Structure of (2 + 1) photodynamics

Ya. I. Kogan and A. Yu. Morozov

Institute of Theoretical and Experimental Physics
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Three-dimensional photodynamics with a quadratic Wess-Zumino term is discussed. The theory has one degree of freedom, which corresponds to a massive photon. However, the interaction of external currents contains a nondispersional pole at zero momentum, describing an instantaneous long-range interaction of Aharonov-Bohm type. This pole appears on the Hamiltonian quantization of the theory with sources from a massless excitation of the unconstrained theory. There is a deep analogy between this theory and the problem of the motion of a charged particle in a constant magnetic field. In this analogy, the massless degree of freedom corresponds to the drift of the Landau orbits across the magnetic field.

INTRODUCTION

Deser, Jackiw, and Templeton[1] noted some time ago that the addition of a Wess-Zumino (WS) term to the Lagrangian of three-dimensional (2 + 1) gauge theory can lead to the appearance of a photon with a gauge-invariant mass (gluon). The possibility of introducing into a Lagrangian expressions that are implicitly invariant with respect to symmetry transformations—being changed by a total derivative, which does not affect the action or the equations of motion—was discussed for the first time by Wess and Zumino[2] in the four-dimensional chiral model. Since then, WS terms have appeared in the most varied places, beginning with fermion determinants[3–5] and ending with supergravity theories.[6] In odd-dimensional Yang-Mills theory, WS terms can be constructed on the basis of topological charge in a space with dimension greater by unity. For example, for $d = 3$ the WS term has the form

$$\int d^3x\, \varepsilon^{\mu\nu\lambda}\left(F_{\mu\nu}{}^a A_\lambda{}^a - \frac{1}{3} f^{abc} A_\mu{}^a A_\nu{}^b A_\lambda{}^c \right) \qquad (1)$$

and is related to the four-dimensional topological charge

$$\int d^4x\, F_{\mu\nu}{}^a \tilde{F}_{\mu\nu}{}^a = \int d^4x\, \partial_\mu K_\mu. \qquad (2)$$

The WS term is equal to the integral over d^3x of the zeroth component of the vector K_μ. WS terms have an interesting topological and field-theory interpretation; in particular, the hierarchy of anomalies in spaces of different dimensions is related to them.[7]

In this paper, we discuss the physical significance of WS terms and their effect on the structure of a theory for the simplest but still fairly nontrivial example, namely, Abelian (2 + 1) photodynamics. In doing this, we also obtain important information about (2 + 1) electrodynamics with fermions. For, as noted in Refs. 4 and 5, the effective action obtained after integration over the Dirac fermions in an odd number of dimensions contains not only a renormalization of the coupling constant in front of the bare Lagrangian $F^2_{\mu\nu}$ and higher terms of the type $F^3_{\mu\nu}$, but also the WS term (1). This contribution is nonzero at energies exceeding the fermion mass and is related to the need for regularization of the theory. In essence it is anomalous and in many ways is analogous to the term $m^2(A^\perp_\mu)^2$ that arises in the same manner in two-dimensional Schwinger electrodynamics (for more details, see Ref. 8). The coefficient H in front of the WS term [see (3)] has the dimensions of mass and is proportional to the square e^2 of the fermion charge (the dimension of e is equal to $(4 - d)/2$ in the d-dimensional theory). In contrast to the usual terms in the effective action which are polynomial in $F_{\mu\nu}$, the appearance of the WS term significantly changes the properties of the theory. Therefore, we believe it is worth considering the model

$$S = \int d^3x \left(-\frac{1}{4} F_{\mu\nu}{}^2 + \frac{1}{4} H \varepsilon_{\mu\nu\lambda} F_{\mu\nu} A_\lambda \right), \qquad (3)$$

which we shall call (2 + 1) photodynamics. This theory is quadratic and can be solved exactly, but this does not prevent it having somewhat unusual and surprising properties [which are also present in the more realistic (2 + 1) electrodynamics].

Among these properties, we include the fact that despite the presence of the photon's gauge-invariant mass the theory (3) has one degree of freedom. Here it should be recalled that in an even number of dimensions we are accustomed to a gauge-invariant massless photon having $d - 2$ degrees of freedom, the 2 arising from the gauge arbitrariness and the constraint condition. An ordinary massive photon is not gauge invariant, but the constraint condition for it remains, so that there are $d - 1$ degrees of freedom. On the other hand, for the gauge-invariant massive photon in Schwinger's electrodynamics there is no constraint condition, so that it also has $d - 1 = 1$ degree of freedom. Not so in the theory (3), since both the gauge invariance and the constraint remain, so that the photon has $d - 2$ degrees of freedom.

A second remarkable property is that although the physical photon is massive not all the Green's functions decrease exponentially. In the momentum representation, the interaction of the sources contains a pole term $1/p^2$. In its origin, it is analogous to the ordinary Coulomb pole or Veneziano ghost in four-dimensional theories. For conserved sources, the pole is nondispersive—the singularity does not in fact exist for $p^2 = 0$ ($p_0^2 = \mathbf{p}^2$) but only for $p_\mu = 0$

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($p_0 = \mathbf{p} = 0$). This means that the pole is not associated with the propagation of any physical massless particle. An undoubted advantage of (2 + 1) photodynamics is that in it the nondispersive poles are not masked by the presence in the spectrum of real massless particles, which, as in four-dimensional theories, also give rise to singularities in the amplitudes when $p_\mu = 0$. As a result, it becomes completely obvious that the presence of only massive particles in the spectrum of a gauge theory does not in general mean that there is no long-range interaction at all. Indeed, in gauge theory there is a set of disconnected sectors, i.e., states that are not carried into each other as the system evolves. In photodynamics, these sectors differ in the number of longitudinal (or scalar) photons—massless "particles" that do not interact with the physical transverse photons. On the gauge-theory states the constraint is imposed, which specifies the sector in which there are no unphysical photons. It is important that the Hamiltonian of a theory with sources (even when they are conserved) does not commute with such a constraint and couples the different sectors of such a theory. The states singled out by the new constraint do contain unphysical (nondispersive) photons, and this leads to an instantaneous Coulomb interaction. At the same time, nondispersive poles appear in the T correlation functions of the external sources. In (2 + 1) photodynamics, the transverse photon acquires a mass by virtue of the WS term, but the longitudinal photon remains massless, and therefore the singularities of the amplitudes at $p_\mu = 0$ are entirely due to this degree of freedom. We note that in the Higgs effect the longitudinal photon becomes massive at the same time as the transfverse, this being a consequence of the "swallowing" of one degree of freedom from the scalar sector.

The paper is arranged as follows. In Sec. 1, we discuss a solution of plane-wave type and show that it corresponds to one degree of freedom. In addition, we show that the nondispersive pole really does exist if $e^2/H\pi$ is not equal to an integer. It is shown that this instantaneous interaction literally describes an Aharanov-Bohm effect[9] [in a (2 + 1)-dimensional world, this is possible for point objects]. It is for this reason that at discrete values of H^{-1} (corresponding to the magnetic flux in the Aharanov-Bohm effect) the effect of the long-range interaction becomes unobservable. Unfortunately, we have not yet succeeded in finding reasons that prohibit a WS term for arbitrary H. (We note that Deser, Jackiw, and Templeton[10] concluded that all Green's functions are well defined at all H.) If arguments for H being discrete nevertheless exist, this would be tantamount to the existence of a new charge quantization principle in an Abelian theory.

In Sec. 2, we consider a quantum-mechanical analog of (2 + 1) photodynamics—the problem of a charge in a homogeneous magnetic field—and we elucidate the origin of the nondispersive pole, which is associated in this case with the possibility of drift of the Landau orbits across the magnetic field. (It is because of this analogy that we have denoted the coefficient in front of the WS term by H.)

In Sec. 3, we discuss the Hamiltonian quantization of the (2 + 1) theory and show that without the constraints it is entirely analogous to the quantum-mechanical problem, while allowance for the constraings causes the massless mode to disappear from the physical spectrum but introduces an instantaneous long-range interaction of the external sources.

To end the Introduction, we should like to note that we know of only two papers, Refs. 10 and 11, in which there has been a detailed discussion of the physical properties of the theory (3). In these, an expression is given for the propagator [see Eq. (8)], and the existence of a pole at $p_\mu = 0$ is noted; however, the analysis is not complete. We hope that our paper will clarify some questions that remained somewhat obscure after Refs. 10 and 11.

1. GENERAL PROPERTIES OF (2 + 1) PHOTODYNAMICS

From the action (3) there follow the equations of motion

$$\partial_\mu F_{\mu\lambda} + {}^1/_2 H \varepsilon_{\mu\nu\lambda} F_{\mu\nu} = 0. \tag{4}$$

The left-hand side is a total derivative, and (4) is equivalent to

$$F_{\mu\lambda} + H \varepsilon_{\mu\nu\lambda} A_\nu = \varepsilon_{\mu\nu\lambda} c_\nu.$$

The curl of the vector c_ν is required to vanish: $\varepsilon_{\mu\nu\lambda}\partial_\mu c_\nu = 0$, and as a result it can be eliminated by a gauge transformation

$$A_\nu \to A_\nu + \frac{1}{H}\partial_\nu \int^x c_\alpha \, dx^\alpha. \tag{5}$$

In other words, for every solution of Eq. (4) there exists a solution gauge-equivalent to it of the equation

$$F_{\mu\nu} = H \varepsilon_{\mu\nu\lambda} A_\lambda. \tag{6}$$

Note that for a theory without a WS term it is impossible to arrive at Eq. (6), since when $H = 0$ the gauge transformation (5) is not defined. However, the explicit plane-wave type solution given below does admit continuous passage to the limit $H = 0$ and goes over into an ordinary plane wave. From (6) the transversality condition $\partial_\lambda A_\lambda = 0$ follows directly as a consequence of the Bianchi identities $\varepsilon_{\mu\nu\lambda}\partial_\mu F_{\nu\lambda} = 0$.

Equation (6) recalls a duality equation; at least it is of first order and, in contrast to non-Abelian theories, all solutions of the equations of motion (4) are here determined by the first-order equation (6) up to gauge transformations.

The plane-wave solution has the form

$$A_\mu = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = a \begin{pmatrix} p_1^2 + p_2^2 \\ \omega p_1 + i p_2 H \\ \omega p_2 - i p_1 H \end{pmatrix} \exp i(\omega t - \mathbf{px}), \quad \omega^2 - \mathbf{p}^2 = H^2. \tag{7}$$

The field (7) is transverse: $\omega A_0 - \mathbf{p}\cdot\mathbf{A} = 0$. Note that this solution describes one degree of freedom—for given p_1 and p_2, the solution is completely determined. Using the gauge invariance, we can reduce the solution (7) to the form $A_0 = 0$ or $\partial_i A_i = 0$; however, it is not possible to achieve both when $H \neq 0$. This means that if $A_0 = 0$ then the physical photon is not transverse, but if it is then there is always a scalar photon. Such behavior is due to the specific nature of the constraint in the Hamiltonian description of the theory and will be discussed in more detail in Sec. 3. Note the presence of the factors of i in (7). They indicate that for true real solutions $A_\mu^{(1)} = ReA_\mu$ and $A_\mu^{(2)} = ImA_\mu$ there is a phase shift between the different components, which becomes equal to $\pi/2$ in the limit $p_1 \ll p_2$, $\mathbf{p} \to 0$. This rotation of the polarization vector corresponds to the picture of the motion of a charge on a plane in a homogeneous magnetic field. In the following sections, we shall see that precisely this picture corresponds to the quantum description of $(2+1)$ photodynamics.

We now consider the propagator. The action (3) is gauge invariant, and therefore to find the propagator it is necessary to fix the gauge, which can be done, for example, by adding to the Lagrangian the term $(\partial_\mu A_\mu)^2/2\alpha$. Then

$$\Pi_{\mu\nu}(p) = -i\int d^3x\, e^{ipx}\langle TA_\mu(x)A_\nu(0)\rangle$$
$$= \frac{g_{\mu\nu} - p_\mu p_\nu/p^2}{p^2 - H^2} + i\frac{H\varepsilon_{\mu\nu\lambda}p_\lambda}{p^2(p^2-H^2)} + \alpha\frac{p_\mu p_\nu}{p^4}. \quad (8)$$

The first question that must be clarified is the extent of the gauge freedom in the expression (8) and whether anything "survives" apart from the massive pole after elimination of this freedom. To answer this question, we calculate the contraction $J_\mu(p)\Pi_{\mu\nu}J_\nu(-p)$ for conserved sources J_μ $(p_\mu J_\mu = 0)$. The manifestly noninvariant final term in (8) drops out immediately, but the axial structure $\varepsilon_{\mu\nu\lambda}p_\lambda$ can be only partly eliminated. We note first that

$$\varepsilon_{\mu\nu\lambda}p_\lambda J_\mu(p)J_\nu(-p) \neq 0,$$

since the interchange of J_μ and J_ν is accompanied by a change in the sign of p_λ. It must now be borne in mind that not all components of the conserved current J_μ are independent. For example, let us express J_0 in terms of J_i:

$$J_0 = (p_i J_i)/\omega = (\mathbf{pJ})/\omega, \quad p^2 = \omega^2 - \mathbf{p}^2.$$

Then

$$J_\mu\Pi_{\mu\nu}J_\nu = \frac{1}{\omega^2 - \mathbf{p}^2 - H^2}\left[\delta_{ij} - \frac{p_i p_j}{\omega^2} - \frac{iH\varepsilon_{ij}}{\omega}\right]J_i J_j. \quad (9)$$

Further, we represent J_i as the sum of transverse and longitudinal currents:

$$J_i = J_i^{\parallel} + J_i^{\perp}, \quad J_i^{\parallel} = p_i(\mathbf{Jp})/\mathbf{p}^2.$$

The adjectives longitudinal and transverse refer only to the spatial components of the momentum, $\mathbf{p}$, and have the same meaning as the adjectives longitudinal and transverse for the photon. We now can rewrite (9) as

$$J_\mu\Pi_{\mu\nu}J_\nu = \frac{1}{\omega^2 - \mathbf{p}^2 - H^2}\left[(J^{\perp})^2 - \frac{2iHJ^{\perp}J^{\parallel}}{\omega} + \frac{\omega^2 - \mathbf{p}^2}{\omega^2}(J^{\parallel})^2\right]. \quad (10)$$

If $\mathbf{p} \neq 0$, then the "pole" at $\omega = 0$ can be readily eliminated by expressing $J^{\parallel}$ in terms of J^0 in the form $J^{\parallel} = \omega J_0/|\mathbf{p}|$; then instead of (10), we obtain

$$J_\mu\Pi_{\mu\nu}J_\nu = \frac{1}{\omega^2 - \mathbf{p}^2 - H^2}\left[(J^{\perp})^2 - \frac{2iHJ^{\perp}J^0}{|\mathbf{p}|} + \frac{\omega^2 - \mathbf{p}^2}{\mathbf{p}^2}(J^0)^2\right]. \quad (11)$$

It can be seen from these arguments that the correlation function of conserved currents does not have a pole at $p^2 = \omega^2 - \mathbf{p}^2 = 0$; however, the singularity at $\omega = \mathbf{p} = 0$ is

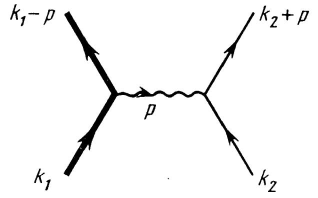

FIG. 1.

not associated with the gauge freedom and is absolutely real and cannot be eliminated. The situation here is exactly the same as in the case of the Coulomb pole. Therefore, we can be sure that here too we have some "instantaneous long-range interaction."

Before we clarify the form of this interaction, let us attempt to dispel any doubts about the reality of the pole at $p_\mu = 0$ that the reader may still have. There may be a fear that the current conservation condition does not eliminate all the spurious poles (although we do not know of such examples, and it would be strange if they existed). To dot the i's, we have calculated the real scattering amplitude of two different massive fermions interacting with the field A_μ. The corresponding diagram is shown in Fig. 1. There is no second (annihilation) diagram, since the fermions are not identical. The result for the cross section is proportional to $[(k_1k_2)^2 - k_1^2k_2^2]/k_1p +$ terms that are finite as $p \to 0$, and the cross section is indeed singular at $p = 0$. (In the derivation, the kinematic relation $k_1p = -k_2p = p^2/2$ must be used.)

Having shown that there is a long-range interaction, we now consider its structure. For this, we consider the field produced by a source J_μ:

$$A_\mu(x) = \int \Pi_{\mu\nu}(x-y)J_\nu(y)\,d^3y,$$
$$A_\mu(p) = \Pi_{\mu\nu}(p)J_\nu(p). \quad (12)$$

In the propagator (8), we are interested only in the term $H^{-1}\varepsilon_{\mu\nu\lambda}p_\lambda/p^2$. The remaining terms describe an interaction that decreases exponentially with the distance. Thus,

$$A_\mu(p) = \frac{1}{H}\frac{\varepsilon_{\mu\nu\lambda}p_\lambda}{p^2}J_\nu(p). \quad (13)$$

For a conserved current, the strength of this field is

$$F_\alpha(p) \equiv \varepsilon_{\alpha\beta\gamma}p_\beta A_\gamma = \frac{1}{H}J_\alpha - \frac{p_\alpha}{H}\frac{p_\mu J_\mu}{p^2} = \frac{1}{H}J_\alpha(p);$$

it vanishes outside the sources:

$$F_\alpha(x) = \frac{1}{H}J_\alpha(x). \quad (14)$$

In other words, outside the sources the field (13) is a pure gauge: $A_\mu(x) = \partial_\mu\chi$. Suppose for simplicity the current J_μ describes a charge at rest at the origin: $J_0(\mathbf{x}) = Q\delta^2(\mathbf{x})$, $\mathbf{J} = 0$. The static field of this charge has the form

$$A_0(\mathbf{x}) = 0,$$
$$A_1(\mathbf{x}) = \frac{Q}{\pi H}\frac{x_2}{x_1^2 + x_2^2} = \partial_1\chi, \quad (15)$$

$$A_2(\mathbf{x}) = -\frac{Q}{\pi H}\frac{x_1}{x_1^2+x_2^2} = \partial_2\chi.$$

Here

$$\chi(\mathbf{x}) = \frac{Q}{\pi H}\tan^{-1}\frac{x_1}{x_2} = \frac{Q}{\pi H}(\varphi-\varphi_0).$$

The angle φ determines the direction of the radius vector $\mathbf{x}$ in the plane. The function χ is well defined in any simply connected domain that does not contain the origin, but it is not single valued in the complete plane, so that globally the field A_μ is not a pure gauge field. The integral $\int \mathbf{A}d\mathbf{x}$ around any contour that circles the origin n times is $(2Q/H)n$. If Q/H is a multiple of π, then $\exp(i\int \mathbf{A}d\mathbf{x})$ is single valued, and the field (15) is unobservable. This then is the charge quantization in Abelian theory mentioned in the Introduction.

We also make a technical remark. We consider what the consequence is in the momentum representation of $\chi(\mathbf{x})$ being multiply valued. The expressions (15) become

$$A_0(\mathbf{p})=0,$$
$$A_1(\mathbf{p}) = \frac{Q}{H}\frac{p_2}{p_1^2+p_2^2} = p_1\chi_1, \tag{16}$$
$$A_2(\mathbf{p}) = -\frac{Q}{H}\frac{p_1}{p_1^2+p_2^2} = p_2\chi_2$$

and two different functions $\chi(\mathbf{p})$ are determined:

$$\chi_1(\mathbf{p}) = \frac{Q}{H}\frac{p_2}{p_1(p_1^2+p_2^2)},$$
$$\chi_2(\mathbf{p}) = -\frac{Q}{H}\frac{p_1}{p_2(p_1^2+p_2^2)} = \chi_1(\mathbf{p}) - \frac{Q}{Hp_1p_2}.$$

On the other hand, it is not difficult to calculate the Fourier transform of $\chi(x)$ directly:

$$\chi(\mathbf{p}) = \frac{iQ}{\pi H}\int\int dx_1\,dx_2\exp(i\mathbf{px})\tan^{-1}\frac{x_2}{x_1}.$$

Calculating the integral over x_1 by parts, we find

$$\chi_1(\mathbf{p}) = \frac{Q}{H}\left[\frac{p_2}{p_1(p_1^2+p_2^2)} + \frac{1}{\pi p_1}\exp(ip_1x_1)\times\int\tan^{-1}\frac{x_2}{x_1}\exp(ip_2x_2)\,dx_2\Big|_{x_1=-\infty}^{x_1=\infty}\right].$$

But if we calculate the integral over x_2 by parts, we obtain

$$\chi_2(\mathbf{p}) = \frac{Q}{H}\left[-\frac{p_1}{p_2(p_1^2+p_2^2)} + \frac{1}{\pi p_2}\exp(ip_2x_2)\int\tan^{-1}\frac{x_2}{x_1}\exp(ip_1x_1)\,dx_1\Big|_{x_2=-\infty}^{x_2=\infty}\right].$$

It is readily seen that interchange of the limits of integration changes the values of the integrals in the expressions for $\chi_{1,2}$ by Q/Hp_1p_2, as must be in accordance with (16).

It is readily seen that these expressions describe the Aharanov-Bohm effect, which is well known in ordinary four-dimensional electrodynamics.[9] In the original effect, there is a field produced by an infinitely long thin solenoid. The magnetic field is entirely within the solenoid, and outside remains only a vector potential that is locally but not globally a pure gauge. Nevertheless, particles that pass the solenoid on different sides acquire a relative phase shift, and this can lead to interference between them, so that the particles interact with the field of the solenoid despite the fact that for the entire time they remain in a region with zero field strength. In three-dimensional space-time, a point charge can play the part of the solenoid, as we have seen above.

We now show how it is that the phase shift between trajectories that pass on different sides of the origin leads to a pole in the scattering amplitude. The point is that the plane wave for which the scattering amplitude is calculated in accordance with the Feynman rules is distributed over the complete space, and half of it passes on one side of the origin and half on the other. Suppose the wave propagates along the x_2 axis. If in the limit $x_2 \to -\infty$ it has the form $\exp(ipx_2)$, then in the limit $x_2 \to +\infty$ it is transformed into

$$\exp(ipx_2)\,[1+\theta(x_1)e^{i\alpha}],\quad \alpha=2Q/H,$$

i.e., it changes by

$$\alpha\theta(x_1)\exp(ipx_2)+O(\alpha^2).$$

It is now necessary to find the Fourier transform of this expression:

$$\int\theta(x_1)e^{i\mathbf{px}}\,d^2x \propto \frac{1}{p_1}\delta(p_2);$$

this is precisely the pole that arises from the propagator (8).

We discuss briefly the connection between the Aharanov-Bohm effect and topology. We have seen that the pole in the scattering amplitude arises when the correction to the plane wave is expanded in powers of the phase shift $\alpha = 2Q/H$. To obtain the total phase factor $\exp(i\alpha)$, it is necessary to sum the infinite number of diagrams that describe the interaction between the particle and the external field in all orders. This summation is conveniently done directly in the effective action for the sources whose interaction we study. This effective action is the generating functional of the quadratic theory and can be readily found. The structure in which we are interested, $J_\mu\varepsilon_{\mu\nu\lambda}p_\lambda J_\nu/Hp^2$, occurs in the generating functional in the form

$$Z[J] = \int\mathscr{D}A_\mu\exp i\left(S[A]+\int J_\mu A_\mu\,d^3x\right) = \exp\left[\frac{2i}{H}\int\frac{J_\mu(x)\,\varepsilon_{\mu\nu\lambda}(x-y)_\lambda J_\nu(y)}{(x-y)^3}d^3x\,d^3y+\ldots\right]. \tag{17}$$

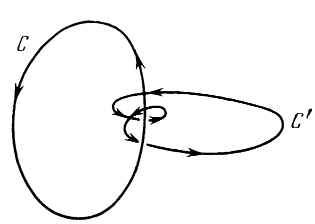

FIG. 2.

Here, $S[A]$ is the action (3). The ellipsis represents the contributions associated with exchange of the massive photon; these decrease rapidly at large distances. If the sources describe closed trajectories (Fig. 2), i.e., they correspond to particles moving along closed trajectories, then

$$J_\mu(x) = \int \delta^3(x-y(\tau))\dot{y}_\mu(\tau)\,d\tau.$$

After this, (17) becomes

$$Z[J] = \exp\left[\frac{2i}{H}\int_c\int_{c'} \varepsilon_{\mu\nu\lambda}\frac{(x-y)_\lambda}{(x-y)^3}dx_\mu dy_\nu + \dots\right]. \tag{18}$$

The integral in the exponential is a topological invariant; it is called the Gauss integral and is equal to the winding index of the two curves, i.e., it measures the number of times the one closed curve passes round the other.[1] It has the same meaning if one of the curves begins and ends at infinity, as would be the case if we were to consider the motion of a test particle in the field of a charge at rest.

We should like to end this section with a fairly general assertion, namely, that the propagator of any local gauge theory is singular at $p_\mu = 0$. This result can be readily understood by noting that the Lagrangian of any local gauge theory is invariant under a shift $A_\mu \to A_\mu + c_\mu$ with constant vector c_μ. Strictly speaking, this is not a gauge transformation, since it does not decrease at infinity. For example, Schwinger's two-dimensional electrodynamics is gauge invariant but does not have this symmetry, which is broken by the nonlocal mass term

$$m^2(A^\perp)^2 = m^2 A_\mu\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)A_\nu.$$

A second example is a theory with the Higgs effect; under the shift $A_\mu \to A_\mu + c_\mu$, it is necessary to rotate the scalar, $\Phi \to \exp(icx)\Phi$, which changes its vacuum expectation. However, if a theory has this shift symmetry, the action does not depend on constant fields A_μ ($p = 0$), and therefore the integral over the fields in the generating functional is not suppressed by an exponential, and the contribution of constant fields is, unless zero for some other special reasons, infinite. It is obvious that the action (3) has this symmetry, and therefore a pole at $p_\mu = 0$ must be present. However, this explanation seems to us too general, and we wish to give below a more detailed analysis of the appearance of the pole in a consistent quantum theory.

2. QUANTUM-MECHANICAL ANALOGY: PARTICLE IN A HOMOGENEOUS MAGNETIC FIELD

A direct $(0+1)$ analog of (3) does not exist, since there is no one-dimensional gauge theory even without the WS term. Therefore, a quantum-mechanical analogy must be sought elsewhere. In (3) we set $A_0 = 0$, and we regard the fields $A_{1,2}$ as constant in space and varying only in time. Then instead of (3) we obtain

$$S = \int dt\left[\frac{1}{2}\dot{A}_i^2 + \frac{H}{2}\varepsilon_{ij}\dot{A}_i A_j\right], \quad i,j = 1,2. \tag{19}$$

This is the action of a two-dimensional particle (not a field!) on the plane A_1A_2 in a magnetic field H at right angles to this plane. The Hamiltonian of the particle is

$$\mathcal{H} = \frac{1}{2}\left(\pi_i - \frac{H}{2}\varepsilon_{ij}A_j\right)^2$$

$$= \frac{1}{2}\left(\pi_1 - \frac{H}{2}A_2\right)^2 + \frac{1}{2}\left(\pi_2 + \frac{H}{2}A_1\right)^2. \tag{20}$$

After canonical quantization, $\pi_i = i\partial/\partial A_i$, the Lagrangian

$$\tfrac{1}{2}\dot{A}_1^2 + \tfrac{1}{2}\dot{A}_2^2 + H\dot{A}_1A_2 \tag{21}$$

is associated with the different Hamiltonian

$$\mathcal{H}' = \tfrac{1}{2}(\pi_1 - HA_2)^2 + \tfrac{1}{2}\pi_2^2, \tag{22}$$

but the equations of motion are the same in the cases (19) and (21):

$$\ddot{A}_i + H\varepsilon_{ij}\dot{A}_j = 0. \tag{23}$$

Accordingly, the propagators (24) (see below) are also identical. Clearly, the Hamiltonians (20) and (22) must be related by a unitary transformation:

$$\mathcal{H} = U^+\mathcal{H}'U; \quad U = \exp(\tfrac{1}{2}iHA_1A_2).$$

We shall say that such transformations are quasigauge, since they describe the possibility of choosing in different ways the vector potential $\mathscr{A}_i$ corresponding to the given magnetic field H: $H = \varepsilon_{ij}\partial_i\mathscr{A}_j$. In the Hamiltonian (20), the vector potential is chosen in the form $\mathscr{A}_i = -1/2H\varepsilon_{ij}A_j$; in (22), in the form $\mathscr{A}_i = -HA_2$, $\mathscr{A}_2 = 0$. The spectrum does not depend on the choice of the vector potential; all that change are the wave functions of the states. The name quasigauge transformation recalls the fact that the freedom in the choice of the vector potential $\mathscr{A}_i$ has nothing to do with the gauge transformations of the theory (3). The quasigauge transformations are linear mappings of the Hilbert space of the physical states into itself, whereas the gauge transformations cause this space itself to change. In particular, the correlation functions [for example, (24)] change under gauge but not under quasigauge transformations.

The physical states of the system (19) are extermely well known—they are Landau levels (see, for example, Ref. 12). The propagator, defined in the usual way as the Green's function of the equations of motion (33), has the form

$$\Pi_{ij}(\omega) = -i\int dt\, e^{i\omega t}\langle TA_i(t)A_j(0)\rangle$$

$$= \frac{\delta_{ij}}{\omega^2 - H^2} + \frac{iH\omega\varepsilon_{ij}}{\omega^2(\omega^2 - H^2)} \tag{24}$$

and is completely analogous to the propagator (8). In it, there are also two poles: a massive one at $\omega = H$ and a massless one at $\omega = 0$, the latter being present only in the nondiagonal correlation function. With what states are these poles associated?

To answer this question, we turn to the Hamiltonian quantum description. The choice of the basis states for a strongly degenerate system such as a charge in a magnetic field is rather arbitrary. We use an axial basis in which the operators of the angular momentum L and the energy $E=(N+\frac{1}{2})H$ are diagonal. In what follows, we shall consider the excitation energy after subtraction of the zero-point fluctuations (in field theory, it is only this energy that is meaningful), i.e., $E=NH$. The true parameter of the wave functions in this basis is not N but $n=N-\frac{1}{2}(|L|-L)$. At the same time,

$$\Psi_{n,L}=\exp(iL\varphi)\psi_{n,L}(r), \tag{25}$$

and the radial functions can be expressed in terms of Laguerre polynomials:

$$\psi_{n,L}(r)=C_{n,L}\exp\left(-\frac{Hr^2}{4}\right)r^{|L|}\mathscr{L}_n^{|L|}\left(\frac{Hr^2}{2}\right), \tag{26}$$

where $C_{n,L}$ is a H-dependent gauge constant, $A_1=r\cos\varphi$, $A_2=r\sin\varphi$, and

$$E_{n,L}=[n+{}^1/_2(|L|-L)]H.$$

Despite the cumbersome form of the wave functions (25), calculations with them are very simple if one uses the generating function of the Laguerre polynomials:

$$\sum_{n=0}^{\infty}\mathscr{L}_n^{|L|}\left(\frac{Hr^2}{2}\right)t^n=\frac{1}{(1-t)^{|L|+1}}\exp\left(-\frac{Hr^2}{2}\frac{t}{1-t}\right). \tag{27}$$

Any matrix element A_1 and A_2 reduces to the product of an angular and a radial element. We choose some ground state, for example,

$$\Psi_{\text{vac}}=\Psi_{n=0,\,L=0}.$$

The operators A_1 and A_2, which are proportional to exp $(i\varphi)\pm\exp(-i\varphi)$, can only have transitions in which L changes by unity. The radial part of the matrix elements $\langle n,L=\pm1|A_{1,2}|0,0\rangle$ is determined by the integral

$$\int r\mathscr{L}_n^1\left(\frac{Hr^2}{2}\right)r\mathscr{L}_0^0\left(\frac{Hr^2}{2}\right)r\exp\left(-\frac{Hr^2}{2}\right)dr,$$

which, as is readily seen by means of (27), is nonvanishing only for $n=0$. Thus, only the intermediate states with $n=0$ and $L=\pm1$ contribute to the propagator of A. The first of these has zero energy, and the second H.

It remains to show why the massless pole is absent in the diagonal correlation function. The reason is trivial, although somewhat unusual. We write down a spectral representation for the correlation function (in quantum field theory, this is the Källén-Lehmann representation):

$$-i\int dt\exp(i\omega t)\langle0|TA_i(t)A_j(0)|0\rangle$$
$$=\frac{1}{i}\int dt\exp(i\omega t)$$
$$\times[\theta(t)\langle0|A_i(t)A_j(0)|0\rangle+\theta(-t)\langle0|A_j(0)A_i(t)|0\rangle]$$
$$=\frac{1}{i}\sum_n\int dt[\theta(t)\exp(i(\omega-E_n)t)\langle0|A_i|n\rangle\langle n|A_j|0\rangle]$$
$$+\frac{1}{i}\sum_n\int dt[\theta(-t)\exp(i(\omega+E_n)t)\langle0|A_j|n\rangle\langle n|A_i|0\rangle]$$
$$=\sum_n\frac{2E_n}{\omega^2-E_n^2}\mathrm{Re}\langle0|A_i|n\rangle\langle n|A_j|0\rangle$$
$$+i\sum_n\frac{2\omega}{\omega^2-E_n^2}\mathrm{Im}\langle0|A_i|n\rangle\langle n|A_j|0\rangle. \tag{28}$$

It is clear that the pole $\omega=0$ is absent in the first term, since the residue is proportional to E_n, but is present in the second. For the diagonal correlation function we have $i=j$, and the term with the imaginary part is simply absent; hence, the pole $1/\omega^2$ is absent in it too. In general, the pole $1/\omega^2$ is present in the nondiagonal correlation function, and it is necessarily imaginary (in Minkowski space); the propagator (24) is constructed in just this manner. Moreover, using the normalized wave functions

$$|0,0\rangle=\left(\frac{H}{2\pi}\right)^{1/2}\exp\left(-\frac{Hr^2}{4}\right),$$
$$|n=0,L=\pm1\rangle=\frac{H}{2\pi^{1/2}}r\exp\left(-\frac{Hr^2}{4}\right)\exp(\pm i\varphi),$$

we can readily verify by a simple calculation that

$$\langle n=0,L=\pm1|A_1|0,0\rangle=\frac{(-1)^L}{i}\langle n=0,L=\pm1|A_2|0,0\rangle$$
$$=\frac{1}{(2H)^{1/2}}, \tag{29}$$

after which the propagator (24) is directly recovered from (28).

From this analysis it is clear that transitions between energy-degenerate states are responsible for the $1/\omega^2$ pole. In the semiclassical description, it is precisely these transitions that cause the drift of the Landau orbits in the plane perpendicular to the magnetic field. We recall that the semiclassical picture of the motion is that of revolution in one direction in a circle. This corresponds to the motion of the polarization vector for the plane wave discussed in Sec. 1.

These arguments may occasion a certain disbelief: Are we saying that the $1/\omega^2$ pole is altogether impossible in diagonal correlation functions? The critical point here is that in the example discussed above the Hamiltonian does not have a continuum near the origin (or indeed anywhere), i.e., the states with zero energy are separated from the others by a mass gap. If this is not the case, a pole can arise in a diagonal correlation function on account of the singular behavior of the matrix element $\langle0|A_i|n\rangle$ as $E_n\to0$, as occurs, for example, in the case of the free theory. The Hamiltonian is $\frac{1}{2}(\pi_1^2+\pi_2^2)$, and the matrix element $\langle|A_i|p\rangle\propto\delta'(p)$. At the same time,

$$\sum_n|\langle0|A_i|n\rangle|^2\frac{E_n}{\omega^2-E_n}\sim\int d^2k[\delta'(k)]^2\frac{k^2}{\omega^2-\dfrac{k^4}{4}}$$
$$\propto\frac{\partial^2}{\partial k^2}\left(\frac{k^2}{\omega^2-\dfrac{k^4}{4}}\right)\Bigg|_{k\to0}\propto\frac{1}{\omega^2}.$$

Of course, this pole is obtained from the massive $1/(\omega^2-H^2)$ one as $H\to0$, and in this limit the matrix elements (29) are, of course, singular.

We have discussed this well-known example here in such detail because it is a direct analog of the (2 + 1) photodynamics in which we are interested. In the following section, we shall obtain a consistent Hamiltonian formulation of this theory and show that without allowance for the constraints it completely parallels the example analyzed above.

3. HAMILTONIAN QUANTIZATION

To obtain the Hamiltonian, it is necessary to find the canonical momenta $\pi_\mu = \delta\mathscr{L}/\delta\dot{A}_\mu$. The Lagrangian from which we find the Hamiltonian is determined by the Lagrangian (3) together with the contribution of the sources that interact with the field A_μ. Therefore,

$$\mathscr{L}=-{}^1/_4F_{\mu\nu}{}^2+{}^1/_4H\varepsilon_{\mu\nu\lambda}F_{\mu\nu}A_\lambda+J_\mu A_\mu. \tag{30}$$

From (30), we can readily find the canonical momenta

$$\pi_0=0,\quad \pi_i=F_{0i}+{}^1/_2H\varepsilon_{ij}A_j=\dot{A}_i-\partial_iA_0+{}^1/_2H\varepsilon_{ij}A_j. \tag{31}$$

The fact that $\pi_0 = 0$ means that the field component A_0 is not a dynamical variable. Knowing the canonical momenta π_i, we can express $\dot{A}_i$ in terms of them and functions of fields that do not contain time derivatives, after which we obtain the Hamiltonian density

$$\mathscr{H}=\pi_i\dot{A}_i-\mathscr{L}={}^1/_2(\pi_i-{}^1/_2H\varepsilon_{ij}A_j)^2+{}^1/_4F_{ij}{}^2-A_0(\partial_i\pi_i+{}^1/_4H\varepsilon_{ij}F_{ij}+J_0)+J_iA_i. \tag{32}$$

Variation of $\mathscr{H}$ with respect to A_0 leads to the constraint

$$-\delta\mathscr{H}/\delta A_0=\partial_i\pi_i+{}^1/_4H\varepsilon_{ij}F_{ij}+J_0=0. \tag{33}$$

Note that the constraint (33) is gauge invariant and is simply the zeroth component of the equations of motion; however, in Hamiltonian quantization it must be regarded, not as an equation of motion for the operators, but as a condition on the physical states (for more detail about the Hamiltonian quantization of systems with constraints, see Ref. 13). In order to work in a physical state space without constraints, we can, using (33), express one of the momenta π_i in terms of the other and the fields A_i. In addition, using Eq. (31), we can eliminate one of the fields (to eliminate the field does not mean to annihilate it but to express it in terms of the remaining dynamical variables and sources) and remove the other by a gauge transformation. As a result, from the three fields and two momenta we are left with one field and one momentum, i.e., one degree of freedom.

But before we discuss correct quantization with allowance for the constraint (33), let us forget the constraint for a moment and regard the Hamiltonian (32) as a system with two degrees of freedom. For this, we ignore the term with the constraint (since it vanishes anyway on physical states) and the contribution of the sources. This means that we artificially "revive" one spurious degree of freedom. After we have found what this extra degree of freedom means, we shall show that imposing the constraint that drives this degree of freedom out of the spectrum leads to an instantaneous interaction of the sources, which behaves as if it proceeded through the intermediate states corresponding to this extra degree of freedom. Note that in removing the constraint from the Hamiltonian we simultaneously removed A_0, i.e., we essentially chose the gauge $A_0 = 0$. The Hamiltonian has (in the momentum representation) the form

$$\mathscr{H}=\int d^2p\left\{\frac{1}{2}\left(\pi_i-\frac{H}{2}\varepsilon_{ij}A_j\right)^2+\frac{1}{2}p_1{}^2A_2{}^2+\frac{1}{2}p_2{}^2A_1{}^2-p_1p_2A_1A_2\right\}. \tag{34}$$

In this representation, the Hamiltonian is diagonal with respect to the 2-vector **p**, which parametrizes the canonical momenta π_i and the fields A_i. Therefore, all the wave functions of the system factor and have the form

$$|\Psi\rangle=\prod_{\mathbf{p}}|\Psi_{\mathbf{p}}\rangle,$$

where $|\Psi_{\mathbf{p}}\rangle$ are the wave functions of the Hamiltonian density (34) for given **p**. Clearly, states with different **p** are orthogonal, this being simply a consequence of momentum conservation—the eigenfunctions of the Hamiltonian can also be eigenfunctions of the momentum operator.

We choose the sector with $p_2 = 0$ and some p_1. By virtue of the rotational symmetry, this does not restrict the generality of the arguments. Then

$$\mathscr{H}(p_1,\ 0)={}^1/_2(\pi_1-{}^1/_2HA_2)^2+{}^1/_2(\pi_2+{}^1/_2HA_1)^2+{}^1/_2p_1{}^2A_2{}^2. \tag{35}$$

Unfortunately, because of the term $p_1^2A_2^2$ the Hamiltonian (35) does not separate in the angular variables

$$A_1=r\cos\varphi,\quad A_2=r\sin\varphi,$$

which we used to discuss the analogous Hamiltonian in Sec. 2. However, applying to (35) the quasigauge transformation

$$U=\exp({}^1/_2iHA_1A_2),$$

which does not change the energy spectrum, we obtain

$$\mathscr{H}'(p_1,\ 0)={}^1/_2(\pi_1-HA_2)^2+{}^1/_2\pi_2{}^2+{}^1/_2p_1{}^2A_2{}^2. \tag{36}$$

This is also the Hamiltonian of a charge in a magnetic field, but, in addition, there is also an oscillator potential along the A_2 axis. Recalling Sec. 2, we readily see that the motion along the A_2 axis corresponds to a discrete spectrum of energies, except that now the energy of the intermediate (excited above the vacuum) state is not H but $(p_1^2 + H^2)^{1/2}$, as it must be in a Lorentz-invariant theory. Along the A_1 axis there is no potential, and the spectrum is degenerate with respect to π_i, which can take any value, this being essentially the position of the center of the Landau orbit. Hence, in this case too there is a soft mode, which leads to the massless pole $1/\omega^2$. One can directly calculate the correlation function, in the same way as was done in Sec. 2 [see (28) and (24)]. In this case, the wave functions are determined by Hermite polynomials, and the calculations simplify strongly if we use the generating function

$$\sum_{n=0}^{\infty}H_n(x)t^n=\exp(2xt-t^2).$$

We shall not go into the details of the calculations, since they are entirely analogous to the ones made above, and we give the result directly for the correlation function:

$$\Pi_{ij}=\frac{1}{\omega^2-\mathbf{p}^2-H^2}\left[\delta_{ij}-\frac{p_ip_j}{\omega^2}-i\frac{H\varepsilon_{ij}}{\omega}\right]. \tag{37}$$

In (37), it is easy to recognize the propagator (9).

Thus, we have obtained the correct propagator, but the theory is not correct, since we did not take into account the constraint. If we do, we eliminate the soft mode, but the important thing is not the propagator itself but its contraction with the conserved external sources. What we wish to demonstrate is the occurrence of contact terms, quadratic in the sources, corresponding completely to the massless pole in (37).

For this, we continue with the temporarily interrupted correct Hamiltonian quantization. It is simplest to use the gauge

$$\partial_i A_i = \operatorname{div} \mathbf{A} = 0. \tag{38}$$

Then from (31) it follows that

$$\partial_i A_i = \partial_i \pi_i + \partial^2 A_0 - {}^1/_2 H \varepsilon_{ij} \partial_i A_j = 0.$$

Comparing this with (33), we obtain

$$\partial^2 A_0 = H \varepsilon_{ij} \partial_i A_j + J_0. \tag{39}$$

Note that even in the absence of sources, $J_0 = 0$, it follows from (39) that it is in general impossible to have in this gauge $A_0 = 0$ (this is possible only in the trivial case $H = 0$). Similarly, in the gauge $A_0 = 0$ it is not possible to make the photon transverse. (This was discussed in the plane-wave example in Sec. 1.)

Solving (39) for A_0, we obtain

$$A_0 = \partial^{-2} (H \varepsilon_{ij} \partial_i A_j + J_0),$$

or, in the momentum representation,

$$A_0 = -iH \varepsilon_{ij} p_i A_j / \mathbf{p}^2 - J_0 / \mathbf{p}^2.$$

In the momentum representation, the constraint becomes

$$p_i (\pi_i + {}^1/_2 H \varepsilon_{ij} A_j) - iJ_0 = 0.$$

It is convenient to go over to transverse and longitudinal components of the fields and sources:

$$A_i = A_i^{\parallel} + A_i^{\perp}, \quad J_i = J_i^{\parallel} + J_i^{\perp}, \quad \mathbf{p}\mathbf{A}^{\perp} = \mathbf{p}\mathbf{J}^{\perp} = 0, \tag{40}$$

$$A_i^{\parallel} = p_i \frac{\mathbf{p}\mathbf{A}}{\mathbf{p}^2}, \quad A_i^{\perp} = \left(\delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2} \right) A_j,$$
$$J_i^{\parallel} = p_i \frac{\mathbf{p}\mathbf{J}}{\mathbf{p}^2}, \quad J_i^{\perp} = \left(\delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2} \right) J_j. \tag{41}$$

In accordance with (38), $A_i^{\parallel} = 0$.

In these variables, the Hamiltonian density has the form

$$\mathscr{H} = \frac{(\pi^{\perp})^2}{2} + \frac{(\mathbf{p}^2 + H^2)}{2} (A^{\perp})^2 + J^{\perp} A^{\perp} + \frac{1}{2} \frac{J_0^2}{\mathbf{p}^2} - \frac{iH}{|\mathbf{p}|} J_0 A^{\perp}. \tag{42}$$

Using current conservation, $\omega J_0 = |\mathbf{p}| J^{\parallel}$, we can rewrite (42) as

$$\mathscr{H} = \frac{(\pi^{\perp})^2}{2} + \frac{(\mathbf{p}^2 + H^2)}{2} (A^{\perp})^2 + J^{\perp} A^{\perp} + \frac{1}{2} \frac{J^{\parallel 2}}{\omega^2} - i \frac{H}{\omega} J^{\parallel} A^{\perp}. \tag{43}$$

We see that the first three terms of the Hamiltonians (42) and (43) describe a massive single-component photon interacting with the transverse source that creates it, while the last two terms describe the instantaneous interaction of the sources J_0 or $J^{\parallel}$ and their interaction with the physical photon $A^{\perp}$. In the case when external photons are absent, it is easy to obtain (6) and (7) from (43) and (42), respectively.

Thus, the correct quantum description of (2 + 1) photodynamics does indeed lead to nondispersive massless poles responsible for an instantaneous long-range interaction of Aharanov-Bohm type.

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