Damping of soliton oscillations in media with a negative dispersion law

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We consider the evolution of small two-dimensional perturbations of a plane soliton. We solve the Cauchy problem of the linearized Kadomtsev-Petviashvili equation. The obtained asymptotic values of the Green function as $t \to +\infty$, which give the damping rate of the soliton oscillations in media with a negative dispersion law.

INTRODUCTION

Kadomtsev and Petviashvili[1] were the first to derive an equation which describes quasi-two-dimensional weakly nonlinear waves in media with a weak dispersion. The equation was applied[1] to the study of soliton stability to small two-dimensional perturbations that are long-wave in the transverse direction. In media with a positive dispersion law the soliton turned out to be unstable, but it turned out to be stable in media with a negative dispersion law. The assumption was made[1] that if one took the finite transverse scale of the oscillations of the soliton in the stable case into account this would lead to a weak damping of the oscillations. Zakharov, using the inverse scattering method, obtained[2] an exact expression for the dispersion law of the soliton oscillations for arbitrary transverse scales and determined thus the damping rate for oscillations in media with a negative dispersion law. Pesenson and Shrira[3] considered the problem of the evolution of transverse perturbations on the front of a soliton and of shock waves, using the multiple-scale asymptotic method. In particular, in the case of the Kadomtsev-Petviashvili (KP) equation they found the same dispersion law for the soliton oscillations (apart from terms of higher order of smallness) as in Ref. 2. However, the fact that the soliton oscillations in Refs. 2 and 3 for media with a negative dispersion law have a weakly growing tail in the longitudinal direction going to infinity in one direction caused criticism and doubts about the validity of the results obtained about the damping of the oscillations with increasing time. It is necessary to note that the spatial growth of the pertubation mode in this case is a natural consequence of the Hamiltonian nature of the linearized KP equation (see §1). The weak damping of the mode with time, in conjunction with a weak growth in the spatial variable towards infinity in one direction, is typical of any Hamiltonian system and follows from the energy conservation law for the system. This property of Hamiltonian systems manifests itself in various fields of physics. One of the examples is the Gamow theory of α-decay of nuclei,[4] in which the α-particle wave function $\psi(r,t)$ which satisfies the Hamiltonian of the Schrödinger equation has exponential damping in time in conjunction with a weak increase as $r \to \infty$. In the present paper the Cauchy problem for the linearized KP equation is solved and we show that two-dimensional perturbations of the soliton are weakly damped in time in media with a negative dispersion law.

§1. INTEGRABILITY OF THE KP EQUATION USING THE INVERSE SCATTERING METHOD. EXACT SOLUTION OF THE LINEARIZED KP EQUATION

We consider the KP equation

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\frac{1}{4}\frac{\partial^3 u}{\partial x^3}+\frac{3}{2}u\frac{\partial u}{\partial x}\right]+\frac{3}{4}\beta^2\frac{\partial^2 u}{\partial y^2}=0, \quad (1)$$

where $\beta^2 = \pm 1$. Equation (1) is the quasi-two-dimensional analog of the Korteweg–de Vries (KdV) equation and describes the balance of two effects, nonlinearity and dispersion, under conditions where the transverse scale (along the y-axis) is much larger than the longitudinal scale (along the x-axis). The KP equation has the same degree of physical universality as the KdV equation. Equation (1) arises in hydrodynamics when one describes long-wavelength waves on shallow water, when one considers ion-acoustic waves in a plasma, and so on. The case $\beta^2 = 1$ corresponds to a medium with a negative dispersion law ($d^2\omega/dk^2 < 0$) and the case $\beta^2 = -1$ to a medium with a positive dispersion law ($d^2\omega/dk^2 > 0$). It was shown in Ref. 5 that the inverse scattering method is applicable to Eq. (1): if we take a function $F(x,z,y,t)$ satisfying the two equations

$$\frac{\partial F}{\partial t}+\frac{\partial^3 F}{\partial x^3}+\frac{\partial^3 F}{\partial z^3}-\nu^2\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\right)=0, \quad (2)$$

$$\beta\frac{\partial F}{\partial y}+\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 F}{\partial z^2}=0, \quad (3)$$

and solve the integral equation

$$F(x,z,y,t)+K(x,z,y,t)+\int_x^\infty K(x,s,y,t)F(s,z,y,t)\,ds=0, \quad (4)$$

for all x,y,z, and t, the quantity $u(x,y,t) = 2dK(x,x,y,t)/dx$ satisfies the equation

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}-\nu^2\frac{\partial u}{\partial x}+\frac{1}{4}\frac{\partial^3 u}{\partial x^3}+\frac{3}{2}u\frac{\partial u}{\partial x}\right]+\frac{3}{4}\beta^2\frac{\partial^2 u}{\partial y^2}=0. \quad (5)$$

Equation (5) is the KP equation written in a system of coordinates fixed in the moving soliton. The soliton

$$u_0(x)=2\nu^2/\mathrm{ch}^2\nu x \quad (6)$$

is a stationary solution of (5). For it

$$F=F_0(x,z)=2\nu e^{-\nu(x+z)},$$

$$K=K_0(x,z)=-2\nu e^{-\nu(x+z)}/(1+e^{-2\nu x}).$$

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The problem of the stability of the soliton (6) is very interesting. It was partially solved in Ref. 1 where it was shown that if we take

$$u=u_0+u_1,\quad u_1\infty e^{-i\Omega t+ipy},\quad u_1\ll u_0,$$

we get for values $p\ll 1$

$$\Omega^2(p)=\beta^2p^2\nu^2+\dots\ . \tag{7}$$

The soliton is thus stable against small perturbations which are long-wave in the transverse direction if $\beta^2=1$, and unstable if $\beta^2=-1$. The inverse scattering method allows us to determine exactly the dispersion of the soliton oscillations for any value of the parameter p. We follow Ref. 2 to consider the simplest way to solve this problem. Let

$$u=u_0+u_1,\quad F=F_0+F_1,\quad K=K_0+K_1.$$

The function $F_1(x,z,y,t)$ then satisfies Eqs. (2) and (3), and the function $K_1(x,z,y,t)$ the linearized Eq. (4):

$$F_1(x,z,y,t)+K_1(x,z,y,t)+\int_x^\infty K_0(x,s)F_1(s,z,y,t)\,ds$$

$$+\int_x^\infty K_1(x,s,y,t)F_0(s,z)\,ds=0; \tag{8}$$

here the quantity $u_1(x,y,t)=2dK(x,x,y,t)/dx$ satisfies the linearized KP equation:

$$\frac{\partial}{\partial x}\left[\frac{\partial u_1}{\partial t}-\nu^2\frac{\partial u_1}{\partial x}+\frac{1}{4}\frac{\partial^3u_1}{\partial x^3}+\frac{3}{2}\frac{\partial}{\partial x}(u_0u_1)\right]$$

$$+\frac{3}{4}\beta^2\frac{\partial^2u_1}{\partial y^2}=0. \tag{9}$$

Let

$$F_1(x,z,y,t)=\exp[-i\Omega t+ipy-nx-kz].$$

It then follows respectively from (2) and (3) that

$$-i\Omega=n^3+k^3-\nu^2(n+k), \tag{10}$$

$$i\beta p+n^2-k^2=0. \tag{11}$$

We determine the function K_1 from (8):

$$K_1(x,z=x,y,t)$$

$$=\exp[-i\Omega t+ipy-(n+k)x]\left[1-\frac{2\nu}{\nu+k}\frac{1}{1+e^{2\nu x}}\right]$$

$$\times\left[1-\frac{2\nu}{\nu+n}\frac{1}{1+e^{2\nu x}}\right]. \tag{12}$$

We consider first the case $\beta^2=-1$. From the condition that the function $u_1(x,y,t)$ decrease as $x\to-\infty$ we get

$$k=\nu, \tag{13}$$

$$\operatorname{Re}(n-\nu)<0, \tag{14}$$

and from the condition that u_1 decrease as $x\to+\infty$ we have

$$\operatorname{Re}(n+\nu)>0. \tag{15}$$

Let $i\beta p=|p|$; we then easily get n satisfying both conditions (14) and (15):

$$n=\pm[\nu^2-|p|]^{1/2}. \tag{16}$$

Let now $\beta=1$. It is clear from an analysis of Eq. (12) that it is impossible to achieve simultaneously the decrease of the function u_1 as $x\to\pm\infty$. Indeed, let

$$n=a+ib. \tag{17}$$

We then easily get

$$a=2^{-1/2}[(\nu^4+p^2)^{1/2}+\nu^2]^{1/2}, \tag{18}$$

$$b=-p/2a. \tag{19}$$

If $p\ll\nu^2$ the function u_1 weakly increases as $x\to-\infty$:

$$u_1(x,y,t)\infty\exp[-(p^2/8\nu^3)x].$$

Further, we get from (10), (11), and (13) the dispersion of the soliton oscillations:

$$-i\Omega=-i\beta pn, \tag{20}$$

where the quantity n is given by Eq. (16) in the unstable case, and by Eqs. (17) to (19) in the stable case. We rewrite the dispersion in closed form:

$$\Omega^2(p)=\beta^2p^2(\nu^2-i\beta p). \tag{21}$$

Expression (21) describes the instability growth rate of the soliton for $\beta^2=-1$ and the spectrum of the damped soliton oscillations for $\beta^2=1$. Considering (17) to (20), we get the damping rate:

$$\operatorname{Re}(-i\Omega)=-2^{-1/2}p^2[(\nu^4+p^2)^{1/2}+\nu^2]^{-1/2}<0.$$

If $|p|\to0$, the Kadomtsev-Petviashvili result (7) follows from (21). The weak growth of the function u_1 as $x\to-\infty$ in the stable case is a natural consequence of the Hamiltonian nature of the linearized KP Eq. (9). We rewrite (9) in the usual Hamiltonian form:

$$\frac{\partial u_1}{\partial t}=\frac{\partial}{\partial x}\frac{\delta H}{\delta u_1},$$

where

$$H=\iint\left[\frac{1}{2}\nu^2u_1^2+\frac{1}{8}\left(\frac{\partial u_1}{\partial x}\right)^2-\frac{3}{4}u_0u_1^2-\frac{3}{8}w^2\right]dx\,dy,$$

$$\frac{dw}{\partial x}=\beta\frac{\partial u_1}{\partial y}.$$

According to the mechanism proposed in Ref. 2 the soliton oscillations are damped due to the emission of small oscillations of the medium. In a medium with a negative dispersion law the soliton is supersonic and the small oscillations will therefore lag behind the soliton and go off to $x=-\infty$. Moreover, the obtained exact solution of Eq. (9)

$$u_1(x,y,t)=2dK_1(x,z=x,y,t)/dx,$$

where the function K_1 is given by Eq. (12), is not a solution of any Cauchy problem and we can let t tend to $-\infty$ in (12) and obtain in the stable case an infinite perturbation on the front of the soliton at $t=-\infty$. As the Hamiltonian of the system is conserved during the emission process, the energy of the infinite perturbation on the front of the soliton at $t=-\infty$ changes at a time $t=+\infty$ to the energy of a tail that increases weakly as $x\to-\infty$.

§2. THE GREEN FUNCTION OF THE LINEARIZED KP EQUATION AND ITS ASYMPTOTIC BEHAVIOR AS $t \to +\infty$

In this section we solve the Cauchy problem for the linearized KP Eq. (9). One easily checks directly that Eq. (9) has the following elementary solution:

$$u_1(x,y,t)=e^{-i\Omega t+ipy}\frac{d}{dx}\left\{e^{i(k+n)x}\left[1-\frac{2\nu}{\nu-ik}\frac{1}{1+e^{2\nu x}}\right]\times\left[1-\frac{2\nu}{\nu-in}\frac{1}{1+e^{2\nu x}}\right]\right\}, \quad (22)$$

where

$$-i\Omega(k,p)=i[n^3+k^3+\nu^2(n+k)], \quad (23)$$

$$i\beta p-n^2+k^2=0. \quad (24)$$

The quantities k and p are free parameters. Using the eigenfunctions $\psi(k,x)$ of the continuous spectrum of the soliton potential (6):

$$\psi_{xx}(k,x)+u_0(x)\psi(k,x)=-k^2\psi(k,x), \quad (25)$$

$$\psi(k,x)=e^{ikx}\left[1-\frac{2\nu}{\nu-ik}\frac{1}{1+e^{2\nu x}}\right], \quad (26)$$

we can rewrite (22) in the compact form:

$$u_1(x,y,t)=e^{-i\Omega t+ipy}\frac{d}{dx}[\psi(k,x)\psi(n,x)].$$

We consider next only media with a negative dispersion law: $\beta=1$. We introduce a parametrization of the quantities n and k using the real variable z:

$$k=k(z,p)=z-ip/4z, \qquad n=n(z,p)=z+ip/4z. \quad (27)$$

The parametrization (27) automatically satisfies Eq. (24). If we substitute (27) into (22) and (23) we obtain small oscillations of the medium which decrease as $|x|\to\infty$. We shall now look for a solution of the Cauchy problem of Eq. (9) in the form of a superposition of parametrized elementary solutions:

$$u_1(x,y,t)=\frac{1}{(2\pi)^{1/2}}\int_{\Gamma_0}dz\int_{-\infty}^{\infty}dp\, e^{-i\Omega t+ipy}r(z,p)\frac{d}{dx}\times[\psi(k(z,p),x)\psi(n(z,p),x)], \quad (28)$$

where

$$-i\Omega(z,p)=2i(z^3+\nu^2 z-3p^2/16z).$$

We get the integration contour Γ_0 from the contour Γ_ρ shown in Fig. 1 by letting $\rho\to 0$. The point $z=0$ is an essential singularity for the integrand in (28). When $t>0$ we must go around it from above.

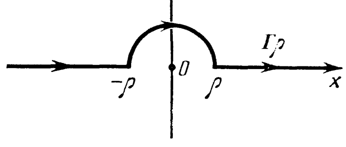

FIG. 1

We proceed to the determination of the unknown function $r(z,p)$ using the initial condition $u_1(x,y,t)|_{t=0}=u_1(x,y,0)$. Putting $t=0$ in (28) and Fourier transforming with respect to y we get

$$\int_{-\infty}^{\infty} r(z,p)\frac{d}{dx}[\psi(k(z,p),x)\psi(n(z,p),x)]dz=u_1(x,p,0), \quad (29)$$

where

$$u_1(x,p,0)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}e^{-ipy}u_1(x,y,0)\,dy.$$

We impose the condition

$$\int_{-\infty}^{\infty}u_1(x,y,0)\,dx=0$$

on the function $u_1(x,y,0)$. For the sake of convenience we introduce the notation

$$\Psi(z,p,x)=\psi(k(z,p),x)\psi(n(z,p),x), \quad (30)$$

$$\Phi(z,p,x)=\frac{1}{2i\pi}\frac{d}{dx}[\psi(k(z,p),x)\psi(n(z,p),x]. \quad (31)$$

The following orthogonality relation

$$\int_{-\infty}^{\infty}\Psi(z,p,x)\Phi(-z_1,p,x)\,dx=-z_1\delta(z-z_1) \quad (32)$$

exists between the functions Ψ and Φ for real z and z_1. (We shall prove Eq. (32) in the Appendix.) If we now scalarly multiply (29) by the function $\Psi(-z,p,x)$ and use (32) we get

$$r(z,p)=\frac{1}{2i\pi z}\int_{-\infty}^{\infty}u_1(x,p,0)\Psi(-z,p,x)\,dx.$$

We stipulate that all integrals over the variable z must be taken in the sense of principal value. Moreover, looking for solutions of the Cauchy problem in the form (28) imposes well defined limitations on the initial value $u_1(x,p,0)$.

the solution of the Cauchy problem of Eq. (9) is thus

$$u_1(x,p,t)=\int_{-\infty}^{\infty}u_1(x_1,p,0)G(x_1,p,x,t)\,dx_1,$$

where the Green function is

$$G(x_1,p,x,t)=\int_{\Gamma_0}e^{-i\Omega(z,p)t}\Psi(-z,p,x_1)\Phi(z,p,x)\frac{dz}{z}. \quad (33)$$

We rewrite Eq. (33) in the form

$$G(x_1,p,x,t)=\frac{1}{2i\pi}\int_{\Gamma_0}e^{-i\Omega(z,p)t}\psi(k(-z,p),x_1)\psi(n(-z,p),x_1)\times\left\{2i\psi(k(z,p),x)\psi(n(z,p),x)+\frac{u_0(x)}{2z}\left[e^{ik(z,p)x}\frac{(\psi(n(z,p),x)}{\nu-ik(z,p)}+e^{in(z,p)x}\frac{\psi(k(z,p),x)}{\nu-in(z,p)}\right]\right\}dz. \quad (34)$$

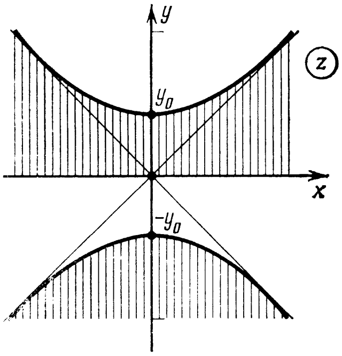

FIG. 2

As the first term in the braces in (34) is some free oscillation of the medium and the second term is localized in the soliton, we shall in what follows drop the first term. "Sitting" on the soliton ($x = 0$) and putting $x_1 = 0$ we get

$$G(x_1{=}0, p, x{=}0, t) = -\frac{\nu^2}{\pi}\int_{\Gamma_0} e^{-i\Omega(z,p)t}\frac{k(z,p)\,n(z,p)}{[\nu^2+k^2(z,p)][\nu^2+n^2(z,p)]}dz. \quad (35)$$

We now evaluate the asymptotic behavior of the Green function (35) as $t \to +\infty$. To simplify the calculations we shall use the approximation $p/\nu^2 \ll 1$. We can easily generalize the results obtained below to the case of an arbitrary ratio of the parameters p and ν^2. We consider first of all the problems connected with the closing of the contour Γ_0 on a contour in the upper half-plane, with the poles of the integrand in (35): $n = \pm i\nu$, $k = \pm i\nu$, and the saddle points of the function $-i\Omega(z,p)$. Let $z = x + iy$; then

$$\operatorname{Re}(-i\Omega) = 2y[-3x^2+y^2-\nu^2-{}^3/_{16}p^2/(x^2+y^2)].$$

The region $Re(-i\Omega) < 0$ in the complex z-plane is shown hatched in Fig. 2.

It follows from Fig. 2 that we can close the contour Γ_0 by any straight line $y = Y$ $(0 < Y < y_0, y_0 = \nu[1 + o(p/\nu^2)])$ provided it is situated in the upper half-plane. The equation $d\Omega/dz = 0$ for the saddle points of the function $-i\Omega(z,p)$ has two roots in the upper half-plane:

$$z_a = \frac{i\nu}{3^{1/2}}\left(1+o\left(\frac{p}{\nu^2}\right)\right), \quad z_b = \frac{3^{1/2}ip}{4\nu}\left(1+o\left(\frac{p}{\nu^2}\right)\right).$$

Since the function $-i\Omega(z = x + iy_b, p)$ has at the point $z = z_b$ a minimum with respect to x, and at the point $z = z_a$ it has a maximum which is the only one on the line $y = y_a$, the steepest-descent contour γ is the line $y = y_a$. From the eight poles determined from the quadratic (in the variable z) equations

$$k(z,p) = \pm i\nu, \quad n(z,p) = \pm i\nu,$$

only four lie in the upper half-plane and two of those lie above the steepest-descent contour γ and two below. Therefore, we close the contour Γ_0 by the steepest descent contour γ as shown in Fig. 3. The point

$$z_1 = {}^1/_2[-i\nu+[(i\nu)^2+ip]^{1/2}] = (p/4\nu)(1+ip/4\nu)[1+o(p/\nu^2)]$$

is the root of the equation $k(z,p) = -i\nu$ and the point

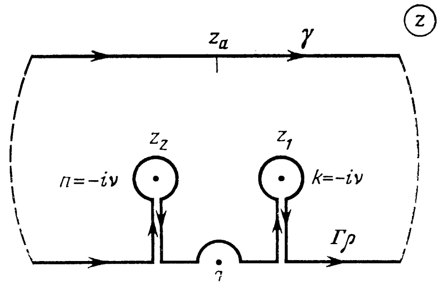

FIG. 3

$z_2 = -z_1^*$ is the root of the equation $n(z,p) = -i\nu$. Let now

$$\tilde{F}(z,p,t) = -\frac{\nu^2}{\pi}e^{-i\Omega(z,p)t}k(z,p)\,n(z,p)/[\nu^2+k^2(z,p)][\nu^2+n^2(z,p)].$$

In that case

$$G(x_1{=}0, p, x{=}0, t) = 2\pi i\left[\operatorname*{res}_{z=z_1}\tilde{F} + \operatorname*{res}_{z=z_2}\tilde{F}\right] + \int_\gamma \tilde{F}\,dz.$$

The asymptotic behavior on the contour γ as $t \to +\infty$ is found by using the steepest-descent method:

$$\int_\gamma \tilde{F}(z,p,t)\,dz = \exp\left[-\frac{4}{3^{3/2}}\nu^3 t\right]\left[\frac{3^{3/2}}{32\pi t}\right]^{1/2} \times\left[1+O\left(\frac{1}{t}\right)\right]\left[1+o\left(\frac{p}{\nu^2}\right)\right]. \quad (36)$$

We now determine the pole contribution to the asymptotic behavior of the Green function. We find first the value of the dispersion in the simple poles of the function $\tilde{F}(z,p,t)$: $z = z_1$ and $z = z_2$. Substituting $k(z,p) = -i\nu$ in (23) and using (24) for $\beta = 1$ we get after simple calculations the expression

$$-i\Omega(z_1, p) = -pn(z_1, p), \quad (35')$$

where

$$n(z_1, p) = 2z_1 + i\nu. \quad (36')$$

Equation (35′) is the same as Eq. (20) apart from the notation. Equations (35′) and (36′) determine the damping in time of the pole contribution. Indeed,

$$\operatorname{Re}(-i\Omega(z_1, p)) = -2px_0 = -2^{-1/2}p^2[(\nu^4+p^2)^{1/2}+\nu^2]^{-1/2} < 0,$$
$$x_0 = \operatorname{Re} z_1. \quad (37)$$

Writing the value of the dispersion $-i\Omega(z_1,p)$ in closed form we get an expression which is the same as Eq. (21) ($\beta = 1$):

$$\Omega^2(p) = p^2(\nu^2 - ip).$$

As $z_2 = -z_1^*$, we have $-i\Omega(z_2,p) = [-i\Omega(z_1,p)]^*$.

Thus, by using the results we can write the pole contribution to the asymptotic behavior of the Green function as $t \to +\infty$:

$$2\pi i\left[\operatorname*{res}_{z=z_1}\tilde{F} + \operatorname*{res}_{z=z_2}\tilde{F}\right] = \frac{\nu^2}{p}z_1\exp[-i\Omega(z_1,p)t] + \frac{\nu^2}{p}z_1^*\exp[i\Omega^*(z_1,p)t]. \quad (38)$$

The two terms in (38) describe waves which have the same damping rate and move along the transverse y-axis in opposite directions. By virtue of (36) to (38) the asymptotic behavior as $t \to +\infty$ of the Green function of the linearized KP equation is determined exclusively by the pole terms describing the weakly damped soliton oscillations.

We also note that for a cnoidal wave which is periodic in x there is no damping effect. This is explained by the fact that perturbations emitted by a single soliton are superimposed on the solitons positioned behind. The dispersion law of the second-sound propagating along the periodic cnoidal wave was evaluated in Ref. 6.

In conclusion the author thanks V. E. Zakharov for supervison of this work.

APPENDIX

The orthogonality relation

Using the stationary Schrödinger equation one easily finds the following relation for the wave functions:

$$\psi(k,x)\psi(n,x)\frac{d}{dx}[\psi(-k_1,x)\psi(-n_1,x)]$$
$$=\frac{1}{2}\frac{d}{dx}\left[\frac{W_{k;-k_1}W_{n;-n_1}}{k^2-k_1^2}+\psi(k,x)\psi(n,x)\psi(-k_1,x)\psi(-n_1,x)\right]$$
$$W_{k;-k_1}=W[\psi(k,x);\psi(-k_1,x)], \tag{39}$$

where W is the Wronskian and the variables n,k and n_1,k_1 satisfy Eq. (24). Let now the wave function $\psi(k,x)$ be an eigenfunction of the continuous spectrum of the soliton potential (26). Introducing the parametrization (27) for the quantities n,k and n_1,k_1 by means of the real variables z and z_1, respectively, considering the functions $\Psi(z,p,x)$ and $\Phi(z,p,x)$ from (30) and (31), and integrating (39) over x, we get

$$\int_{-\infty}^{\infty}\Psi(z,p,x)\Phi(-z_1,p,x)\,dx$$
$$=\frac{1}{4\pi i}\{W_{k(z,p);k(-z_1,p)}W_{n(z,p);n(-z_1,p)}/[k^2(z,p)-k^2(-z_1,p)]+\psi(k(z,p),x)\psi(k(-z_1,p),x)$$
$$\times\psi(n(z,p),x)\psi(n(-z_1,p),x)\}\Big|_{-\infty}^{+\infty}. \tag{40}$$

By virtue of the Riemann-Lebesque lemma, we can neglect the second term in (40). Using now the following relation from the theory of generalized functions:

$$\lim_{x\to\pm\infty}\mathrm{P}\frac{e^{ikx}}{ik\pi}=\pm\delta(k),$$

where the symbol P indicates the principal part, we get the required relation:

$$\frac{1}{4i\pi}\left(\lim_{x\to+\infty}-\lim_{x\to-\infty}\right)$$
$$\times\{W_{k(z,p);k(-z_1,p)}W_{n(z_1,p);n(-z_1,p)}/[k^2(z,p)-k^2(-z_1,p)]\}$$
$$=-z_1\delta(z-z_1).$$

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