New classes of exact solutions of the Schrödinger equation and potential-field description of spin systems

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New classes of exact solutions of the steady-state, one-dimensional Schrödinger equation are found for fields corresponding to potential wells or periodic potentials. These solutions refer to the $2S+1$ lowest energy levels (S is an integer or half-integer which appears in the potentials) which correspond to the eigenvalues of the spin Hamiltonian describing an anisotropic paramagnet of spin S in an external magnetic field. The potentials found are one- and two-parameter potentials (at a fixed value of S). Their form and the structure of the energy spectrum vary substantially with the parameter values. In particular, a symmetric well and an asymmetric one with two minima are found, as is a well with a fourfold minimum. There are simple and exact analytic expressions for the wave functions and the energies in, for example, the single-parameter case with $S = 0$, 1/2, 1, 3/2, and 2. The coordinate and spin systems are related because the coordinate Hamiltonian can be written as a combination of linear differential operators which satisfy commutation relations for the spin components.

A description of the dynamic interaction which is widely used in the theory of magnetism involves only the coordinate degrees of freedom in terms of spin variables. This description is generally only approximate and involves physical assumptions of some sort: that an average can be taken over the orbital variables, that perturbation theory is applicable, etc. (a good example is the Heisenberg Hamiltonian[1–3]). There are, on the other hand, cases of a spin-coordinate correspondence with a rigorous meaning, in which the coordinate Hamiltonian can be expressed directly in terms of differential operators which actually serve as effective-spin operators (§4 of this paper).

It was shown in Ref. 4 that the energy spectrum of the spin Hamiltonian describing a paramagnet with an easy-axis anisotropy in an external magnetic field directed perpendicular to the easy axis reproduces the first $2S+1$ energy levels of a particle moving in a potential well of a certain type (S is the spin). In this case we are dealing with the inverse transformation—from a discrete spin space to coordinate space—and an extremely unusual example of the effective-field method[5] (which proves useful here in a study of the low-temperature properties of a paramagnet with $S\gg 1$). For the coordinate system we therefore find a new class of exact solutions of the Schrödinger equation. Of particular interest are cases in which the spin is not very large, so that the characteristic equations in spin space have simple explicit solutions. There is a similar spin-coordinate correspondence for an anisotropic paramagnet in a magnetic field in an arbitrary direction.

We proceed to a specific analysis of the exact solutions, which, as we will see, fall naturally into three classes.

§1. CLASS OF EXACT SOLUTIONS FOR SYMMETRIC POTENTIALS

We consider the steady-state Schrödinger equation for a particle which is moving in a one-dimensional potential field:

$$\Psi''+[\varepsilon-U(\xi)]\Psi=0 \quad (1)$$

(for simplicity, we use dimensionless energies and a dimensionless coordinate ξ), where the potential is constructed from hyperbolic functions and has two parameters (B and S),

$$U(\xi)=\frac{B^2}{4}\,\mathrm{sh}^2\,\xi-B\left(S+\frac{1}{2}\right)\mathrm{ch}\,\xi. \quad (2)$$

Exact solutions of this equation cannot in general be found. However, let us assume that we have an integer or half-integer value $S\geqslant 0$, and let us assume $B>0$. Direct substitution then shows that Eq. (1) has solutions of the form

$$\Psi(\xi)=\exp\left(-\frac{B}{2}\,\mathrm{ch}\,\xi\right)\sum_{\sigma=-S}^{S}\frac{c_\sigma}{((S-\sigma)!\,(S+\sigma)!)^{1/2}}\exp(\sigma\xi), \quad (3)$$

where the coefficients c_σ satisfy the system of finite-order linear equations

$$(\varepsilon+\sigma^2)c_\sigma+\frac{B}{2}[((S-\sigma)(S+\sigma+1))^{1/2}c_{\sigma+1}+((S+\sigma)(S-\sigma+1))^{1/2}c_{\sigma-1}]=0,$$
$$c_{S+1}=c_{-S-1}=0;\quad \sigma=-S,\,-S+1,\ldots,\,S, \quad (4)$$

and the energy levels are found from the corresponding characteristic equation. We can conclude from the oscillation theorem that such solutions correspond to only the first $2S+1$ energy levels for Schrödinger equation (1).

It can be seen that the problem of solving system (4) is equivalent to that of finding the eigenvalues of the dimensionless spin Hamiltonian $H=-S_z^2-BS_x$, where the operators S_x and S_z represent the corresponding projections of the spin S, c_σ is the wave function in the S_z representation, and B is proportional to the magnetic field.[1] (The questions of an algebraic nature which arise here are discussed in §4.)

For each fixed integer or half-integer value of S, expression (2) determines a family of potential fields which depend on the one parameter B; a variation of this parameter leads to an extremely important deformation of the potential profile.

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If $B > B_0 \equiv 2S + 1$, the potential is a single well with a simple minimum; if $B < B_0$, the potential becomes a well with two minima; and at $B = B_0$ we find a well with a fourfold minimum.

We mentioned above that in problems of this case there are exact solutions for only a limited number of low-lying energy levels. If S is not too large, the wave functions and the energies can be described by simple explicit expressions. These cases are of particular interest, since it is usually the low-lying states which are the subject of a purely quantum analysis (while good results on high-lying states can be found by a semiclassical approach).

System of linear equations (4) can be split up into two simpler systems: The states which are of even parity in σ separate from those of odd parity. For an integer spin and even states, for example, there are $S + 1$ instead of $2S + 1$ equations. We will discuss some examples of the simple, explicit analytic expressions which can be found for the energy levels and wave functions of the stationary states.

If $S = 0$ (a trivial case for a spin system, but in the coordinate picture the potential undergoes changes in shape which are also typical of other values of S), the ground-state energy $\varepsilon_0 = 0$ is independent of B, and

$$\Psi_0(\xi) = A_0 \exp\left(-\frac{B}{2}\operatorname{ch}\xi\right).$$

In the case $S = 1/2$ the corresponding quantities are

$$\begin{aligned} \varepsilon_0 &= -\frac{1}{4} - \frac{B}{2}; \quad \Psi_0(\xi) = A_0 \exp\left(-\frac{B}{2}\operatorname{ch}\xi\right)\operatorname{ch}\frac{\xi}{2}; \\ \varepsilon_1 &= -\frac{1}{4} + \frac{B}{2}; \quad \Psi_1(\xi) = A_1 \exp\left(-\frac{B}{2}\operatorname{ch}\xi\right)\operatorname{sh}\frac{\xi}{2}. \end{aligned} \tag{5}$$

For $S = 1$ we find, combining states with the same parity,

$$\begin{aligned} &\varepsilon_{0;2} = -{}^1/_2 \mp (B^2 + {}^1/_4)^{1/2}; \\ &\Psi_{0;2}(\xi) = A_{0,2} \exp\left(-\frac{B}{2}\operatorname{ch}\xi\right)\left(1 - \frac{\varepsilon_{0;2}}{B}\operatorname{ch}\xi\right); \\ &\varepsilon_1 = -1; \quad \Psi_1(\xi) = A_1 \exp\left(-\frac{B}{2}\operatorname{ch}\xi\right)\operatorname{sh}\xi \end{aligned} \tag{6}$$

(here and below, the upper sign corresponds to the index at the left).

The case $S = 3/2$ is of particular interest since it corresponds to the maximum number of exact solutions for the energy which are simple, explicit expressions and which comprehensively convey the characteristic features of the energy spectrum. In this case we have

$$\begin{aligned} &\varepsilon_{0;2} = -\frac{5}{4} - \frac{B}{2} \mp (B^2 - B + 1)^{1/2}; \\ &\Psi_{0;2}(\xi) = A_{0;2} \exp\left(-\frac{B}{2}\operatorname{ch}\xi\right) \\ &\qquad \times\left[\operatorname{ch}\frac{3\xi}{2} - \frac{2}{B}\left(\varepsilon_{0;2} + \frac{9}{4}\right)\operatorname{ch}\frac{\xi}{2}\right]; \\ &\varepsilon_{1;3} = -\frac{5}{4} + \frac{B}{2} \mp (B^2 + B + 1)^{1/2}; \\ &\Psi_{1;3}(\xi) = A_{1;3} \exp\left(-\frac{B}{2}\operatorname{ch}\xi\right) \\ &\qquad \times\left[\operatorname{sh}\frac{3\xi}{2} - \frac{2}{B}\left(\varepsilon_{1;3} + \frac{9}{4}\right)\operatorname{sh}\frac{\xi}{2}\right]. \end{aligned} \tag{7}$$

If $S = 2$, simple results can be found for the odd-parity stationary states, since the characteristic equation in this case is quadratic (cubic for the states of even partiy):

$$\begin{aligned} &\varepsilon_{1;3} = -\frac{5}{2} \mp \left(B^2 + \frac{9}{4}\right)^{1/2}; \quad \Psi_{1;3}(\xi) \\ &= A_{1;3} \exp\left(-\frac{B}{2}\operatorname{ch}\xi\right)\left[\operatorname{sh}2\xi - \frac{2}{B}(\varepsilon_{1;3} + 4)\operatorname{sh}\xi\right]. \end{aligned} \tag{8}$$

The energy spectrum has some general properties which are not restricted to integer or half-integer values of S. The deformation of the potential profile which we mentioned earlier is accompanied by a substantial change in the spectral structure. At $B \gtrsim B_0$ the spectrum is a fan of levels which, in a first approximation, depend linearly on B, while at $B = B_0$ the spectrum is that of an oscillator with a pronounced (fourfold) nonlinearity. Curiously, although we know that there are no exact solutions for a potential $U(\xi) = \beta\xi^4$, exact solutions can be found for the more complicated function $B_0^2 \sinh^4(\xi/2)$ (for the low-lying states).

At small values of B the spectrum consists of a set of levels of spin origin (for integer of half-integer values of S) which move closer together in pairs and which correspond to two symmetrically positioned solitary (in the limit $B = 0$) Morse potential wells and "superspin" levels which form a quasicontinuous part of the spectrum and which become more closely spaced toward a zero energy.

These changes in the shape of the potential and in the nature of the energy spectrum have such physical consequences for the corresponding spin system as the existence of a maximum in the low-temperature magnetic susceptibility as a function of the magnetic field.[4]

§2. CLASS OF ASYMMETRIC POTENTIALS

The results derived in §1 are generalized to the case of potentials of the form

$$U(\xi) = \frac{B^2}{4}\left(\operatorname{sh}\xi - \frac{C}{B}\right)^2 - B\left(S + \frac{1}{2}\right)\operatorname{ch}\xi, \tag{9}$$

which, in contrast with (2), are asymmetric and contain two varying parameters $B > 0$ and $C > 0$, at a fixed value of S. The wave functions of the first $2S + 1$ levels differ from (3) by a factor of $\exp(C\xi/2)$, and the quantities c_σ satisfy an equation similar to (4), differing only by an additional term σC in the coefficient of c_σ.

In this case the potential profile differs substantially, depending on whether the point on the B,C plane corresponding to the system lies inside, outside, or on the astroid $B^{2/3} + C^{2/3} + = B_0^{2/3}$. In the first of these cases, the potential is a well with two minima; in the second, it is a well with a single minimum; and in the third the maximum and the nearest minimum merge, forming an inflection point with a horizontal slope. The point $C = 0$, $B = B_0$ on the astroid corresponds to the critical field for the symmetric case.

Interestingly, the deformation of the potential profile which we mentioned above is a typical example of the transformations which are analyzed in catastrophe theory, namely a cusp catastrophe, the simplest realization of which is in a so-called Ziman's machine.[7]

Potential (9) corresponds to the dimensionless spin Hamiltonain $H = -S_z^2 - BS_x - CS_z$, which describes an anisotropic easy-axis paramagnet, where B and C are proportional to the transverse and longitudinal components of the magnetic field, respectively. This astroid equation is a quantum generalization of the equation which separates the region of metastable stables for a paramagnet of this type in the classical case.[8]

The even states do not separate from the odd states in the characteristic equation now, in contrast with (4), since the order of the equation is $2S + 1$. We can give some simple exact solutions for the cases $S = 0$ [in which the potentials in (6) and (2) take a form associated with a supersymmetric quantum mechanics[9]] and $S = 1/2$.

In the former case, the solution is

$$\varepsilon_0=0;\quad \Psi_0(\xi)=A_0\exp\left(-\frac{B}{2}\operatorname{ch}\xi+\frac{C}{2}\xi\right),$$

and in the latter it is

$$\varepsilon_{0;1}=-\frac{1}{4}\mp\frac{(B^2+C^2)^{1/2}}{2};\quad \Psi_{0;1}(\xi)$$
$$=A_{0;1}\exp\left(-\frac{B}{2}\operatorname{ch}\xi+\frac{C}{2}\xi\right)\left(e^{\xi/2}-\frac{C\mp(B^2+C^2)^{1/2}}{B}e^{-\xi/2}\right).$$

Among the special functions, the spheroidal Coulomb functions[10] are the closest approximations of these solutions with $C = 0$. The case $C \neq 0$ gives us one generalization of this class of special functions.

The energy spectrum has several extremely interesting properties, not found in the case of symmetric potential (2), which can be seen best in the limit $B = 0$. For example, there is the unusual behavior of the levels $\varepsilon_n(C)$ as a function of C. In the region $C \leqslant 2S$, where the potential takes the form of two solitary Morse wells of different depths, the energy levels turn out to be doubly degenerate for certain integer values of C. As a result, we find a discontinuous behavior for all the levels $\varepsilon_n(C)$ except the ground level, whose energy falls off monotonically with increasing C: $\varepsilon_0 = -S^2 - CS$. Since the points of discontinuity, $\varepsilon_n(C)$, pertain to excited states, there are no discontinuities in the magnetization at low temperatures (in the case of an easy-plane spin Hamiltonian, in contrast, there are discontinuities,[11] since the behavior of the ground level is discontinuous).

If $B \neq 0$ the degeneracy is lifted, and the corresponding levels "repel each other."

Typical of the behavior of the spectrum in strong "fields" is a linear dependence of the energy levels on the "intensity" $(B^2 + C^2)^{1/2}$.

We wish to emphasize the following distinctions between the classes of potential fields discussed above and the standard, exactly solvable quantum-mechanical models.[12,13] In the first place, the latter are actually one-parameter models and can be written in the form $U(x) = U_0 f(x/a)$. As the parameter U_0 is varied there is a change in the "intensity" but not the shape of the potential. Examples of these models are the Eckart and Morse potentials

$$-U_0/ch^2\, x/a,\ U_0[\exp(-2x/a)-2\exp(-x/a)].$$

The power-law models (the simple harmonic oscillator, the quaternary oscillator, and a linear potential), in contrast, contain essentially no parameter, since the Schrödinger equation in dimensionless variables is of the form $\Psi'' + (\varepsilon - \xi^m)\Psi = 0$.

On the other hand, the models of spin origin in which we are interested in the present paper are far richer in possibilities. The shape of the potential, for example, can change substantially even in the one-parameter case, i.e., with $C = 0$. The models with $C \neq 0$, in contrast, which give rise to a two-parameter potential (for a given value of S), apparently have no analogs of any sort among simple models of potential fields. Exact solutions for asymmetric double wells were found in Ref. 14, but the potential was expressed in a complicated way in terms of the confluent hypergeometric function, although the spectrum is analogous to the energy levels of a simple harmonic oscillator.

Second, in the case at hand the spectrum is completely discrete, and the potentials have no singularities. In this regard the symmetric potentials which have been found could be compared with only one of the standard, exactly solvable models: the simple harmonic oscillator. As we have already emphasized, however, the profile of the simple harmonic oscillator is fixed, and the energy spectrum has an unambiguous structure.

§3. PERIODIC POTENTIALS WITH EXACT SOLUTIONS

Up to this point we have studied the Schrödinger equation for well potentials; the discrete spectrum has been found as a consequence of the decay of the wave function at infinity. It turns out that there also exists a class of periodic potentials which allow exact solutions and which have a direct relationship with a spin system. The corresponding Schrödinger equation is

$$\frac{d^2\Psi}{d\varphi^2}+\left[\varkappa-\frac{B^2}{4}\sin\varphi-B\left(S+\frac{1}{2}\right)\cos\varphi\right]\Psi=0. \tag{10}$$

This equation can be derived from (1) and (2) through the formal substitution $\xi \to i\varphi$. The wave functions for the energies which belong to the spectrum of the corresponding spin system are

$$\Psi(\varphi)=\exp\left(-\frac{B}{2}\cos\varphi\right)\sum_{\sigma=-S}^{S}\frac{c_\sigma}{((S-\sigma)!\,(S+\sigma)!)^{1/2}}\exp(i\sigma\varphi)$$

(see §1), where the c_σ satisfy relations (4). These solutions obey periodic or antiperiodic conditions,[2] depending on S:

$$\Psi(\varphi+2\pi)=(-1)^{2S}\Psi(\varphi). \tag{11}$$

The energies ε of the spin system studied in §2 (an easy-axis paramagnet) differ only in sign from the corresponding eigenvalues of Eq. (10): $\varepsilon = -\varkappa$. In other words, they appear in exactly the opposite order in the spectra. For $S = 0$, 1/2, 1, 3/2, and 2 the explicit expressions for the eigenvalues $\varkappa$ and the wave functions $\Psi(\varphi)$ are similar to those in §1, with changes in the index and the sign of the energy, and with the replacement of the hyperbolic functions by trigonometric functions. On the other hand, these eigenvalues agree in both magnitude and sign with the spectrum of an easy-plane spin Hamiltonian $H = S_z^2 - BS_x$.

By virtue of the symmetry of the potential we can transform from (11) to more-customary boundary conditions. For example, for odd states and integer spin we find the problem of the motion of a particle with $0 \leqslant \varphi \leqslant \pi$ in a potential well with the boundary conditions $\Psi(0) = \Psi(\pi) = 0$ (an analogous procedure can be followed in the other cases). This reformulation of the boundary conditions allows us to classify the energy levels on the basis of an oscillation theorem, from which it follows that the form of the wave function implies that the spin system corresponds to the first $2S + 1$ values of $\varkappa$.

The solutions found here can be interpreted quite simply in terms of an energy-band diagram: They correspond to quasimomentum values $k = 0$ in the case of an integer spin and $k = 1/2$ in the case of a semi-integer spin; in the reduced-band diagram, the "energies" correspond to the alternating bottom and top of different bands. For $B = 0$ we find the usual problem of a free plane rotator for integer S or a rotator with antiperiodic boundary conditions for semi-integer S.

The spectrum of Schrödinger equation (1) contains an infinite number of levels; the energies ε which are not pertinent to the spin system lie above ε_{2S}. The spectrum of Eq. (10) with these boundary conditions also contains an infinite number of energy levels, but in the opposite order because of the relation $\varepsilon = -\varkappa$. The ground-state energy (ε_0) of the spin system (for an easy-axis paramagnet) corresponds to $\varkappa_{2S}$, and the "extra" energies lie below ε_0. The $2S + 1$ levels which are the intersection of the two sets correspond to the spin system.

An equation with a complex periodic potential corresponds to a system with an oblique field ($C \neq 0$), and the condition

$$\Psi(\varphi + 2\pi) = \exp[i(C/2 - S)2\pi]\Psi(\varphi)$$

singles out real values of the energy in the band solution. With $S = 0$ and $S = 1/2$ we find particularly simple exact solutions, similar to those in §2.

4. ALGEBRAIC STRUCTURE OF THESE HAMILTONIANS

The exact solutions of the Schrödinger equation for the potentials considered here have been found on the basis of the correspondence between the coordinate and spin systems. Up to this point, this correspondence has been used only in terms of the correspondence between spectra; the energy levels of the spin system have been embedded in a semi-infinite set of levels of the coordinate system. It is also interesting to determine the algebraic meaning of this correspondence between spaces of quite different natures, one finite-dimensional and the other infinite-dimensional. As we will now see, this correspondence exists because the coordinate Hamiltonian is a combination of differential operators which satisfy commutation relations for the spin components.

A direct check shows that the Hamiltonian of the Schrödinger equation with potential (2) can be written in the form

$$\tilde{H} = -\tilde{S}_z^2 - B\tilde{S}_x,$$

where

$$\begin{aligned} \tilde{S}_+ &\equiv \tilde{S}_x + i\tilde{S}_y = \left(S - \frac{B}{2}\operatorname{sh}\xi\right)e^{\xi} - e^{\xi}\frac{d}{d\xi}, \\ \tilde{S}_- &\equiv \tilde{S}_x - i\tilde{S}_y = \left(S + \frac{B}{2}\operatorname{sh}\xi\right)e^{-\xi} + e^{-\xi}\frac{d}{d\xi}, \\ \tilde{S}_z &= \frac{B}{2}\operatorname{sh}\xi + \frac{d}{d\xi}, \end{aligned} \qquad (12)$$

and where

$$[\tilde{S}_z, \tilde{S}_\pm] = \pm\tilde{S}_\pm, \ [\tilde{S}_+, \tilde{S}_-] = 2\tilde{S}_z \text{ and } \tilde{S}^2 = S(S+1).$$

The set of functions of the type in (3) which decay at infinity forms a subspace which is invariant under the action of operators (12). There is a similar situation when a "longitudinal field" is present ($C \neq 0$).

The operators in (12), which look slightly unusual, can be found (within a similarity transformation) from the expresison for the generators of the spinor representation of the rotation group,[15]

$$S_+ = 2Sz - z^2\frac{d}{dz}, \quad S_- = \frac{d}{dz}, \quad S_z = z\frac{d}{dz} - S \qquad (13)$$

through the substitution $z = \exp\xi$. It can be shown that Eqs. (12) and (13) correspond to the representation of coherent spin states,[16] for which the use of these differential operators in the corresponding spin subspace is equivalent to the use of ordinary finite-dimensional spin matrices. The wave functions in (3), on the other hand, are (within a weight factor) the eigenvectors of the spin Hamiltonian in the representation of coherent spin states.

If we choose $z = \exp(i\varphi)$ (where φ is real) in (13), i.e., if we choose z to vary along a circle of unit radius, we arrive at periodic potentials (10). Accordingly, from a single picture with a complex z, both a potential well and a periodic potential emergy as two different, topologically nonequivalent cases. We recall that for each of these cases there is a semi-infinite set of discrete levels, and the intersection of these sets gives us the energy spectrum of the spin system describing an easy-axis paramagnet.

From the algebraic standpoint, the cases we have considered here differ from the known exactly solvable models in two regards. First, the role of the algebra on whose basis the spectrum is found is played by a Lie algebra on whose basis the spectrum is found is played by a Lie algebra corresponding to the compact group $SU(2)$ [while for most of the known exactly solvable models, it is the Lie algebra of the noncompact group $SU(1,1)$ which "generates the spectrum"[17]]. Second, the Hamiltonian itself does not enter this algebra because of the term which is quadratic in the operators, and the possibility of finding the exact energies stems from the finite dimensionality of the corresponding invariant subspace. The set of generators relates in a single irreducible representation only those states which correspond to this spin subspace.

If a potential differs only slightly from those considered here, a perturbation theory can be constructed from these results, even though there will generally be no invariant, finite-dimensional subspace, and it will not be possible to find exact solutions. Although exact solutions of the unperturbed problem are known for only the first $2s+1$ stationary states, the appropriate corrections will produce a perturbation theory (see ref. 18, for example) in which it is sufficient to have information on only the particular level of the unperturbed system which is under consideration.

Up to this point in our study of the correspondence between the spin and coordinate systems we have used a representation in which S_z is diagonal, i.e., the quadratic term in the spin Hamiltonian. By working from the same spin system, but transforming to a representation in which S_x or S_y is diagonal, we can find other, less graphic differential equations.

CONCLUSION

Exact solutions of the Schrödinger equation are known for only an extremely few problems, so that finding new exactly solvable cases is of fundamental interest in its own right. Furthermore, the illustrative potential found here have several important and extremely unusual problems which distinguish them from other known exactly solvable models.

1) For all three classes, the shape of the potential can vary substantially with the parameter values. A particularly interesting result is that double wells—symmetric and asymmetric—and a well with a fourfold minimum can be found.

2) Furthermore, there is a direct relationship between problems involving motion in potential-well fields and in fields with periodic potentials. The fourfold minimum in the ξ representation corresponds to a fourth maximum in the φ representation. We wish to emphasize that among all of the previously known periodic fields there are apparently none with an exact analytic expresison for the energy levels except the trivial case of a free rotator (e.q., there is no explicit expression for the energies in the Kronig-Penney model).

3) The classes of potential fields found here have a specific physical meaning, describing the behavior of an anisotropic paramagnet in an external magnetic field. A similar pseudospin Hamiltonian arises in one of the models of interacting fermions[19] which is used in nuclear physics. It is particularly interesting to note that in this manner we find cases in which potentials with a fourth minimum and a fourth maximum have direct applications.

4) In these problems there is definite interest in not only the exact solutions themselves but also in the algebraic nature of the Hamiltonians and in the particular method (corresponding to this algebraic nature) which is used to seek these solutions with the apparatus of generalized coherent states.

Another point which deserves attention is the very method used to study spin systems with the help of effective potential fields.

The results derived here may also prove useful in a variety of physical situations in which the problem reduces to studying the motion of a particle in potential fields similar to those discussed here, especially if the field profile is a double potential well. This comment applies not only to the example mentioned above[4] but also to use of effective potentials in the quantum theory of molecular vibrations (in, say, a description of the inversion splitting in ammonia[20]), in the theory of metals (in a study of self-intersecting trajectories and nearby trajectories in phase space for an arbitrary dispersion law in a magnetic field[21]), in field theory, where the anharmonic oscillator is a simple model case[22] (in particular, in the study of systems with spontaneous symmetry breaking), etc. (see Refs. 23 and 24, for example).

Finally, the exact solutions can be used to test the effectiveness of various analytic approximations and numerical methods for studying the Schrödinger equation.

We wish to emphasize that the restriction to low-lying states for these classes of potential fields with exact solutions is completely justified by the rich set of properties of these solutions.

These is the question of whether the results and methods of this study can be generalized, in particular, to multidimensional cases by using Lie algebras corresponding to compact groups more complex than $SU(2)$.

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[1)]In the transformation from a spin system to a coordinate system, the approach of introducing a generating function like (3) is analogous to, for example, the transformation of Ref. 6, used to study the Dicke model.

[2)]For brevity we refer to simply "a periodic potential," without specifying the form of the boundary conditions. In the case $S=-1/2$, Eq. (10) reduces to the Mathieu equation.

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