assumes a constant value at low temperatures.

In Sec. VI of the paper we investigate in detail the magnetic properties of the system. The variation of the susceptibility with temperature depends on the relative values of the anisotropic interactions. It is then possible to distinguish between three characteristic regimes, which we shall consider in order of increasing intensity of the anisotropic forces: 1) monotonic increase of $\chi(T)$ with decrease of temperature; 2) smooth maximum of $\chi(T)$ in the region $\tau\sim\tau^*\ll 1$ followed by a decrease to $\chi(T=0)$; 3) a sharp break at $T=T_c$. For the second regime we investigate in detail the behavior of the nonlinear susceptibility $\widetilde{\chi}=\partial^2\chi/\partial h^2$ and of the differential susceptibility $\widetilde{\chi}(T,h)$ in finite fields h. The function $\widetilde{\chi}(T)$ has at $\tau\sim\tau^*$ a sharp maximum that is not connected with a true singularity, but stems from a rapid crossover between the regions of "strong" $\langle(\delta\theta)^2\rangle\gg 1$ and "weak" fluctuations of the phase variable $\theta(x)$. In any of the aforementioned three regimes, the behaviors of the longitudinal and transverse (relative to the spin rotation plane) susceptibilities differ greatly. For example, in the regime of strongly developed fluctuations the transverse susceptibility $\chi_\perp$ changes little with lowering of the temperature, whereas the longitudinal susceptibility $\chi_\parallel$ increases in accordance with the Curie law to not too low temperatures, but with an additional coefficient 3/2, so that the susceptibility χ averaged over all the directions is purely paramagnetic; in other words, the longitudinal susceptibility coincides in this case with the paramagnetic susceptibility of a planar magnet. The behavior of $\chi_\perp$ changes little with change of the anisotropic interactions, whereas $\chi_\parallel$ depends strongly on them and characterizes the state of the system, and will therefore be the main object of our study. The value of $\chi_\parallel(T,h)$ changes rapidly when an external magnetic field is turned on. In the region of the maximum of the linear susceptibility $\chi(T)$ the value of $\chi(T,h)$ is determined by the values of $\chi(T)$ and $\widetilde{\chi}(T)$, and decreases with the field, but at lower temperatures the initial decrease of $\chi(T,h)$ gives way to an increase even in weak fields $h\sim h_0$ (where h_0 is determined by the intensity of the dipole forces in the case of XY spins or by the spatial anisotropy of the initial interaction), and is eventually governed by the Curie law. In other words, the susceptibility measured in a field $h\gtrsim h_0$ has no temperature maximum and increases monotonically with decreasing temperature. This nonlinear effect is due to a competition between the dipole forces that suppress the fluctuations, on the one hand, and the magnetic field that enhances them, on the other (the action of a uniform magnetic field on a randomly distorted helicoid is analogous to the action of a random magnetic field on a ferromagnet.

The last section of the paper is devoted to a discussion of the results and of the possibilities of experimentally observing the predicted spin-glass phase with helical structure.

II. CONFIRMATION OF MODEL

We shall consider dilute solutions of magnetic atoms in a carrier matrix. The interaction between the spins of the magnetic atoms is assumed paired, and the spins themselves are assumed to be of the classical planar or Heisenberg type:

$$H=\frac{1}{2}\sum_{ij}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_jV_{ij}. \qquad (2.1)$$

The interaction V_{ij} is assumed to depend on $\mathbf{x}_i-\mathbf{x}_j$, i.e., $V_{ij}=V(\mathbf{x}_i-\mathbf{x}_j)$, and the magnetic atoms are assumed located at random points $\mathbf{x}_i$. This class includes the most frequently considered model of classical spin glasses $Cu_{1-x}Mn_x$, $Au_{1-x}Fe_x$, etc. The magnetic moments of the Mn and Fe atoms interact in these glasses in accordance with the RKKY law given, in the approximation in which the Fermi surfaces of Cu and Au are spherical, by

$$V(r)=-V_0 r^{-3} e^{-r/l} \cos 2p_F r, \quad (2.2)$$

where p_F is the Fermi momentum and l is the mean free path of the electrons in the matrix, $p_F l \gg 1$. We consider hereafter the usual situation when $c^{1/3} l \gg 1$ (c is the density of the magnetic atoms) and the exponential factor can be neglected. The characteristic properties of the interaction (2.2) are: 1) fast oscillations of $V(r)$; 2) divergence of the integral $I=\int V^2(r) d^3 r$; 3) spherical anisotropy of $V(r)$. The first and second properties are possessed by all interactions effected indirectly via the conduction electrons (if there is no special proximity to a phase transition into a helicoidal structure). The analysis that follows in this section is based only on these properties. The spherical isotropy of $V(r)$, on which the properties of the low-temperature phase depend strongly, will be useful to us in the succeeding sections.

The interaction (2.2) is singular as $r \to 0$, so that diverging (near $r=0$) integrals are encountered when an attempt is made to obtain the high-temperature expansion. This might seem to mean that the principal role is played by closely spaced atoms. This, however, is not so: the magnetic moments of these atoms are strongly bound (the interaction energy is much higher than the temperature) and make no contribution whatever to the thermodynamics. A more correct method is that of the virial expansion,[10] which shows that the main contribution is made by atoms separated by distances on the order of $r(T) \sim (V_0/T)^{1/3}$, i.e., those for which the interaction energy is of the order of the temperature. We verify this using as the example the first nontrivial term of the virial series for the free energy:

$$F^{(2)}=-\frac{c^2 T}{2} \int \ln \left\{ \mathrm{sh} \left[\frac{\cos 2p_F r}{T r^3} \right] \frac{T r^3}{\cos 2p_F r} \right\} d^3 r. \quad (2.3)$$

(We have put $V_0=1$ and assumed Heisenberg spins.) Calculating the next terms of the virial expansion, we can verify[10] that all the correlation corrections to the free energy become of the same order at $T \sim c$. Unfortunately, the virial expansion is of little use for calculations in that the region of strongly developed multiparticle correlations in which a diagram technique is preferable. The diagram expansion is not suitable for a total interaction of $V(r)$, as explained above. We therefore write the potential (2.2) as the sum of two parts:

$$\begin{aligned} V(r)&=V_1(r)+V_2(r), \\ V_1(r)&=V(r) \exp(-a T^{2/3} r^2), \quad (2.4) \\ V_2(r)&=V(r)[1-\exp(-a T^{2/3} r^2)]. \end{aligned}$$

Here $V_1(r)$ is the short-range potential, for which we can construct a convergent virial series; we handle $V_2(r)$ by a diagram technique. Since we confine ourselves in the virial expansion to a finite number of terms, this calculation method is approximate and its results depend on the subdivision parameter a. For the best choice of a we use a variational principle, i.e., independence of the free energy of a:

$$\partial F/\partial a|_{a=a_0}=0. \quad (2.5)$$

Our purpose is to derive an effective interaction of slow degrees of freedom, so that we actually require that the parameters of this interaction be independent of a. It is clear beforehand, however, that $a_0 \sim 1$, inasmuch as at $a \ll 1$ the contribution of the virial series is small and the high-temperature expansion terms are large, whereas at $a \gg 1$ the terms of the virial series are large.

We rewrite the partition function with the interaction (2.1) in a way that makes possible calculation of the terms of the virial series with the potential $V_1(r)$. We add to this end an additional molecular field $\mathbf{S}(\mathbf{x})$ [$\mathbf{S}(\mathbf{p})$ is its Fourier transform, $r_{ij}=|\mathbf{x}_i-\mathbf{x}_j|$]:

$$Z=\sum_{\{\sigma\}} \int D\mathbf{S} \exp\left\{-\frac{1}{T}\left[\frac{1}{2}\sum_{ij} \sigma_i \sigma_j V_1(r_{ij}) + \sum_i \sigma_i \mathbf{S}(\mathbf{x}_i) + \frac{1}{2}\int V_2^{-1}(p) |\mathbf{S}(\mathbf{p})|^2 \frac{d^3 p}{(2\pi)^3}\right]\right\},$$

$$V_2(p)=\int V(r) e^{i\mathbf{pr}} d^3 r. \quad (2.6)$$

We calculate the partition function over σ_i with the aid of the virial expansion, and $\mathbf{S}(\mathbf{x}_i)$ plays the role of the external field:

$$F\{\mathbf{S}\}=-T \ln \sum_{\{\sigma\}} \exp\left\{-\frac{1}{2T}\sum_{ij} \sigma_i \sigma_j V_1(r_{ij}) - \frac{1}{T}\sum_i \mathbf{S}(\mathbf{x}_i) \sigma_i\right\}. \quad (2.7)$$

The first term of the virial series can be calculated directly:

$$F_1=-T\sum_i \ln\left[\frac{T}{|\mathbf{S}_i|} \mathrm{sh} \frac{|\mathbf{S}_i|}{T}\right] \approx \sum_i \left[-\frac{S_i^2}{6T}+\frac{(S_i^2)^2}{180 T^3}\right], \quad (2.8)$$

where $\mathbf{S}_i \equiv \mathbf{S}(\mathbf{x}_i)$. When only F_1 is taken into account, the effective interaction $H[\mathbf{S}]$ takes the form

$$H_1[\mathbf{S}]=\frac{1}{2}\int V_2^{-1}(p) |\mathbf{S}(\mathbf{p})|^2 \frac{d^3 p}{(2\pi)^3} - \sum_i \frac{S_i^2}{6T} + \sum_i \frac{(S_i^2)^2}{180 T^3}. \quad (2.9)$$

We have confined ourselves only to those terms of the expansion of H_1 in powers of S^2 which are needed at high temperatures. In the mean-field approximation we see that the high-temperature phase becomes unstable at $T=T_{c1}=1/3c \min[V_2^{-1}(p)]$. The properties of the low-temperature phase and its instability will be investigated in later sections of the paper; here we obtain the effective interaction $H[\mathbf{S}]$. At temperatures $T\sim T_{c1}$ small changes of the temperature T_c (defined formally as the point where that part of $H[\mathbf{S}]$ which is quadratic in $\mathbf{S}$ becomes unstable in the mean-field approximation) alter strongly the free energy of the system. As the variational principle for finding a_0 we use therefore in lieu of (2.5) the condition

$$dT_c/da|_{a=a_0}=0. \qquad (2.10)$$

This allows us to retain only terms quadratic in $\mathbf{S}$ when calculating the next term of the virial series:

$$F_2=\frac{1}{6T}\sum_{ij}\left[\operatorname{cth}\frac{V_1(r_{ij})}{T}-\frac{T}{V_1(r_{ij})}\right]\mathbf{S}_i\mathbf{S}_j. \qquad (2.11)$$

In this approximation, the part of $H[\mathbf{S}]$ that is quadratic in $\mathbf{S}$ takes the form

$$H^{(2)}[\mathbf{S}]=\frac{1}{2}\int V_2^{-1}(p)\,|\mathbf{S}(\mathbf{p})|^2\frac{d^3p}{(2\pi)^3}-\sum_i\frac{S_i^2}{6T}$$
$$+\frac{1}{6T}\sum_{ij}\left[\operatorname{cth}\frac{V_1(r_{ij})}{T}-\frac{T}{V_1(r_{ij})}\right]\mathbf{S}_i\mathbf{S}_j. \qquad (2.12)$$

Averaging formally over the impurity locations and neglecting the fluctuations of the impurity density, we obtain

$$\overline{H^{(2)}[\mathbf{S}]}=\frac{1}{2}\int\left[V_2^{-1}(p)+W(p)-\frac{c}{3T}\right]|\mathbf{S}(\mathbf{p})|^2\frac{d^3p}{(2\pi)^3},$$
$$W(\mathbf{p})=-\frac{c^2}{3T}\int\left\{\operatorname{cth}\left[\frac{\cos 2p_Fr}{Tr^3}\exp(-aT^{2/3}r^2)\right]\right.$$
$$\left.-\frac{Tr^3\exp(aT^{2/3}r^2)}{\cos 2p_Fr}\right\}e^{i\mathbf{pr}}\,d^3r. \qquad (2.13)$$

The parameter a, the temperature T_c, and the momentum p_0 at which the quadratic form becomes unstable are determined from the system of equations

$$\frac{\partial}{\partial p}[V_2^{-1}(p,a_0)+W(p,a_0)]=0, \qquad (2.14)$$

$$\frac{\partial}{\partial a}[V_2^{-1}(p_0,a)+W(p_0,a)]=0, \qquad (2.15)$$

$$V_2^{-1}(p_0,a_0)+W(p_0,a_0)=c/3T_c. \qquad (2.16)$$

We reduce this system to dimensionless form. We introduce the new variables $\Theta=T/c$, $k=pT^{-1/3}$, $q=2p_FT^{-1/3}$. The quantity q is the only dimensionless parameter that determines the solution (a,Θ_c,k_0) of the system (2.14)–(2.16). It is convenient to transform the functions W and V_2 in this system into

$$W(k)=-\frac{4\pi}{3\Theta^2k}\int_0^\infty r\,dr\sin kr\left\{\operatorname{cth}\left[\frac{\cos qr}{r^3}\exp(-ar^2)\right]\right.$$
$$\left.-\frac{r^3\exp(ar^2)}{\cos qr}\right\}, \qquad (2.17)$$

$$V_2(k)=\frac{\pi}{k}\left\{(k-q)\left[\ln\frac{4a}{(k-q)^2}-\Phi_\alpha'\left(0,\frac{3}{2},-\frac{(k-q)^2}{4a}\right)\right]\right.$$
$$\left.+(k+q)\left[\ln\frac{4a}{(k+q)^2}-\Phi_\alpha'\left(0,\frac{3}{2},-\frac{(k+q)^2}{4a}\right)\right]\right\}. \qquad (2.18)$$

Here $\Phi_\alpha'(\alpha,\beta\,z)$ is the derivative of the confluent hypergeometric function with respect to the first index.

We consider first the limiting case $q\ll1$ (or, equivalently, large densities c). In this case the main contribution to the integral in (2.17) is made by the region $r\sim1$, so that $\cos qr$ and $(\sin kr)/kr$ can be replaced by unity, making W independent of k. Retaining only the leading terms in q^2/a, we obtain in place of (2.18)

$$V_2(k)=\frac{\pi}{k}\left\{(k-q)\ln\frac{4a}{(k-q)^2}+(k+q)\ln\frac{4a}{(k+q)^2}\right\}. \qquad (2.19)$$

We note that $a\,dV_2^{-1}/da\ll V_2^{-1}$, whereas $a\,dW/da\sim W$, and it follows thus from (2.15) that $W\ll V_2^{-2}$. From (2.16) we get

$$\Theta_c={}^1/_3V_2(k_0,a_0). \qquad (2.20)$$

Solving numerically the transcendental equation (2.14), we obtain

$$k_0=C_0q,\quad C_0=1{,}1997. \qquad (2.21)$$

Substituting this value in (2.20), and then (2.20) and (2.19) in (2.15), we obtain a transcendental equation for the cutoff parameter a:

$$\int_0^\infty r^2\,dr\{\operatorname{cth}[\exp(-ar^2)r^{-3}]-r^3\exp(ar^2)\}=\frac{1}{6}. \qquad (2.22)$$

Solving it numerically we obtain the final answer:

$$a=1{,}12,\quad \Theta_c={}^1/_3[\ln(4a/q^2)+C_1],\quad C_1\approx-1{,}177. \qquad (2.23)$$

To investigate the properties of the low-temperature phase outside the framework of the mean-field theory we need the unrenormalized form of the fluctuation spectrum $S(\mathbf{p})$ with momenta p close to p_0. Substituting in (2.13) the expressions (2.19), (2.21), and (2.23) we have, at the required accuracy,

$$H^{(2)}[\mathbf{S}]=\frac{1}{2}\int\frac{1}{3\Theta_c}\left[\frac{(p-p_0)^2}{\varkappa^2}+\tau\right]|\mathbf{S}(\mathbf{p})|^2\frac{d^3p}{(2\pi)^3},$$
$$\varkappa^2=\frac{8}{C_2}p_F^2\left(\ln\frac{4a}{q^2}+C_1\right),\quad C_2\approx4.553,\quad \tau=\frac{T-T_c}{T_c}. \qquad (2.24)$$

In the opposite limiting case, $q\gg1$, we can also obtain the solution (2.14)–(2.16) but, as will be shown below, $\mathbf{S}(\mathbf{x})$ is now an incorrectly chosen slow variable, and accordingly the point of occurrence of the instability $H^{(2)}[\mathbf{S}]$ does not manifest itself in any way in the physical properties of the system, so that the use of condition (2.10) is not valid. The structure of the low-temperature phase at $q\gg1$ will not be dealt with in this paper.

We determine now the form of the interaction of the variables $\mathbf{S}(\mathbf{p})$, confining ourselves to the first term of the virial expansion:

$$H^{(int)}[\mathbf{S}]=\int\left\{\left[\sum_i \delta(\mathbf{x}-\mathbf{x}_i)-c\right]\frac{S^2(\mathbf{x})}{6T}+c\frac{S^4(\mathbf{x})}{180T^3}\right\}d^3x. \tag{2.25}$$

We have retained here only the leading terms relative to the random field (in this case, the impurity locations $\mathbf{x}_i$) and terms of fourth order in $\mathbf{S}$: this is valid at not too low temperatures ($T\sim T_c$). An interaction energy of the same form as in (2.24) and (2.25) can be obtained directly if the initial spin interaction differs somewhat from the RKKY form:

$$V(r)=W_0(\varkappa p_0/2\pi r)\sin p_0 r e^{-\varkappa r}. \tag{2.26}$$

It follows from the reasoning above that the RKKY interaction is a particular case of the interaction (2.26), the parameters $\varkappa$ and p_0 can be determined by using Eqs. (2.21) and (2.24), while $W_0=3\Theta_c$. A direct interaction of this type can be expected between spins located in a host metal close to a helicoidal transition, so that exchange of the virtual helicon leads to (2.26) (see Ref. 6). The interaction (2.26) contains two dimensionless parameters and describes a larger group of phenomena than (2.2). We shall therefore consider hereafter just this interaction. The parameter $\varkappa$ has the meaning of the reciprocal interaction radius; we assume that $\varkappa\ll c^{1/3}$; the results will be valid, in order of magnitude, also at $\varkappa\lesssim c^{1/3}$.

The behavior of the system at low temperature depends very strongly on the parameter $\gamma=\varkappa p_0^2/4\pi c$. At $\gamma\ll 1$ and at sufficiently low temperatures ($-\tau\gtrsim\gamma^{2/3}$), a phase[5] of the spoiled-helix type is produced. We shall investigate its low-temperature processes in the sections that follow. In the opposite limiting case $\gamma\ll 1$ the potential $V(r)$ oscillates rapidly, so that the correlations between V_{ij} and the different parameters (j, i) can be neglected, and an analog of the Edwards-Anderson model[1] with a large but finite interaction radius can be obtained:

$$\overline{V_{ij}V_{kl}}={}^1/_2[\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}]V_0^2e^{-2\varkappa r},\quad \overline{V}_{ij}=0. \tag{2.27}$$

The question of the low–temperature properties and of the phase transition in this model has not yet been completely investigated (it is the subject of Refs. 1, 7, and 11–13). We shall not deal with it here, and examine only how the transition takes place (when γ is decreased) from the Edwards-Anderson model to the spoiled-helix structure. We consider the low-temperature expansion of the correlator $\overline{\langle \mathbf{S}_x\mathbf{S}_{x'}\rangle^2}$ in the model (2.27) and compare it with the expansion for the potential (2.26). Let us draw the diagrams corresponding to these expansions. In the model (2.27), each line (potential) should enter twice, as shown in Fig. 1a, whereas in model (2.26) a diagram of type of Fig. 1b is possible. Corresponding to these diagrams are the expressions

$$\langle \mathbf{S}_0\mathbf{S}_\mathbf{R}\rangle_a^2\sim c\int V^2(\mathbf{R}-\mathbf{r})V^2(\mathbf{r})d^3r,$$
$$\overline{\langle \mathbf{S}_0\mathbf{S}_\mathbf{R}\rangle_б^2}\sim c^2\int V(\mathbf{R}-\mathbf{r}_1)V(\mathbf{r}_1)V(\mathbf{R}-\mathbf{r}_1+\mathbf{r}_2)V(\mathbf{r}+\mathbf{r}_2)d^3r_1d^3r_2. \tag{2.28}$$

We compare these integrals, substituting the expression (2.26) for $V(r)$. We obtain the ratio $\overline{\langle \mathbf{SS}\rangle_b^2}/\overline{\langle\mathbf{SS}\rangle_a^2}\approx\gamma^{-1}$, so that at $\gamma\gg 1$ the diagrams of the second type can be neglected, and the high-temperature expansion with the potential (2.26) coincides with the expansion in the Edwards-Anderson model. This proves the equivalence of these models, at least at not too low temperatures $T\sim T_c\sim(c\varkappa p_0^2)^{1/2}$.

III. DERIVATION OF LONG-WAVE EFFECTIVE HAMILTONIAN

1. We consider the classical statistical mechanics of a system of Heisenberg spins $\boldsymbol{\sigma}_i$ that are randomly scattered in three-dimensional space at an average density c. We choose the interaction between the spins, as in the preceding section of the paper, in the form

$$H=\frac{1}{2}\sum_{ij}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_jV_0(\mathbf{x}_i-\mathbf{x}_j), \tag{3.1}$$

$$V_0(r)=W_0(\varkappa p_0/2\pi r)\sin p_0 r e^{-\varkappa r},\quad \varkappa\ll c^{1/3}. \tag{3.2}$$

The behavior of the system is determined by the value of the parameter $\gamma=\varkappa p_0^2/4\pi c$. (We confine ourselves to the region $\gamma\ll 1$.) In this case, at sufficiently low temperatures, the Heisenberg spins from, as do the planar ones, a distorted spatial helix (helicoid). Let us prove this and derive the effective Hamiltonian of slow deformations of this structure at intermediate scales (larger than the length over which the helix is formed and smaller than the length at which the long-range order is lost).

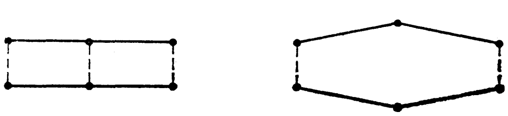
It is convenient to rewrite the partition function $Z=\mathrm{Tr}\exp(-H/T)$ by introducing the continuous field $\mathbf{S}(\mathbf{x})$ (the molecular field acting on the spin at the point $\mathbf{x}$):

$$Z=\int D\mathbf{S}(\mathbf{x})\exp\left[-\frac{1}{2TW_0}\int\left\{\left[\left(\frac{\nabla^2+p_0^2}{2p_0\varkappa}\right)\mathbf{S}\right]^2+S^2-\sum_i\ln\left[\frac{T}{|\mathbf{S}+\mathbf{h}|}\operatorname{sh}\frac{|\mathbf{S}+\mathbf{h}|}{T}\right]\delta(\mathbf{x}-\mathbf{x}_i)\right\}d^3x\right], \tag{3.3}$$

where $\mathbf{x}_i$ are the locations of the spins $h=q_L\mu_B\mathscr{H}$, and $\mathscr{H}$ is the external magnetic field. We confine ourselves to not too low temperatures and expand the argument of the exponential in powers of S/T. We obtain (at $h=0$)

$$Z=\int D\mathbf{S}(\mathbf{x})\exp\{-H[\mathbf{S}]\},$$

$$H[\mathbf{S}]=\int d^3x\left\{\frac{1}{2TW_0}\left[\left(\frac{\nabla^2+p_0^2}{2p_0\varkappa}\right)\mathbf{S}\right]^2+\frac{1}{2TW_0}\left(1-\frac{W_0}{3T}\sum_i\delta(\mathbf{x}-\mathbf{x}_i)\right)S^2\right.$$



a b

FIG. 1. Part of high-temperature diagram series in the Edwards-Anderson model (a) and in the model (2.26) that competes with it (b).

$$+\frac{1}{180T^4}(S^2)^2\sum_i\delta(\mathbf{x}-\mathbf{x}_i)\Big\}. \tag{3.4}$$

In the last term of (3.4) we can directly replace $\Sigma\delta(\mathbf{x}-\mathbf{x}_i)$ by c. In the mean-field approximation, at $T=T_c=cW_0/3$ (we put hereafter $W_0=1$), a phase transition takes place into an inhomogeneous helix-like state: $S=\mathbf{e}_1\cos\mathbf{p}_0\cdot\mathbf{x}+e_2\sin\mathbf{p}_0\cdot\mathbf{x}$, where $\mathbf{e}_1\cdot\mathbf{e}_2=0$. In the temperature region $\tau=1-c/3T\ll1$, the functional H takes the form

$$H=\frac{1}{2T}\int\Big\{\Big[\Big(\frac{\nabla^2+p_0^2}{2p_0\varkappa}\Big)\mathrm{S}\Big]^2+\tau S^2+\frac{3}{10c^2}(S^2)^2+\Big[1-\frac{1}{c}\sum_i\delta(\mathbf{x}-\mathbf{x}_i)\Big]S^2\Big\}d^3x. \tag{3.5}$$

We shall consider the temperature region $|\tau|\gg\gamma^{2/3}$, where short-range order of helical type obtains (see below, as well as Ref. 5). The last term of (3.5) leads to large-scale distortions of the helix. Let us derive an expression for the free energy of such deformations. We take first into account the influence of the thermal fluctuations that take place over scales much smaller than those of the distortions connected with the random fields. The last term of (3.5) can then be neglected.

2. It is convenient to represent the field S_α in the form

$$S_\alpha=a_\alpha(\mathbf{x})\exp(i\theta)+a_\alpha^*(\mathbf{x})\exp(-i\theta)+\varphi_\alpha(\mathbf{x}), \tag{3.6}$$

where $\theta=\mathbf{p}_0\cdot\mathbf{x}$ and a_α is a complex vector that varies "slowly" with the coordinate, $a_\alpha^2=0$ (It is convenient to think of a_α in the form $a_\alpha=e_{1\alpha}+ie_{2\alpha}$), while φ_α are "fast" fluctuations with momenta $p\sim p_0$, $|\mathbf{p}\pm\mathbf{p}_0|\sim p_0$. The Green's function φ of the fluctuations was calculated in Ref. 14. In the region $\gamma^{2/3}\ll|\tau|\ll1$ of interest to us we have

$$g_{\alpha\beta}(p)=\langle\varphi_\alpha(\mathbf{p})\varphi_\beta(-\mathbf{p})\rangle=g_1(p)\rho_{\alpha\beta}+g_2(p)(\delta_{\alpha\beta}-\rho_{\alpha\beta}),$$
$$\rho_{\alpha\beta}=\rho^{-1}(a_\alpha a_\beta^*+a_\beta a_\alpha^*),\quad \rho=|a_\alpha|^2,$$
$$g_1=T\Big[\Big(\frac{p-p_0}{\varkappa}\Big)^2+|\tau|\Big]^{-1}, \tag{3.7}$$
$$g_2=T\Big[\Big(\frac{p-p_0}{\varkappa}\Big)^2+\Big(\frac{6}{5}\gamma\Big)^{2/3}\Big]^{-1}.$$

Let us clarify these equations. The tensor $\rho_{\alpha\beta}$ picks out only fluctuations in the plane of rotation of the formed spiral, so that g_1, which corresponds to such (longitudinal) fluctuations has (at $|\tau|>\gamma^{2/3}$) a gap $|\tau|$. In the mean-field approximation the transverse fluctuations have no gap at all, but when their interaction with one another is taken into account it turns out that they acquire a gap $\sim\gamma^{2/3}$.

We consider now the slow deformations of $a_\alpha(x)$, which do not change the modulus $\rho=|a_\alpha|^2$:

$$a_\alpha(\mathbf{x})=a_\alpha^{(0)}+\tilde{a}_\alpha(\mathbf{x}).$$

The vector a_α defines the plane in which the spins rotate; the normal to this plane is $\mathbf{n}=i\mathbf{a}\times\mathbf{a}^*/\rho$. We represent the deformation $\tilde{\mathbf{a}}(\mathbf{x})$ in the form of the sum $\tilde{\mathbf{a}}=\tilde{\mathbf{a}}_\perp+\tilde{\mathbf{a}}_\|$, such that $\tilde{\mathbf{a}}\perp$ changes the direction $\mathbf{n}$, and the deformation produced by $\tilde{\mathbf{a}}_\|$ is equivalent to a slow change of the angle $\theta(\mathbf{x})$ [see (3.6)], i.e., to a shift of the helical without changing the spin-rotation plane. Substituting (3.6) in (3.5), we obtain that part of thee free energy which depends only on the $\tilde{\mathbf{a}}_\|$ deformations, or, equivalently, on $\theta(\mathbf{x})$:

$$H_0[\theta]=\frac{\rho}{T\varkappa^2}\int\Big\{\Big[\frac{(\nabla\theta)^2-p_0^2}{2p_0}\Big]^2+\frac{(\nabla^2\theta)^2}{4p_0^2}\Big\}d^3x,$$
$$\rho=\frac{5}{6}|\tau|c^2. \tag{3.8}$$

It can be shown that the fluctuations $\varphi_\alpha(\mathbf{x})$ do not lead to a substantial change of $H_0[\theta]$ at $|\tau|>\gamma^{2/3}$. Matters are different with the transverse deformations. Their energy, disregarding the thermal fluctuations, is

$$H_0^{(0)}(\tilde{\mathbf{a}}_\perp)=\frac{1}{8T\varkappa^2p_0^2}\int\{|\nabla^2a_\alpha|^2+4|\mathbf{p}_0\nabla a_\alpha|^2+2i[(\mathbf{p}_0\nabla a_\alpha)\nabla^2a_\alpha^*-\mathrm{H.a.}]\}d^3x, \tag{3.9}$$

from which it can be seen that $H_0^{(0)}(\tilde{\mathbf{a}}_\perp)$ vanishes for all fluctuations with momenta $\mathbf{p}$ such that $(\mathbf{p}_0\cdot\mathbf{p})+\mathbf{p}^2=0$. Allowance for the thermal fluctuations φ_α leads to a nonzero energy of all the deformations with nonzero momenta, and therefore alters $H_0(\tilde{\mathbf{a}}_\perp)$ substantially. Let us carry out this calculation.

We substitute the expression (3.6) for S_α in (3.5) and separate that part of the free energy which depends on the product $\bar{a}_\alpha\varphi_\alpha$:

$$H_{int}=\frac{9}{5c^3}\int\varphi_\alpha\varphi_\beta(a_\alpha a_\beta^*+a_\beta a_\alpha^*)d^3x. \tag{3.10}$$

We have neglected here the terms that are linear in φ_α (they yield zero after averaging over the fluctuations φ_α) and the terms of third order in φ_α (they contribute only in the higher orders of perturbation theory, i.e., they will contain extra powers of γ), as well as terms containing the oscillating factors $\exp 2i\theta(\mathbf{x})$. We average over φ_α in second-order perturbation theory:

$$\Delta H(\tilde{a}_{\perp\alpha})=-\ln\int D\varphi\exp\Big[-\Big(H_{int}+\frac{1}{2}\int\varphi_\alpha g_{\alpha\beta}^{-1}\varphi_\beta\,d^3x\Big)\Big]$$
$$=-\frac{1}{2}\Big(\frac{9}{5c^3}\Big)^2\iint(a_\alpha a_\beta^*+a_\beta a_\alpha^*)_{\mathbf{x}}(a_\gamma a_\delta^*+a_\delta a_\gamma^*)_{\mathbf{x}'}(g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma})_{\mathbf{x}-\mathbf{x}'}d^3x\,d^3x'. \tag{3.11}$$

The functions $a_\alpha(\mathbf{x})$ vary little over the characteristic scales of the variation of $g(\mathbf{x}-\mathbf{x}')$, so that we can replace the $\mathbf{a}(\mathbf{x})$ in Eq. (10) by their Taylor-series expansions about the point $\mathbf{x}'$ and integrate with respect to the difference $\mathbf{x}-\mathbf{x}'=\mathbf{y}$:

$$\Delta H(\tilde{a}_{\perp\alpha})=\frac{1}{3}\Big(\frac{9}{5c^3}\Big)^2\int\partial_\mu(a_\alpha a_\beta^*)\partial_\mu(a_\gamma a_\delta^*)d^3x\times\int(g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma})_y y^2\,d^3y. \tag{3.12}$$

We use now the orthogonality relation

$$(\partial_\mu \tilde{\mathbf{a}}_\perp)\,\mathbf{a}=(\partial_\mu \tilde{\mathbf{a}}_\perp)\,\mathbf{a}^*=0$$

and find that a contribution to the second integral is made only by terms of the type $g_1 g_2$:

$$\Delta H(\tilde{a}_{\perp\alpha})=5|\tau|\left(\frac{3}{5c^2}\right)^2\int(\partial_\mu\tilde{a}_{\perp\alpha})(\partial_\mu\tilde{a}_{\perp\alpha}^*)\,d^3x \int g_1(y)g_2(y)y^2\,d^3y. \tag{3.13}$$

Expressing $(\partial_\mu \tilde{\mathbf{a}}_\perp)$ in this equation in terms of $(\partial_\mu \mathbf{n})^2$, calculating the integral, and using Eq. (3.9) for the free energy of the transverse deformations in the mean-field approximation, we obtain ultimately:

$$H_0\{\mathbf{n}\}=\int\left\{\frac{p_0^2}{12\pi\varkappa}\left(\frac{6}{5}\gamma\right)^{-1/3}(\partial_\mu\mathbf{n})^2+\frac{5}{8}\frac{|\tau|c}{\varkappa^2}\left[\frac{(\mathbf{p}_0\nabla)\mathbf{n}}{p_0}\right]^2\right\}d^3x. \tag{3.14}$$

The spectrum of the long-wave fluctuations of the direction of the normal n turns out to be quadratic and strongly anisotropic: the ratio of the coefficients of the first and second terms of (3.14) is of the order of $\gamma^{2/3}|\tau|\ll 1$. The peculiarity of $H_0[\mathbf{n}]$ is that the spectrum of the fluctuations with momenta $\mathbf{q}\perp\mathbf{p}_0$ does not become harder with decreasing temperature and with increasing $|\tau|$ (the growth of the structure amplitude ρ is offset by the weakening of the "fast" fluctuations $\varphi_\alpha(\mathbf{x})$, which were the ones that led to the appearance of the first term of (3.14).

3. We have shown that slow deformation of the helical structure in a Heisenberg magnet are described by two vectors, $\mathbf{Q}=\nabla\theta$ and $\mathbf{n}$, and the corresponding deformation energies are given by (3.8) and (3.14). These equations were obtained neglecting the spatial fluctuations of the density of the spin-location points. Allowance for these fluctuations (i.e., for the last term of (3.5)) leads (in the language of the vectors $\mathbf{Q}$ and $\mathbf{n}$) to the appearance of terms of the random-anisotropy type. The general form of these terms can be obtained from symmetry considerations: the initial isotropy of the spin Hamiltonian (3.1) requires invariance of the total effective Hamiltonians of the fields θ, $\mathbf{n}$ relative to the shift $\theta\to\theta+\text{const}$ and to a uniform rotation $\mathbf{n}\to\hat{O}\mathbf{n}$, where $\hat{O}$ is an orthogonal matrix. Those expressions that are of lower order in the derivatives and satisfy these requirements, are of the form

$$\tilde{H}[\theta]=\int f_\mu(\mathbf{x})\,\partial_\mu\theta\,d^3x, \tag{3.15}$$

$$\tilde{H}[\mathbf{n}]=\int g_{\mu\nu}(\mathbf{x})\,\partial_\mu\mathbf{n}\,\partial_\nu\mathbf{n}\,d^3x, \tag{3.16}$$

where $f_\mu(\mathbf{x})$ and $g_{\mu\nu}(\mathbf{x})$ are random functions with small correlation radii. A simple dimensionality analysis shows that the Hamiltonian $H_0[\mathbf{n}]$ is stable relative to $\tilde{H}[\mathbf{n}]$ in the sense that weak random fields $g_{\mu\nu}(\mathbf{x})$ do not destroy the long-range order of the vectors $\mathbf{n}$ (just as the thermal fluctuations in a three-dimensional system with a quadratic spectrum). A substantially different situation obtains for the field $\theta(\mathbf{x})$ and the perturbation $\tilde{H}[\theta]$: the unrenormalized field correlator is of the form

$$G_0(q)=\langle\delta\theta(q)\,\delta\theta(-q)\rangle_0=2Tp_0^2\varkappa^2/\rho[q^4+4(\mathbf{q}\mathbf{p}_0)^2], \tag{3.17}$$

so that the thermal fluctuations

$$\langle(\delta\theta)^2\rangle_0=\frac{1}{(2\pi)^3}\int G_0(q)\,d^3q$$

diverge logarithmically, and the deformations due to the random field have a power-law divergence

$$\overline{\langle\delta\theta\rangle^2}=\frac{1}{(2\pi)^3}\int_{q_1}\overline{f_\mu^2(\mathbf{q})}\,q^2G_0^2(\mathbf{q})\,d^3q\sim O(q_1^{-2}) \tag{3.18}$$

(we assume, and will prove below, that $\overline{f_\mu^2(\mathbf{q})}\to\text{const}$ as $\mathbf{q}\to 0$). Expression (3.18) shows that the disorder-induced deformations of the phase θ of a helical structure are large and can lead to destruction of the long-range order over sufficiently large scales.

In the next section we investigate in detail the long-wave properties of a system with an effective "phase" Hamiltonian $H[\theta]=H_0[\theta]+\tilde{H}[\theta]$ and show that there is indeed no long-range order, the local mean values of the spins $|\langle\sigma_\alpha(\mathbf{x})\rangle|\sim\langle\cos\theta(\mathbf{x})\rangle$ are also equal to zero [in the absence of spin-orbit forces and of anisotropy of the initial interaction $V(r)$], but the low-temperature phase does nontheless exist. We shall not consider the fluctuations of the field of the normal $n(x)$ (we put $n(x)=n0$), since these fluctuations are finite and small, and therefore do not influence the stability of the low-temperature phase.

We note that in the case of planar XY spins the direction of $\mathbf{n}$ is fixed from the very outset, so that slow structure distortions are described by the Hamiltonian $H[\theta]$. The analysis that follows, of the effect of the disorder $\tilde{H}[\theta]$ on the properties of the helix pertains therefore to both Heisenberg and XY spins.

IV. DESTRUCTION OF HELICAL LONG-RANGE ORDER

1. We derive the effective Hamiltonian $\tilde{H}[\theta]$ [Eq. (3.15)], which was written out above from symmetry considerations. Averaging the last term of (3.5) over the disposition of the points $\mathbf{x}_i$ by the replica method,[1] we obtain the Hamiltonian in the form

$$H_R[\mathbf{S}^a]=-\int d^3x\frac{9}{8c^3}\sum_{ab}(\mathbf{S}^a)^2(\mathbf{S}^b)^2. \tag{4.1}$$

The summation over the replica indices (a, b) is from 1 to N, and at the end of the calculations one must put $N=0$. In the derivation of (4.1) we have regarded $1-c^{-1}\Sigma_i\delta(\mathbf{x}-\mathbf{x}_i)$ as a random Gaussian field, a good approximation at $\gamma\ll 1$. The term $H_R[\mathbf{S}^a]$ makes no contribution to $H[\theta]$ in first order, since $\overline{\mathbf{S}_0^2}=2\varphi$ does not depend on θ for a helical structure [see (3.6)]. The contribution of interest to us, of second order in $H_R[\mathbf{S}^a]$ to $H[\theta]$, is given by

$$\tilde{H}[\theta_a]=-\frac{1}{2}\iint d^3x\,d^3x'\left(\frac{9}{8c^3}\right)^2$$

$$\sum_{abcd} \langle\langle (S^a(\mathbf{x}))^2 (S^b(\mathbf{x}))^2 (S^c(\mathbf{x}'))^2 (S^d(\mathbf{x}'))^2 \rangle\rangle_0, \tag{4.2}$$

where the irreducible mean value is taken over the "fast" Gaussian fluctuations $\varphi_\alpha(\mathbf{x})$ [see (3.6) and (3.7)]. The terms significant in (5.2) are those with $a = c$, $b = d$ or $a = d$, $b = c$; it is clear beforehand that a contribution to the irreducible correlator (4.2) is made only by the longitudinal Green's function $g1$, because only longitudinal fluctuations alter the value of S, so that only they enter in the irreducible correlator (4.2). Substituting $S(x)$ in the form (3.6), we have

$$\tilde{H}[\theta_a] = -\frac{36\rho^2}{c^6} \sum_{ab} \iint d^3x\, d^3x' g_1^2(\mathbf{x}-\mathbf{x}') \cos \mathbf{Q}_a\mathbf{x} \cos \mathbf{Q}_b\mathbf{x}', \tag{4.3}$$

where $\mathbf{Q}_a = \nabla\theta_a$. The integral in (4.3) can be calculated at $|\mathbf{Q}_a \pm \mathbf{Q}_b| \ll p_0$:

$$\tilde{H}[\theta_a] = -\int d^3x \frac{25}{36\pi} \tau p_0^2 \varkappa \sum_{ab} \left[\frac{1}{|\mathbf{Q}_a - \mathbf{Q}_b|} \operatorname{arctg} \frac{|\mathbf{Q}_a - \mathbf{Q}_b|}{2\varkappa\tau^{1/2}} + \frac{1}{|\mathbf{Q}_a + \mathbf{Q}_b|} \operatorname{arctg} \frac{|\mathbf{Q}_a + \mathbf{Q}_b|}{2\varkappa\tau^{1/2}} \right]. \tag{4.4}$$

We shall need hereafter only the term of (4.4) that is leading at $|\mathbf{Q}_a - \mathbf{Q}_b| \ll \varkappa\tau^{1/2}$, so that we ultimately have [see (3.8)]

$$H[\theta_a] = \int \left\{ \frac{5|\tau|c}{2\varkappa^2} \sum_a \left[\left(\frac{(\nabla\theta_a)^2 - p_0^2}{2p_0} \right)^2 + \frac{(\nabla^2\theta_a)^2}{4p_0^2} \right] - \frac{25 p_0^2}{432\pi\varkappa\tau^{1/2}} \sum_{ab} \nabla\theta_a \nabla\theta_b \right\} d^3x. \tag{4.5}$$

The second term of (4.5) is equivalent to the random Gaussian δ-correlated field $f_\mu(\mathbf{x})$ [see (3.15)]. The fact that $\tilde{H}[\theta_a]$ depends on the angle between $\mathbf{Q}_a$ and $\mathbf{Q}_b$ means the onset (owing to the disorder of the system) of a locally preferred direction of $\mathbf{Q} = \nabla\theta$. We note that terms of the type $(\mathbf{Q}_a \cdot \mathbf{Q}_b)^2$ arise in the description of ferromagnets with random second-order anisotropy axis,[15] but the vector $\mathbf{Q}_a$ is then an independent variable and is not equal to $\nabla\theta_a$. In our case $\tilde{H}[\theta_a]$ can be represented as a series in even Legendre polynomials in $(\mathbf{Q}_a \cdot \mathbf{Q}_b)/Q_a Q_b$, i.e., anisotropies of all order are present.

The foregoing derivation of the Hamiltonian $H[\theta_a]$ (4.5) is valid at temperatures near the transition point. The Hamiltonian retains its form also at lower temperatures, but the equations for its parameters are somewhat different. In particular, in the region $T \ll T_c = c/3$ we obtain, in analogy with the foregoing,

$$H[\theta_a] = \int \left\{ \frac{3cT_c}{2\varkappa^2 T} \sum_a \left[\left(\frac{(\nabla\theta_a)^2 - p_0^2}{2p_0} \right)^2 + \frac{(\nabla^2\theta_a)^2}{4p_0^2} \right] - \frac{9T_c^2 p_0^2}{192\pi T^2 \varkappa} \sum_{ab} \nabla\theta_a \nabla\theta_b \right\} d^3x. \tag{4.5}$$

2. A Hamiltonian of type (4.5) for long-wave fluctuations was investigated in detail by us[6] in connection with an Ising spin-glass model. It was shown that the bare Green's function of the long-wave fluctuations

$$G_0(\mathbf{p}) \infty [(\mathbf{pQ})^2 + p^4]^{-1}$$

is substantially renormalized by the presence of H_R and takes the form $G(\mathbf{p}) \propto p^{-7/2}$. Let us describe briefly the derivation given in Ref. 6. The change of variables $x = \tilde{x}/2p_0$, $\theta = \tilde{\theta}/2$ transforms $H[\theta_a]$ into

$$H[\tilde{\theta}_a] = \int d^3\tilde{x} \left\{ \frac{1}{2t} \sum_a \left[\frac{A}{4} ((\tilde{\nabla}\tilde{\theta}_a)^2 - 1)^2 + B(\tilde{\nabla}^2\tilde{\theta}_a)^2 \right] - \frac{g}{t^2} \sum_{ab} (\tilde{\nabla}\tilde{\theta}_a \tilde{\nabla}\tilde{\theta}_b) \right\}, \tag{4.6}$$

where at $|\tau| \ll 1$ we have

$$A = B = 1, \quad t = \frac{8\varkappa^2 p_0}{5c|\tau|}, \quad g = \frac{\varkappa^3 p_0^3}{54\pi c^2 |\tau|^{5/2}},$$

and at $T \ll T_c$

$$t = \frac{8\varkappa^2 p_0 T}{3cT_c}, \quad g = \frac{\varkappa^3 p_0^3}{24\pi c^2}.$$

The bare Green's function corresponding to (4.6) is of the form ($N = 0$)

$$G_{ab}^{(0)} = t\delta_{ab} G_0(\mathbf{p}) + gp^2 G_0^2(\mathbf{p}), \tag{4.7}$$

where

$$G_0(\mathbf{p}) = [A(\tilde{\mathbf{Q}}\tilde{\mathbf{p}})^2 + Bp^4]^{-1}, \tag{4.8}$$

$\tilde{\mathbf{Q}} = \tilde{\nabla}\tilde{\theta}$ is the "bare" wave vector, $\tilde{\nabla} = 2p_0\nabla$, $\tilde{\mathbf{Q}}^2 = 1$. We note that the fluctuations of a field $\theta(\mathbf{x})$ with a momentum $\mathbf{p}$ correspond to fluctuations of the total field $\mathbf{S}(\mathbf{x})$ with momenta $\mathbf{Q} + \mathbf{p}$. Equations (4.6)–(4.8) for the Hamiltonian and for the Green's functions are valid over rather large scales, where the fluctuations of the order-parameter amplitude can be neglected. These fluctuations are small [see (3.7)] at $|\mathbf{p} - \mathbf{p}_0| \ll \varkappa\tau^{1/2}$, so that in the dimensionless variables used in (4.6)–(4.8) we have

$$(\tilde{\mathbf{Q}}\tilde{\mathbf{p}}, \tilde{\mathbf{p}}^2) \ll \varkappa\tau^{1/2}/p_0 = q_0^2.$$

The first term of $B_{ab}(\mathbf{x})$ corresponds physically to the averaged irreducible correlator $\overline{\langle\langle \theta(0)\theta(\mathbf{x}) \rangle\rangle}$ that characterizes the thermodynamic properties of the system; the second term $\overline{\langle\theta(0)\rangle\langle\theta(\mathbf{x})\rangle}$ describes the disorder-induced deformation of the "bare" structure with $\theta(\mathbf{x}) = \mathbf{p}_0 \cdot \mathbf{x}$. At $g = 0$ the theory (4.6) is logarithmically renormalizable,[16] and the renormalization changes the parameters A and B:

$$A(\mathbf{p}) = \left(1 + \frac{5t}{64\pi}\xi\right)^{-4/5}, \quad B(\mathbf{p}) = \left(1 + \frac{5t}{64\pi}\xi\right)^{2/5}, \quad \xi = \ln\frac{q_0}{p} \tag{4.9}$$

The renormalization is caused by the strong thermal fluctuations that are typical of a $3D$ system with one-dimensional periodicity. It is convenient to carry out the calculations by representing the field $\tilde{\theta}(\mathbf{x})$ as a sum of a "slow" part $\tilde{\theta}_0(\mathbf{x})$ and a small "fast" part $\tilde{\theta}_1(\mathbf{x})$. As a result, the coefficients A and B acquire increments proportional to

$$\sum_b \frac{1}{t} \int G_{ab}(\mathbf{p}) G_{ba}(\mathbf{p}) p^4 \frac{d^3p}{(2\pi)^3} \propto t\xi. \quad (4.10)$$

At $g \neq 0$ the first-order corrections to A and B also take the form (4.10), but the most singular contribution comes from the crossover product of the first term of (4.7) by the second:

$$-\delta A^{(1)} = 6\delta B^{(1)} = \frac{g}{(2\pi)^3} \int p^6 G_0^3(\mathbf{p})\, d^3p = \frac{3g}{64\pi q^2}. \quad (4.11)$$

The integral (4.11) diverges quadratically at the lower limit, and q is the cutoff momentum. The corrections (4.11) are due not to thermal fluctuations but to deformations produced in the one-dimensionally periodic ground state by the disorder of the system. It is convenient to begin the summation of these strongly diverging corrections with a formal consideration of a model such as (4.6) in a space of dimensionality $5 - \varepsilon$ (at $d = 5$ the integral (4.11) would diverge logarithmically). This was done by us in Ref. 6, where the Hamiltonian (14) was investigated in detail in connection with Ising spin glass. It was shown that at $d = 5 - \varepsilon$ there is a stable fixed point at which the parameters A and B are power-law functions:

$$A(\mathbf{p}) \propto (\mathbf{pQ})^{6\varepsilon/11}, \quad B(\mathbf{p}) \propto p^{-2\varepsilon/11}$$

(the exponents are given in first order in ε). This means that the disorder-induced structure deformations lead to partial isotropization of the spectrum. Of course, the exponents obtained in the first ε-approximation cannot be used at $\varepsilon = 2$. In Ref. 6 are advanced arguments (which we shall not repeat here) according to which for $d = 3$ the Green's function has at $p \ll q_1 \approx g^{1/2}/8$ the form

$$G^{ab}(\mathbf{p}) = tG(\mathbf{p})\delta_{ab} + gp^2G^2(\mathbf{p}), \quad G(\mathbf{p}) \approx [\bar{A}(\mathbf{pQ})^{3/2} + \bar{B}p^{7/2}]^{-1}, \quad (4.12)$$

where the coefficients $\bar{A} \approx q_1^{-3}$ and $\bar{B} \approx q_1^{1/2}$ are determined, in order of magnitude, by joining (4.12) to (4.8) at $\tilde{p} \sim q_1$, $(\tilde{\mathbf{p}}\tilde{\mathbf{Q}}) \sim q_1^2$ (here $q_1 \approx g^{1/2}/8$ is the scale over which the correction (4.11) becomes of the order of unity). Equation (4.12) gives the correlator of the small fluctuations relative to the disorder of the ground state (we note that the parameter g is not renormalized, so that Eq. (4.7) with $G_0(\mathbf{p})$ replaced by $G(\mathbf{p})$ remains valid).

For the foregoing calculations to be self-consistent it is necessary to satisfy the condition $q_1 \ll q_0$ (where $q_0 = (\varkappa|\tau|^{1/2}/p_0)^{1/2}$ is the cutoff momentum for the long-wave Hamiltonian $H[\theta]$. Substituting $q_1 \approx g^{1/2}/8$ and g from (4.6), we get

$$q_1/q_0 \approx 0{,}1\gamma|\tau|^{-3/2} \ll 1, \quad (4.13)$$

as required.

3. Equation (4.12) for the correlator of the phase fluctuations shows that the rms deformations of the phase $\theta(\mathbf{x}) = \bar{\theta}(\mathbf{x})/2$ increase quadratically with distance:

$$\overline{(\langle\theta(0)\rangle - \langle\theta(\mathbf{x})\rangle)^2} = \frac{1}{2(2\pi)^3} \int gp^2G^2(\mathbf{p})\, d^3p\, (1 - \cos \mathbf{px}) \propto x^2 q_1^2, \quad (4.14)$$

therefore the helical long-range order is completely destroyed at $x \gtrsim q_1^{-1}$. The direction of the wave vector $\mathbf{Q} = \nabla\theta$ of the helix varies slowly in space:

$$C(\mathbf{x}) = \overline{\langle \mathbf{Q}(0)\mathbf{Q}(\mathbf{x})\rangle} \propto x^{-\Delta}, \quad \Delta \approx 2.5 q_1^2 \quad (4.15)$$

(see Eq. (30) of Ref. 6 and the discussion that follows it). Averaged over the system, there is therefore no preferred direction of $\mathbf{Q}$, and the correlator $G(\mathbf{p})$ at the smallest momenta should become fully isotropic ($A = B$). We, however, shall not investigate this region of extremely large scales, and confine ourselves to distances in the interval $q_1^{-1} \ll x \ll e^{1/\Delta}$, where a local direction of $\mathbf{Q}$ is defined and the correlator is given by (4.12).

We consider now the thermal fluctuations of the phase $\delta\theta(\mathbf{x}) = \theta(\mathbf{x}) - \langle\theta(\mathbf{x})\rangle$. They are determined by the first term of (4.12) and turn out to be strongly divergent

$$\langle(\delta\theta)^2\rangle = \frac{t}{4(2\pi)^3} \int G(\mathbf{p})\, d^3p \approx 0{,}03t(q_1L)^{1/4}, \quad (4.16)$$

where L is the scale of the long-wave cutoff (in a direction transverse to $\mathbf{Q}$). It is most important that the fluctuations can nevertheless be regarded as Gaussian. The point is that $A(\mathbf{p}) \sim \bar{A}p^{3/2}$, and therefore the interaction vertex Γ that stems from the term $A(\nabla\theta)^4$ contains a high power of the momentum: $\Gamma(\mathbf{p}) \propto p^{11/3}$, and the interaction of the long-wave fluctuations is small (see the detailed discussion of a situation of this kind in the recent paper[17] as applied to smectic liquid crystals). This means that mean values such as $\langle\cos(\delta\theta)\rangle$ can be calculated from the formula $\langle\cos(\delta\theta)\rangle = \exp[-1/2\langle(\delta\theta)^2\rangle]$. The divergence of $\langle(\delta\theta)^2\rangle$ at $L = \infty$ shows that the mean value of the "molecular field" $|\langle\mathbf{S}(\mathbf{x}_i)\rangle| = \langle\rho\cos\theta(\mathbf{x}_i)\rangle$ is zero, and with it also the Edwards-Anderson parameter $q_{EA} = \langle m_i\rangle^2 = 0$.

We have thus an equilibrium low-temperature spin-glass phase with $q_{EA} = 0$ and consequently with a "paramagnetic" linear susceptibility $\chi = c/3T$. It would be quite difficult to distinguish between such a spin glass and a paramagnet by magnetic measurements, although formally a difference between them is ensured by the existence of the slowly decreasing correlator (4.15). These conclusions are based essentially on the absence of a long-wave cutoff of the functions ($L = \infty$) in the isotropic Hamiltonian (3.1), (3.2). The weak anisotropy, which is always present in a real system, leads, as we shall show below, to the appearance of a finite cutoff scale L and to a substantial change of the results obtained for the physical quantities.

V. EFFECT OF INTERACTION ANISOTROPY

1. Dipole forces and anisotropy of the easy plane type

Dipole-dipole interaction of the magnetic moments always takes place in real magnets. Its low intensity (compared with the exchange interaction) notwithstanding, its long range can lead to substantial effects in strongly fluctuating systems (e.g., in two-dimensional ferromagnets[18]). We shall show that the dipole forces couples of the helix vector $\mathbf{Q} = \nabla\theta$ to the normal vector n. (The vectors $\mathbf{Q}$ and $\mathbf{n}$ turn out to be parallel.) If, furthermore, the crystal has easy-plane anisotroy that fixes the direction of $\mathbf{n}$, the direction of $\mathbf{Q}$ cannot change freely, and the long-wave fluctuations of the phase θ turn out to be suppressed.

The dipole-interaction energy is given by

$$E_D = {}^1/_2 (g_L\mu_B)^2 \sum_{ij} [\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - 3(\boldsymbol{\sigma}_i\mathbf{n}_{ij})(\boldsymbol{\sigma}_j\mathbf{n}_{ij})] r_{ij}^{-3}, \tag{5.1}$$

where $\mathbf{n}_{ij} = \mathbf{r}_{ij}/r_{ij}$, and the summation is over the sites of the spins $\boldsymbol{\sigma}_i$. For our pupose it suffices to find the contribution of the dipole forces to the long-wave effective Hamiltonian (4.6). To this end we must substitute in (5.1) $\boldsymbol{\sigma}_i = T^{-1}\mathbf{S}^{(0)}(\mathbf{x}_i)$, where $\mathbf{S}^{(0)}(\mathbf{x}_i)$ is given by the first two terms of (3.6) (the unperturbed helix), and replace the summation by integration. Let

$$\mathbf{Q} = p_0(\sin\alpha, 0, \cos\alpha), \quad \theta_{\mathbf{x}} = \theta_0 + \mathbf{Q}\mathbf{x},$$

Then

$$E_D = \frac{(g_L\mu_B)^2 c^2 \rho}{T^2} \iint d^3x\, d^3R R^{-3} [\cos\mathbf{QR} - 3\sin^2\psi \times \cos(\Phi - \mathbf{Qx})\cos(\Phi - \mathbf{Q}(\mathbf{x}+\mathbf{R}))]. \tag{5.2}$$

The integration over d^3x reduces to averaging of the integrand over $\mathbf{Q}\cdot\mathbf{x}$, so that we have for the dipole energy $(T \approx T_c)$

$$E_D = 9V(g_L\mu_B)^2\rho \int_0^\infty \frac{dR}{R} \int_0^\pi d\psi \sin\psi \int_0^{2\pi} d\Phi \left(1 - \frac{3}{2}\sin^2\psi\right) \times \cos[(\sin\alpha\sin\psi\cos\Phi + \cos\alpha\cos\psi)QR] = 18\pi V(g_L\mu_B)^2\rho({}^1/_3 - \cos^2\alpha),$$
$$\cos\alpha = \mathbf{Qn}/p_0, \tag{5.3}$$

where V is the volume of the system. Thus, the dipole forces tend to orient the vector $Q = \nabla\theta$ parallel (or antiparallel) to $\mathbf{n}$. We assume for the sake of argument that the easy-plane anisotropy that sets the direction of $\mathbf{n}$ is stronger than the dipole interaction (5.3). The direction of $\mathbf{n}$ is then rigidly prescribed, and the dipole energy produces in the long-term Hamiltonian an additional term in the form

$$\tilde{H}[\tilde\theta] = \int d^3\tilde{x} \left\{ \frac{1}{2t} \sum_a \left[\frac{A}{4}((\tilde\nabla\tilde\theta_a)^2 - 1)^2 + B(\tilde\nabla^2\tilde\theta_a)^2 - \mu(\tilde\nabla\tilde\theta_a\mathbf{N})^2 \right] - \frac{g}{t^2} \sum_{ab} (\tilde\nabla\tilde\theta_a)(\tilde\nabla\tilde\theta_b) \right\}, \tag{5.4}$$

where $\mathbf{N} = \mathbf{n}$, $\mu = 18\pi(g_L\mu_B)^2\varkappa^2/p_0^2 \ll 1$. The functional $\tilde{H}[\tilde\theta]$ reaches a minimum at $\nabla\tilde\theta = \tilde{Q}\mathbf{n}$, $\tilde{Q}^2 = 1 + 2\mu/A$. The unrenormalized Green's function takes now the form

$$G_0(\mathbf{p}) = [A(\tilde{\mathbf{Q}}\tilde{\mathbf{p}})^2 + \tilde{B}\mathbf{p}^4 + \mu(\tilde{p}^2 - (\tilde{\mathbf{p}}\mathbf{n})^2)]^{-1}. \tag{5.5}$$

Therefore the fluctuation with the smallest momenta $(p \lesssim q_2)$ are found to be suppressed. We determine q_2, assuming that $q_2 \ll q_1 \approx g^{1/2}/8$. In this case the Green's function takes in the region $q_2 \ll p \ll q_1$ the form [see (4.12)]

$$G^{ab}(\mathbf{p}) = t\delta^{ab}G(\mathbf{p}) + gp^2G^2(\mathbf{p}),$$
$$G(\mathbf{p}) = [\bar{A}(\widetilde{\mathbf{np}})^{7/2} + \bar{B}\tilde{p}^{7/2} + \mu\tilde{\mathbf{p}}_\perp^2]^{-1} \tag{5.6}$$

(it can be shown that the parameter μ is not renormalized). Comparing the third term of $G^{-1}(p)$ with the second, we get

$$q_2 \approx \mu^{2/3}\bar{B}^{-2/3} \approx \mu^{2/3} q_1^{-1/3}. \tag{5.7}$$

The quantity q_2^{-1} assumes the role of the long-wave cutoff L in (4.16). Calculating $\langle(\delta\tilde\theta)^2\rangle$ with the aid of the correlator (5.6), we obtain $(\theta = \tilde\theta/2)$:

$$\langle(\delta\theta)^2\rangle \approx 0.05 t q_1^{2/3} \mu^{-1/3}. \tag{5.8}$$

The mean value of the molecular field differs thus from zero and is equal to

$$|\langle S\rangle| = (2\rho)^{1/2}\langle\cos(\delta\theta)\rangle = c({}^5/_3|\tau|)^{1/2}\exp[-{}^1/_2\Pi|\tau|^{-11/6}], \tag{5.9}$$

where

$$\Pi \approx 1\cdot 10^{-3}\varkappa^{7/3}p_0^{8/3}c^{-5/3}(g_L^2\mu_B^2)^{-1/2}. \tag{5.10}$$

The value of Π can be both large and small in the considered parameter range $\gamma = \varkappa p_0^2/4\pi c \ll 1$. The magnetic properties of the system depend strongly on the value of Π. Before we proceed to study them, we consider an alternate mechanism that leads to the appearance of terms of the type μp^2 in $G^{-1}(\mathbf{p})$, i.e., to cutoff of the fluctuations.

2. Spatial anisotropy of initial interaction

The foregoing analysis is based on the spin-spin interaction (3.2) that depends only on the distances between the spins; its Fourier transform has a maximum on the sphere $|\mathbf{p}| = p_0$:

$$V_0(\mathbf{p}) = W_0\{[(p^2 - p_0^2)/2p_0\varkappa]^2 + 1\}^{-1}. \tag{5.11}$$

Real spin glass is as a rule a disordered solution of magnetic atoms in a crystal matrix whose axes define preferred directions in space. The spin interaction must therefore depend not only on $|\mathbf{p}|$ but also on the direction of $\mathbf{l} = \mathbf{p}/p_0$.

We shall assume hereafter this anisotropy to be weak (an exact criterion will be given below). In some cases the anisotropy of $V(\mathbf{p})$ can be described in the form of a "slow" dependence of the parameters p_0 and $\varkappa$ on $\mathbf{l}$:

$$V_1(\mathbf{p}) = W_0\{[(p^2 - p_0^2(\mathbf{l}))/2p_0(\mathbf{l})\varkappa(\mathbf{l})]^2 + 1\}^{-1}, \tag{5.12}$$

i.e., the maximum of $V_1(\mathbf{p})$ is reached on a nonspherical surface, but the value of $V_{1\,\max}(\mathbf{p}) = W_0$ is constant on this surface. No substantial changes from the foregoing take place in this case, since the main fact, that an infinite number of modes are simultaneously unstable at $T = T_c$, remains in force, and the resultant helical structure can be continuously transformed without a change of energy (in the zeroth approximation in the fluctuations it is necessary to put for this purpose $\mathbf{Q} = \mathbf{l}\, p_0(\mathbf{l})$, where $\mathbf{l}$ is an arbitrary unit vector). The effect of the dependence of W_0 on $\mathbf{l}$ is more substantial. In this case the instability at $T = T_c$ sets in at individual points rather than on the entire surface, and the minimum of the free energy of the helical phase is reached on a discrete set of directions of $\mathbf{Q}$. As a result, the effective long-wave Hamiltonian assumes a form similar to (5.4), but the parameter μ and the vector $\mathbf{N}$ are determined by the $W_0(\mathbf{l})$ dependences. We obtain them by regarding p_0 and $\varkappa$ as constants, and $W_0(\mathbf{l})$ as a weak function of $\mathbf{l}$:

$$W_0(\mathbf{l}) = 1 + w(\mathbf{l}), \quad \langle w(\mathbf{l})\rangle = 0, \quad |w(\mathbf{l})| \ll 1,$$

so that the interaction takes the form

$$V(\mathbf{p}) = (1 + w(\mathbf{l}))\{[(p^2 - p_0^2)/2p_0\varkappa]^2 + 1\}^{-1} \tag{5.13}$$

and leads to the Hamiltonian (3.5) in which the parameter τ is replaced by $\tau(\mathbf{l}) = \tau - w(\mathbf{l})$. Under the condition $|w(\mathbf{l})| \ll \gamma^{2/3}$ the properties of the short-wave fluctuations $\varphi_\alpha(\mathbf{x})$ do not change. In the region $\tau < 0$, $|\tau(\mathbf{l})| \gg \gamma^{2/3}$ the minimum of the free energy is reached for a helical ordering with $\mathbf{l} = \mathbf{l}_0$, where $\mathbf{l}_0$ is one of the maxima of the function $w(\mathbf{l})$, and is equal to $5/12\tau^2(\mathbf{l}_0)c^2$. [see (3.5) and (3.6). The dependence of the effective-Hamiltonian density on $\mathbf{l}$–$\mathbf{l}_0$ is therefore determined by the expression

$$H_A = {}^5/_{12}(c^2/T)[\tau^2(\mathbf{l}_0) - \tau^2(\mathbf{l})] = {}^5/_6(c^2/T)|\tau|[w(\mathbf{l}_0) - w(\mathbf{l})]. \tag{5.14}$$

Assuming the maximum of $w(\mathbf{l})$ to be isotropic and expanding in powers of $\mathbf{l} - \mathbf{l}_0$, we get

$$H_A = \frac{5}{12}\frac{|\tau|c^2}{T} w_2\left(\frac{\nabla\theta}{p_0} - \mathbf{l}_0\right)^2, \quad w_2 = \left.\frac{\partial^2 w}{\partial \mathbf{l}^2}\right|_{\mathbf{l}=\mathbf{l}_0}. \tag{5.15}$$

Using (5.15) in place of (5.3) we obtain an effective Hamiltonian of type (5.4), in which the third term in the square brackets takes the form $\mu_1(\tilde{\nabla}\tilde{\theta} - \mathbf{l}_0)^2$, where

$$\mu_1 = {}^1/_2(\varkappa/p_0)^2 w_2. \tag{5.16}$$

(Equation (5.16) was derived for temperatures $T \approx T_c$, $|\tau| \ll 1$. At $T \ll T_c$ the result will be practically the same, with a slight change of the numerical coefficient.) The bare correlator $G_0(\mathbf{p})$ takes then the form

$$G_0(\mathbf{p}) = [A(\mathbf{l}_0\tilde{\mathbf{p}})^2 + B\tilde{p}^4 + \mu_1\tilde{p}^2]^{-1}, \tag{5.17}$$

so that all the results obtained for the case of a dipole interaction in an easy-plane magnet hold also for a system of Heisenberg spins with the weakly (spatially) anisotropic interaction (5.13) (substituting $\mu \to \mu_1$).

VI. MAGNETIC PROPERTIES

1. For the standard phase of a Heisenberg magnet a distinction must be made between the responses to magnetic fields that lie in the spin-rotation plane (the latter defined by its own normal $\mathbf{n}$) and perpendicular to it. We have shown (see Sec. III) that the fluctuations (both thermal and those due to disorder) do not destroy the long-range order of the directions of $\mathbf{n}$. This means that such a spin glass has the anisotropy of the magnetic properties. The response to a longitudinal magnetic field (relative to the spin-rotation plane) depends on the fluctuations of the soft phase variable $\theta(\mathbf{x})$ and has therefore a complicated structure even in fields that are weak compared with the exchange energy; this is the response that we shall investigate in the main. The response to the transverse field does not depend on the $\theta(\mathbf{x})$ fluctuations and does not have such a structure. We present therefore simply the results for the linear transverse susceptibility near T_c ($|\tau| \ll 1$):

$$\chi_{\alpha\beta} = \chi_\perp(\delta_{\alpha\beta} - \rho_{\alpha\beta}) + \chi_\parallel \rho_{\alpha\beta},$$
$$\chi_\perp = \frac{1}{T}\left(\frac{1}{3} - \frac{2}{45}\frac{\rho}{T_c^2}\right) = \frac{1}{3T_c} + O(\tau^2). \tag{6.1}$$

We have substituted $\rho = 5/6|\tau|c^2$, $T_c = c/3$ (see Sec. III). The tensor $\rho_{\alpha\beta}$ was defined in (3.7).

We proceed to calculate the longitudinal linear susceptibility

$$\chi_\parallel = \frac{\rho_{\alpha\beta}}{2}\frac{T}{N}\frac{\partial^2 \ln Z}{\partial h_\alpha \partial h_\beta} = \frac{\rho_{\alpha\beta}}{2TN}\left[T\sum_i\left\langle \frac{\partial}{\partial S_\alpha} Y_\beta\left(\frac{\mathbf{S}_i}{T}\right)\right\rangle + \sum_{ij}\left\langle\!\left\langle Y_\alpha\left(\frac{\mathbf{S}_i}{T}\right) Y_\beta\left(\frac{\mathbf{S}_j}{T}\right)\right\rangle\!\right\rangle\right], \tag{6.2}$$

where $Y_\alpha = x_\alpha(x^{-1}\coth x - x^{-2})$.

We consider in succession the temperature regions $|\tau| \ll 1$ and $T \ll T_c$. In the first region, expanding (6.2) in powers of $\mathbf{S}_i/T \ll 1$, we get

$$\chi_\parallel = 1/3T + (\rho/45T_c^3)[1 - 5\overline{\langle\cos(\delta\theta)\rangle^2}]. \tag{6.3}$$

Using (5.9), we obtain ultimately

$$\chi_\parallel = \frac{1}{3T_c}\left\{1 + \frac{3|\tau|}{2}\left[1 - \frac{5}{3}\exp(-\Pi|\tau|^{-11/6})\right]\right\}. \tag{6.4}$$

We note that the terms proportional to ρ and not containing $\langle\cos(\delta\theta)\rangle$ cancel out in the susceptibility averaged over the directions

$$\chi = \frac{1}{3}(2\chi_\parallel + \chi_\perp) = \frac{1}{3T_c}\left\{1 - \frac{5|\tau|}{3}\exp(-\Pi|\tau|^{-11/6})\right\}. \tag{6.5}$$

This means that χ, in contrast to $\chi_\parallel$ and $\chi_\perp$, is expressed through the mean values of the Edwards-Anderson order parameter. At low temperature we use the asymptotic form of the functions $Y_\alpha(x)$:

$$\chi_\parallel = \frac{1}{2T}\left[1 - \frac{\overline{\langle S\rangle^2}}{\langle S^2\rangle}\right] = \frac{1}{2T}[1 - \overline{\langle\cos(\delta\theta)\rangle^2}]. \tag{6.6}$$

In the derivation of (6.3) and (6.6) we have retained only terms with $i = j$ from the first term of (6.2), since $\langle\!\langle \mathbf{S}_i \mathbf{S}_j \rangle\!\rangle|_{i\neq j}$ is small relative to the parameter $\varkappa/p_0$.

Calculating $\langle\cos(\delta\theta)\rangle$ at $T \ll T_c$ in analogy with the derivation of (5.9), we have

$$\chi_\parallel = \frac{1}{2T}\left[1 - \exp\left(-\tilde{\Pi}\frac{T}{T_c}\right)\right],$$
$$\tilde{\Pi} \approx 2\cdot 10^{-3}\varkappa^{7/3}p_0^{8/3}c^{-5/3}(g_L\mu_B)^{-4/3} \approx 2\Pi. \tag{6.7}$$

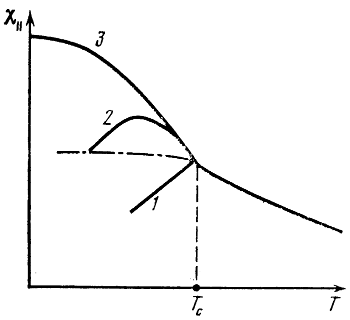


FIG. 2. Temperature dependence of the linear susceptibility. The dashed line shows the transverse susceptibility $\chi_\perp$. Lines 1, 2, and 3 represent the longitudinal susceptibility for different values of the parameter Π: $1-\Pi \ll \gamma^{11/9}$, $2-\gamma^{11/9} \ll \Pi \ll 1$, $3-\Pi \gg 1$.

The temperature behavior of the susceptibility depends substantially on the value of Π and is shown in Fig. 2. We note that only the first term of the expansion in T/T_c was retained in the argument of the exponential in (6.7). At $\Pi \ll 1$ this equation is therefore correct only for $\chi(T=0)$. It is interesting that the behavior of $\chi_\parallel$ is different at high temperatures even in the region of long phase fluctuations, when $\langle\cos(\delta\theta)\rangle \ll 1$: it can be seen from (6.4) and (6.7) the susceptibility $\chi_\parallel$ follows the paramagnetic law, but with the coefficient 1/2 typical of the planar XY spins. In other words, the formation of a local helical structure makes the Heisenberg spins effectively planar.

2. Nonlinear longitudinal susceptibility $\infty\chi_\parallel = (\partial^2\chi_\parallel/\partial h^2)_{h=0}$

The quantity $\tilde{\chi}_\parallel$ can be expressed in terms of the initial spin variables σ_i^α (we calculate hereafter only the longitudinal response and neglect the inessential fluctuations of the spins, so that the indices α and β take on the values 1 and 2):

$$\tilde{\chi}_\parallel = \frac{3}{8}\frac{1}{T^3 N}\sum_{ijkl}\overline{\langle\langle\sigma_i^\alpha\sigma_j^\beta\sigma_k^\alpha\sigma_l^\beta\rangle\rangle}. \tag{6.8}$$

The quantity σ_i^α oscillates rapidly, therefore the main contribution to the sum (6.8) is made by the terms with $i=j$, $k=l$; $i=k, j=l$ or $i=l, j=k$:

$$\tilde{\chi}_\parallel = \frac{3}{8NT^3}\sum_{i,j}[K_1(\mathbf{x}_i-\mathbf{x}_j)+2K_2(\mathbf{x}_i-\mathbf{x}_j)],$$
$$K_1=\overline{\langle\langle\sigma_i^2\sigma_j^2\rangle\rangle},\quad K_2=\overline{\langle\langle\sigma_i^\alpha\sigma_i^\beta\sigma_j^\alpha\sigma_j^\beta\rangle\rangle}. \tag{6.9}$$

We represent the tensor $\sigma_i^\alpha\sigma_i^\beta$ as a sum of irreducible parts:

$$\sigma_i^\alpha\sigma_i^\beta = {}^1/_2\delta^{\alpha\beta}\sigma_i^2 + (\sigma_i^\alpha\sigma_i^\beta - {}^1/_2\delta^{\alpha\beta}\sigma_i^2) = {}^1/_2\delta^{\alpha\beta}\sigma_i^2 + O_i^{\alpha\beta}.$$

The correlator K_2 breaks up then into the sum

$$K_2(\mathbf{x}_i-\mathbf{x}_j) = {}^1/_2K_1 + \overline{\langle\langle O_i^{\alpha\beta}O_j^{\alpha\beta}\rangle\rangle}. \tag{6.10}$$

The second term oscillates rapidly as a function of $\mathbf{x}_i$ and $\mathbf{x}_j$, so that after averaging over the impurity disposition the second term of (6.10) becomes small compared with K_1, and we get $K_2 = 1/2K_1$. Using the definition of the irreducible correlator and the identity $\boldsymbol{\sigma}_i^2 = 1$, we get

$$K_1 = -2\overline{\langle\langle\sigma_i^\alpha\cdot\;\rangle\rangle^2} + 4\overline{\langle\langle\sigma_i^\alpha\sigma_j^\beta\rangle\rangle\langle\sigma_i^\alpha\rangle\langle\sigma_j^\beta\rangle}. \tag{6.11}$$

The main contribution to the sum (6.9) over i and j is made by pairs with $|\mathbf{x}_i - \mathbf{x}_j| \gg c^{-1/3}$, so that the contribution from the pairs $i=j$ can be neglected. We express K_1 (at $i\neq j$) in terms of the mean value of the field variable $\mathbf{S}(\mathbf{x})$:

$$K_1 = -2\overline{\left\langle\!\left\langle Y^\alpha\left(\frac{\mathbf{S}_i}{T}\right)Y^\beta\left(\frac{\mathbf{S}_j}{T}\right)\right\rangle\!\right\rangle^2}$$
$$+4\overline{\left\langle\!\left\langle Y^\alpha\left(\frac{\mathbf{S}_i}{T}\right)Y^\beta\left(\frac{\mathbf{S}_j}{T}\right)\right\rangle\!\right\rangle\left\langle Y^\alpha\left(\frac{\mathbf{S}_i}{T}\right)\right\rangle\left\langle Y^\beta\left(\frac{\mathbf{S}_j}{T}\right)\right\rangle}. \tag{6.12}$$

Just as in the calculation of the linear susceptibility, we consider two temperature regions, $|\tau| \ll 1$ and $T \ll T_c$. Using (6.9)–(6.12), we have

$$\tilde{\chi}_\parallel = -\frac{3}{2NT^3}\sum_{i,j}[\overline{\langle\langle S_i^\alpha S_j^\beta\rangle\rangle^2} - 2\overline{\langle\langle S_i^\alpha S_j^\beta\rangle\rangle\langle S_i^\alpha\rangle\langle S_j^\beta\rangle}]\cdot\left\{\begin{matrix}(3T)^{-4}\\(3T_c)^{-4}\end{matrix}\right.. \tag{6.13}$$

The upper and lower lines refer to $T \approx T_c$ and the $T \ll T_c$, respectively.

To calculate the correlators in (6.13), we represent $\mathbf{S}$ in the form $(2\rho)^{1/2}(\cos\theta, \sin\theta)$ and use the replica method $(a \neq b \neq c \neq d)$:

$$\overline{\langle\langle S_i^\alpha S_j^\beta\rangle\rangle^2} = 4\rho^2[\langle\cos(\theta^a-\theta^b)_{\mathbf{x}_i}\cos(\theta^a-\theta^b)_{\mathbf{x}_j}\rangle$$
$$-2\langle\cos(\theta^a-\theta^b)_{\mathbf{x}_i}\cos(\theta^a-\theta^c)_{\mathbf{x}_j}\rangle$$
$$+\langle\cos(\theta^a-\theta^b)_{\mathbf{x}_i}\cos(\theta^c-\theta^d)_{\mathbf{x}_j}\rangle],$$
$$\overline{\langle\langle S_i^\alpha S_j^\beta\rangle\rangle\langle S_i^\alpha\rangle\langle S_j^\beta\rangle} = 4\rho^2[\langle\cos(\theta^a-\theta^b)_{\mathbf{x}_i}\cos(\theta^a-\theta^c)_{\mathbf{x}_j}\rangle$$
$$-\langle\cos(\theta^a-\theta^b)_{\mathbf{x}_i}\cos(\theta^c-\theta^d)_{\mathbf{x}_j}\rangle]. \tag{6.14}$$

In the calculation of the quantities of type $\langle\cos(\theta^a(0) - \theta^b(\mathbf{x}))\rangle$ we must average over those thermal fluctuations and deformations of the field $\theta(\mathbf{x})$ which are connected with the disorder. It was shown above that the thermal fluctuations with momenta $q \ll q_1$ are asymptotically Gaussian. In the calculation of the averages over the random deformations, the contribution of the non-Gaussian terms is of the order of $t\ln q_1|\mathbf{x}_i - \mathbf{x}_j|$. It will be shown below that the main contribution to $\chi_\parallel$ comes from the region $|\mathbf{x}_i - \mathbf{x}_j| \lesssim q_1^{-1}t^{-2}$, so that in the region $t \ll 1$ considered by us the contribution of the non-Gaussian terms is of the order of $t\ln t^{-1}$ and is small. Assuming thus all the fluctuations to be Gaussian and using expression (5.6) for $\langle\langle\delta\theta_a(0)\delta\theta_b(\mathbf{x})\rangle\rangle = G^{ab}(\mathbf{x})/4$, we obtain $\tilde{\chi}_\parallel$ in the form

$$\tilde{\chi}_\parallel \approx -c\int\exp\left[-\frac{t}{2}G(0)\right]\mathrm{sh}^4\left(\frac{tG(\tilde{\mathbf{x}})}{8}\right)d^3\tilde{x}\left\{\begin{matrix}25\tau^2/12T_c^3\\3/4T^3\end{matrix}\right.. \tag{6.15}$$

In the calculation of the integral (6.15) a distinction must be made between the cases of strong $\langle tG(0)/4 \rangle \gg 1$ [or, equivalently, $t \gg t_1 \approx 20(\mu/q_1^2)^{1/3})$] and weak ($t \ll t$ 1) thermal fluctuations. In the former case (corresponding to $\langle S_i \rangle^2 \ll \rho$) we have

$$\mathrm{sh}\,[tG(\mathbf{x})/8] \approx {}^1/_2 \exp\,[tG(\mathbf{x})/8];$$

in addition, we can neglect the term μp^2 in $(G^{-1}(\mathbf{p})$. As a result we obtain

$$\int \exp\left[-\frac{tG(0)}{2}\right] \mathrm{sh}^4\left[\frac{tG(\tilde{\mathbf{x}})}{8}\right] d^3\tilde{x} = \frac{a_1}{16} \bar{A}^2 \bar{B}^4 \left(\frac{t}{4}\right)^{-6}, \tag{6.16}$$

where a_1 is obtained by numerical integration:

$$a_1 = \iint dz\, d^2y \exp\left[\frac{1}{2\pi^2} \iint \frac{q\, dq\, dk}{k^{7/2}+q^{7/2}} (J_0(qy)\cos kz - 1)\right] \approx 3.9 \cdot 10^6.$$

If the anisotropy of the interaction is large enough, and the thermal fluctuations are correspondingly small, we get

$$\mathrm{sh}[tG(\mathbf{x})/8] \approx tG(\mathbf{x})/8, \quad \exp[-tG(0)/2] \approx 1.$$

Thus

$$\int \exp\left[-\frac{tG(0)}{2}\right] \mathrm{sh}^4\left[\frac{tG(\tilde{\mathbf{x}})}{8}\right] d^3\tilde{x} = \frac{a_2}{16} \bar{A}^{-6/7} \bar{B}^{4/21} \left(\frac{t}{4}\right)^4 \mu^{-10/3},$$

$$a_2 = \iint dz\, d^2y \left[\frac{1}{4\pi^2} \iint \frac{q\, dq\, dk J_0(qy)}{q^{7/2}+k^{7/2}+q^2} \cos kz\right]^4 \approx 7.9. \tag{6.17}$$

The behavior of the nonlinear susceptibility $\tilde{\chi}_\parallel$ (just as that of the linear one) depends strongly on the parameter Π [see (5.10)]. We consider first the case $\gamma^{11/9} \ll \Pi \ll 1$. In this region $|\tilde{\chi}_\parallel|$ has a sharp maximum near the transition point at $|\tau| \sim \tau^* = \Pi^{6/11}$:

$$\tilde{\chi}_\parallel = -60 \frac{c\tau^2}{T_c^3} \mu^{-2} f\left(\frac{|\tau|}{\tau^*}\right), \quad f(x) \approx \begin{cases} x^{11} & x \ll 1, \\ x^{-22/3}, & x \gg 1 \end{cases} \tag{6.18}$$

[the function $f(x)$ at arbitrary x can be obtained by numerical integration in (6.15)]. This peaking of the maximum is due to the large exponents in the asymptotic forms of $f(x)$ and not to a true singularity.

In the region $\Pi \gg 1$ the nonlinear susceptibility decreases rapidly with decreasing temperature, right up to $T^* = T_c/\Pi$, after which it decreases:

$$\tilde{\chi}_\parallel = -12 \frac{c\mu^{-2}}{T^3} \tilde{f}\left(\frac{T}{T^*}\right), \quad \tilde{f}(x) \approx \begin{cases} x^4, & x \ll 1, \\ x^{-6}, & x \gg 1. \end{cases} \tag{6.19}$$

3. Magnetic susceptibility in finite longitudinal fields

We show in this section that at sufficiently low temperatures the differential magnetic susceptibility $\chi_\parallel(h) = \partial M/\partial h$ increases with decreasing field and becomes paramagnetic in fields of the order of the anisotropy parameter μ. The reason for this effect is that the uniform magnetic field applied at random points to the helical structure acts in analogy with a random field in a ferromagnet, softens the spectrum, and decreases $\overline{\langle \mathbf{S} \rangle^2}$ accordingly because of the thermal fluctuations of the phase θ.

The increase of the thermal fluctuations affects strongly the susceptibility at $tG(0;h=0) \ll 1$, i.e., at $\overline{\langle \mathbf{S} \rangle^2}|_{h=0} = 2\rho$. Let us find the change of the Hamiltonian of the long-wave fluctuation in the presence of a finite magnetic field $\mathbf{h}$. We average over the disorder in the Hamiltonian (3.3), using $S \gg h$, and use the representation of $\mathbf{S}$ in the form $(2\rho)^{1/2}(\cos\tilde{\theta}, \sin\tilde{\theta})$. We get

$$H_h = \frac{ch^2}{16\rho_0{}^3 T_c{}^2} \int \sum_{a,b} \cos\frac{\tilde{\theta}_a}{2} \cos\frac{\tilde{\theta}_b}{2} d^3\tilde{x} \begin{cases} 5|\tau|/3, & |\tau| \ll 1, \\ (T_c/T)^2, & T \ll T_c. \end{cases} \tag{6.20}$$

We shall be interested hereafter in the influence of H_h on the spectrum of the long-wave oscillations. We can therefore neglect the rapidly oscillating part of H_h and replace $\cos(\tilde{\theta}_a/2)\cos(\tilde{\theta}_b/2)$ by $1/2 \cos(\tilde{\theta}_a - \tilde{\theta}_b)/2$. The condition $tG(0, h=0) \ll 1$ allows us to neglect the thermal fluctuations in the estimate of the characteristic values of h that alter the spectrum. To calculate the corrections that must be introduced into the Green's function on account of the magnetic field, it suffices to retain the first term of the expansion of H_h in terms of $\tilde{\theta}_a - \tilde{\theta}_b$. As a result we obtain the h-dependent contribution to the effective Hamiltonian (4.6), in the form

$$H_h[\tilde{\theta}_a] = \int d^3\tilde{x} \left(-\frac{u}{t^2}\right) \tilde{\theta}_a \tilde{\theta}_b,$$
$$u = \frac{h^2 \varkappa^4}{10\rho_0 c T_c{}^2} \begin{cases} 1/3|\tau|, & |\tau| \ll 1, \\ {}^5/_9 (T_c/T)^2, & T \ll T_c. \end{cases} \tag{6.21}$$

The "replica" Hamiltonian (6.21) means physically the presence in the system of random fields that are directly connected with the phase $\tilde{\theta}(\mathbf{x})$:

$$H_h[\tilde{\theta}] = \int d^3\tilde{x} \eta(\mathbf{x}) \tilde{\theta}(\mathbf{x}), \quad \overline{\eta(\mathbf{x})\eta(\mathbf{x}')} = 2u\delta(\mathbf{x}-\mathbf{x}') \tag{6.22}$$

(in contrast to $\tilde{H}[\tilde{\theta}]$ of Eq. (3.15), where only the gradient of the phase $\partial_\mu \theta(\mathbf{x})$ was connected with the random field). Random fields such as (6.22) produce structure deformations that increase most strongly at large distances and can therefore influence strongly the system properties even at very small u. We shall consider hereafter the region $u \ll g^2$, in which noticeable corrections due to $H_h[\tilde{\theta}]$ appear at scales $L \gg q_1^{-1}$, and must therefore be calculated using the Hamiltonian (5.4) with renormalized parameters A and B:

$$A(\mathbf{p}) = \bar{A}(\tilde{p}Q)^{3/2}, \quad B(\mathbf{p}) = \bar{B}\tilde{p}^{-1/2}, \tag{6.23}$$

where $\bar{A}$ and $\bar{B}$ are defined in (4.12). The first correction to $A(\mathbf{p})$ from H_h is of the form

$$\delta A(\mathbf{p}) \approx -\frac{u}{(2\pi)^3} \int_{\tilde{p}}^{\infty} d^3p_1 A^2(\tilde{\mathbf{p}}_1) \tilde{p}_1{}^4 G^3(\tilde{\mathbf{p}}_1) \tag{6.24}$$

[cf. the analogous expression (4.11)]. Substituting (6.23) and (5.6) in (6.24) we obtain

$$\frac{\delta\bar{A}(\mathbf{p})}{\bar{A}(\mathbf{p})} \approx -\frac{u}{g}\min\left(\frac{1}{\tilde{p}^2},\frac{1}{q_2^{\,2}}\right). \qquad (6.25)$$

The correction to $\bar{B}$ is similar, whereas the relative correction to μ is substantially smaller. It can be seen from (6.25) that with increasing scale the value of $\bar{A}$ ($\tilde{\mathbf{p}}$) decreases down to the dipole length q_2^{-1}. At sufficiently small q^2 the correction $\delta\bar{A}$ can become of the same order as $\bar{A}$, and this would mean a substantial softening of the spectrum an enhancement of the thermal fluctuations. Let us estimate the value of h at which this occurs, using the expressions (4.6), (5.4), (5.17), (5.16), (6.21), and (6.25):

$$h_0 \approx 4\varepsilon^{2/3} T_c \left(\frac{c}{p_0^{\,3}}\right)^{1/6} \gamma^{1/3} \begin{cases} |\tau|^{-1/3}, & |\tau| \ll 1, \\ T/T_c, & T \ll T_c, \end{cases} \qquad (6.26)$$

where ε can be replaced either by the parameter w_2 of the spatial anisotropy of the interaction $V(\mathbf{p})$, or by the characteristic relative value of the dipole energy $6\pi(g_L\mu)^2 c/T_c$. A quantitative examination of the properties of the spectrum and of the thermal-fluctuation intensity at $h \gtrsim h_0$ is a complicated problem, with which we shall not deal now. We expect the rms thermal fluctuation of the phase

$$\langle(\delta\theta)^2\rangle = \frac{t}{4(2\pi)^3}\int G(\mathbf{p})\,d^3p$$

to increase rapidly in this region, so that the local mean values $|\langle S_i\rangle|$ of the spins tend to zero, and the susceptibility $\chi_\parallel$ increases and eventually obeys the Curie law. We note that the field h_0 is substantially weaker than the characteristic exchange field ($\sim T_c$) and decreases with decreasing temperature. The behavior of the differential susceptibility $\chi_\parallel(T, h)$ is shown qualitatively in Fig. 3. The susceptibility $\chi_\perp$ remains unchanged in fields of the order of h_0.

VII. DISCUSSION OF RESULTS

1. We have shown that a system of Heisenberg spins randomly arrayed in space and interacting in accordance with Eq. (1.1) have a low-temperature phase with helical short-range order (at densities high enough so that $\gamma = \varkappa p_0^2/4\pi c \ll 1$). The helix is locally specified by a vector $\mathbf{Q} = \nabla\theta$ and by the direction of normal $\mathbf{n}$ to the spin-rotation plane. The span of the short-range order in terms of the variable θ is characterized by a length

$$L_\perp \approx (p_0\varkappa)^{-1/2} q_1^{-1} \approx 5\varkappa^{-1}\gamma^{-1} \quad (T \lesssim T_c,\ |\tau| \sim 1) \qquad (7.1)$$

in the direction transverse to the vector Q and by a length $L_\parallel \sim \varkappa L_\perp^{\,2}$ in the longitudinal direction. The lengths $L_\perp$ and $L_\parallel$ are large compared with the interaction radius $\varkappa^{-1}$, and it is this which allows the system to be described by the slow variables. In the parameter region where $\gamma \gg 1$ there are no slow variables and the problem is equivalent to the Edwards-Anderson model.

The $\approx$ symbol in Eq. (7.1) and those similar to it denotes equality to within a number of the order of unity. To facilitate comparison with experiment we have nevertheless retained the large numerical factors in the right-hand sides of these equations. Of course, these factors must not be taken too literally.

2. In the absence of any anisotropy of the interaction, the spin-rotation phase in the helix is subject to strong thermal fluctuations [see Eq. (4.16)], so that the local mean values $\overline{\langle\boldsymbol{\sigma}_i\rangle}$ and the Edwards-Anderson parameter $q_{EA} = \overline{\langle\boldsymbol{\sigma}_i\rangle^2}$ are zero. That this is not a paramagnetic state is indicated by the following: 1) the slow decrease of the correlator $C(\mathbf{x}) = \overline{\langle\mathbf{Q}(0)\mathbf{Q}(\mathbf{x})\rangle}$ of the helix directions [see (4.15)]; 2) the long range order with respect to the variable $\mathbf{n}$, which leads to anisotropy of the magnetic response ($\chi_{\alpha\beta} = \chi_\parallel\,\rho_{\alpha\beta} + \chi_\perp(\delta_{\alpha\beta} - \rho_{\alpha\beta})$). The second circumstance makes it relatively easy to distinguish in experiment between such a spin glass from the completely random one described by the Edwards-Anderson model. One can hardly expect a true long-range order with respect to $\mathbf{n}$ in a real system, since the system is as a whole isotropic; the corresponding correlation length (the size of the "domain") will be determined by sparse large fluctuations and must therefore be much larger than the microscopic lengths $L_\perp$ and $L_\parallel$.

3. The weak deviation of the interaction from the spherically isotropic form (1.1) singles out certain directions of $\mathbf{Q}$ in space (these directions are connected with the nonmagnetic-matrix crystal-lattice axes). As a result, the thermal fluctuations $\delta\theta(\mathbf{x})$ are suppressed, and a nonzero q_{AE} results. A similar effect is produced also by dipole interactions of the spins that couple the vectors $\mathbf{Q}$ and $\mathbf{n}$, in conjunction with the easy-plane anisotropy that fixes the direction of $\mathbf{n}$. In all the experimental realizations known to us, the host has a wide enough symmetry group, so that there exist several equivalent $\mathbf{Q}$ and $\mathbf{n}$ vector-pair directions relative to the crystal axes, and this should lead to domain formation. The dipole interaction always present will make $\mathbf{n}$ collinear with $\mathbf{Q}$ in each of the domains, i.e., the long-range order of $\mathbf{n}$ is destroyed.

The spatial scale L_A at which the thermal fluctuations are cut off is determined by the formula (see (5.7), $L_A = (p_0\varkappa)^{-1/2} q_2^{-1}$):

$$L_A \approx L_\perp (0.2\gamma)^{4/3} (p_0/\varkappa)^{2/3} \varepsilon^{-2/3}, \qquad (7.2)$$

where ε should stand either for the relative nonsphericity of the interaction w_2 [see (5.15)] or the ratio $6\pi(g_L\mu_B)^2 c/T_c$ of

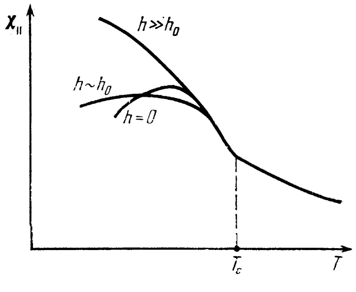


FIG. 3. Differential longitudinal susceptibility $\chi_\parallel(T, h)$ in various fields h. The quantity h_0 is defined in (6.26). The parameter Π is assumed to be in the region $\gamma^{11/9} \ll \Pi \ll 1$.

the characteristic dipole energy to the exchange energy. In our analysis we have assumed $L_A \gg L_\perp$, i.e., not too large a value of ε:

$$\varepsilon \ll (0.2\gamma)^2 p_0/\varkappa. \tag{7.3}$$

In the opposite limit (which is possible if the interaction is highly nonspherical) the directions of **Q** are fixed from the very beginning, the thermal fluctuations $\delta\theta(\mathbf{x})$ can be neglected, and the system breaks up into large domains inside of which there is a rigid helical order. In this case, just as in the one discussed in Sec. IV, the inhomogeneous arrangement of the spins produces a random field that acts on the vector **Q** (i.e., the term $\mathbf{h}(\mathbf{x})\cdot\mathbf{Q}$ in the energy). Thus, the equilibrium behavior of the system at the largest scales is described by the Potts q-component model[19] in a random field, and q is determined by the symmetry group of the crystal. This model is the subject of a large number of recent contradictory papers.[20–22] We shall not discuss this question here, all the more since a realization of any equilibrium situation in a real experiment on systems of this type, with large domains, seems unlikely to us.

4. The intensity of the thermal fluctuations is determined by the parameter Π [See Eqs. (5.9), (5.10), and (6.7)], which can be conveniently written in the form

$$\Pi \approx (0.2\gamma)^{5/3} (\varkappa/p_0)^{2/3} \varepsilon^{-1/3}. \tag{7.4}$$

Depending on the value of Π, the longitudinal susceptibility $\chi_\parallel(T)$ can behave in three ways, illustrated in Fig. 2. At $\Pi \ll \gamma^{11/9}$ the thermal fluctuations of the phase are small in the entire region where the helical phase exists ($|\tau| \gg \gamma^{2/3}$), and the susceptibility has a cusp at $T = T_c$. If $\gamma^{11/9} \ll \Pi \ll 1$ the behavior of the longitudinal susceptibility is described by Eq. (6.4); in the region $\gamma^{2/3} \ll |\tau| \ll \Pi^{6/11}$ it grows in accord with the Curie law for a planar spin, i.e., with an additional coefficient 3/2; at $|\tau| \sim \tau^* = \Pi^{6/11}$ there is a smooth maximum. At $\Pi \gg 1$ the longitudinal susceptibility increases as the temperature is lowered all the way to $T^* = T_c/\Pi$, after which it assumes a constant value of the order of $1/T^*$.

The transverse susceptibility $\chi_\perp$ does not depend on the parameter Π and "freezes" at a value $(3T_c)^{-1}$ [at least at $|\tau| \ll 1$, see (6.1)]. We emphasize that $\chi_\parallel - \chi_\perp > 0$ also in the region of strong thermal fluctuations of the phase, where $q_{AE} \approx 0$ (since these fluctuations do not alter the direction of the normal **n**). At the same time, the susceptibility $\chi = (\chi_\perp + 2\chi_\parallel)/3$ averaged over the directions is determined only by the value of q_{EA} [see (6.5)]:

$$\chi = (1 - q_{EA})/3T. \tag{7.5}$$

If the direction of **n** is fixed by an easy-plane anisotropy, the values of $\chi_\parallel$ and $\chi_\perp$ can be measured independently. In the absence of spin anisotropy the direction of n is determined either by the direction of **Q** via the dipole interaction), or by an external magnetic field **h** that tends to set **n** parallel (perpendicular) to the vector **h** at $\chi_\perp > \chi_\parallel$ ($\chi_\perp < \chi_\parallel$).

5. In the case of greatest interest, $\gamma^{11/9} \ll \Pi \ll 1$, the maximum of $\chi_\parallel(T, h)$ decreases rapidly with increasing h, so that the nonlinear susceptibility $\tilde{\chi}_\parallel(T) = \partial^2\chi_\parallel/\partial h^2|_{h=0}$ is large here [see (6.18), (6.19)]. With increasing magnetic field, the effective value of the disorder increases and the average spin (q_{AE}) decreases correspondingly. At $h \gg h_0$, where h_0 [see (6.26)] is much less than the characteristic exchange energy T_c, the parameter $q_{AE} \to 0$, therefore $\chi_\parallel(T, h)$ becomes purely paramagnetic in these fields, and the Curie constant has an additional coefficient 3/2 compared with the $T > T_c$ region (see Sec. III). The subdivision into $\chi_\perp$ and $\chi_\parallel$ is preserved, however, also in this case. Just as in weak fields, $\chi_\perp$ assumes a constant value when the temperature is lowered, and the entire paramagnetic growth of χ is due to the growth of $\chi_\parallel$.

6. The model considered by us can be realized in several classes of systems. First, in rare earth alloys with Y or Sc as the host.[23] In the pure state, the spectrum of the magnetic excitations has a substantial dip at a finite value of the momentum and moreover[24] this dip is almost symmetrical about the momentum directions. Inclusion of "magnetic" atoms in such a host leads to a glass-type low-temperature phase (see, e.g., Refs. 25 and 26), where the magnetic atoms were Er, Gd, and Tb).

The second class comprises classical spin glasses such as $Cu_{1-x}Mn_x$ with large magnetic-atom concentrations. Our calculations in Sec. II show that the appearance of a magnetic helical structure should be expected in the concentration region $x \gtrsim 0.1$. A similar structure was indeed observed in experiment,[8] but at such high Mn densities the copper Fermi surface assumes a highly irregular form, made up of necks, and the interaction becomes anisotropic. This is evidenced, in particular, by the large period of the structure compared with πp_F^{-1} (this period depends furthermore on the density c). Most probable for systems of this class is formation of a "soft" helix in alloys based on Ag, which has the most symmetric Fermi surface.

The third group comprises alloys such as chromium spinel $Zn_xCd_{1-x}Cr_2Se_4$, which was investigated in Refs. 27. At $x \geqslant 0.5$ the low-temperature phase of the chromium spinel is a helical antiferromagnet, and at $0.3 < x < 0.5$ is observed a phase of the spin-glass type. The possibility of applying our results to this substance is not quite clear, since the anisotropy energy of the produced helical structure is unknown.

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[1)]We consider only equilibrium thermodynamics, and the terms "slow" and "fast" pertain to the dependence of the fields on the spatial coordinates.

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