Anomalously fast convergence of a doubling-type bifurcation chain in systems with two saddle equilibria

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(Submitted 31 May 1984)
Zh. Eksp. Teor. Fiz. **87**, 1696–1699 (November 1984)

The restructuring of stable periodic regimes with a varying parameter is investigated in systems with two coupled unstable equilibria. The asymptotic properties of the bifurcation sequence is investigated, the super-exponential rate of its convergence is noted, and the law of transformation of the scale of the changing structures is described.

One widely used method of changing from regular to random behavior in different physical situations (hydrodynamics, nonlinear optics, chemical kinetics, radiophysics) is via a succession of doubling the period of self-oscillating regular regimes. Feigenbaum[1] has shown that, independently of the particular problem on hand, this procedure is described by solitary scaling laws. In particular, the series of bifurcation values of a parameter converges as a geometric progression at a universal rate 4.660201.... .

Widely used systems are invariant to a symmetry transformation whose repeated application constitutes the identity transformation. These properties are possessed, for example, by the equations of thermal convection (of course, subject to appropriate boundary conditions). In such situations any regime is either invariant to this transformation or has a "double" symmetric to it.

Bifurcation sequences that lead to chaos were previously investigated[2,3] in dissipative dynamic systems that possess the property mentioned above. It was assumed that a symmetric unstable equilibrium exists, that perturbation of only one symmetry-breaking mode can grow, and that the mode with the most slowly damped perturbations is symmetric. The behavior of the phase trajectories in the vicinity of the saddle equilibrium is determined by the growth rate $\lambda_1 > 0$ and $\lambda_2 < 0$ of these monotonic modes, or more accurately, by the saddle index, defined as the ratio $\nu = (\lambda_1 + \lambda_2)/\lambda_1$.

At $\nu < 0$ the bifurcations are connected with the onset of stable periodic motions (cycles) of homoclinic orbits—fine phase trajectories that are doubly asymptotic to the saddle (i.e., that tend to it as $t \to \pm\infty$). At each bifuraction, two stable cycles generate one cycle that is twice as long; the time period at the bifurcation point becomes infinite. Scaling rules exist also for these infinite bifurcation sequences.[2,4] A one-parameter family on universality classes (each class has its own convergence rate and scale factor), and the parameter is the saddle index ν.

We shall consider the structures that occur in a dynamic system with a pair of saddles, when the system has a symmetry that transforms one saddle into the other. Let the dimensionality of the system phase space be not less than three, let the unstable manifold of saddles be one-dimensional, and let trajectories exist from the vicinity of one saddle to the vicinity of the other. Such situations, for example, precede an abrupt loss of equilibrium stability. Figure 1 shows schematically the parameter-plane region contiguous with the stability region (line h). On the left of h there is one saddle (the case described in Ref. 3), and neighboring on the right is a stable node and a pair of saddles. The lines l_1 and l_2 correspond to formation of two homoclinic loops, and line l_3 to the formation of a contour made up of two heteroclinic trajectories (going from one saddle to the other). The corresponding phase trajectories are symbolically shown on the same figure.

The properties of attractors are simulated by the Poincaré map—by a discrete relation that connects the coordinates of two successive intersections of the phase trajectories with a suitably chosen surface. In the situation considered, the usual procedure[4,3] leads to a one-dimensional relation of the form

$$x_{i+1}=f(x_i),\quad f(x)=\{a(|x|-1)^{1-\nu}-\mu\}\cdot\operatorname{sign} x, \qquad (1)$$

which is defined at $|x| \geqslant 1$. The parameters μ, a, and ν (the saddle index) are continously dependent here on the parameters of the initial system, points $x = \pm 1$ correspond to its saddles (unstable stationary regimes), and the segment $(-1, 1)$ and the union of its inverse images correspond to the attraction region of a stable node (trivial attractor).

Analysis of the model relation (1) shows that when μ is varied (a and ν are specified) bifurcations of three types alternate in the infinite sequence. In bifurcations of the first type, the disruption of the heteroclinic contour generates a symmetric stable cycle. In bifurcations of the second type this regime becomes unstable, and a pair of cycles with the same period branch off from it (symmetry crisis). In bifurcations of the third type this pair "gets caught" in the homoclinic

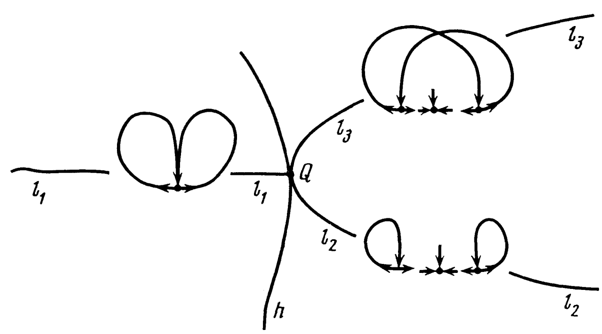

FIG. 1.

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orbits and vanishes. In the parameters intervals between the bifurcations of the first and second type there coexists with the trivial attractor a single stable cycle; between the values of the second and third type there exists a pair of stable cycles, and between the values of the third and succeeding values of the first type, only a trivial regime is stable. The boundaries of the regions of attraction of the coexisting attractors are stable manifold of saddle cycles.

In analogy with the case of systems with a single saddle point, the scenario described is accompanied by complications of the period regimes. The first rearrangements of the attracting trajectories are shown in Figs. 2a–f.

The character of the convergence of such a sequence of bifurcations differs from a geometric progression of the schemes, of Refs. 1–3. Calculations show that the progression is formed in this case not by the differences $\mu_{n+1}-\mu_n$ of like-type bifurcation values of the parameter, but by logarithms of these differences:

$$\lim_{n\to\infty}\frac{\ln(\mu_{n+1}-\mu_n)}{\ln(\mu_n-\mu_{n-1})}=\varkappa(\nu)>1, \tag{2}$$

where $\varkappa(\nu)$ depends monotonically on ν and tends to unity as $\nu\to-\infty$. For $\nu=-1$ Eq. (2) is equivalent to that obtained in an analysis[5] of bifurcations in a mapping with two quadratic extrema. At $\nu=0$, when the bifurcation values of the first type, which mark the formation of 2^n-turn contours, can be written out in explicit form,[5] it can be easily seen that $\varkappa(0)=2$.

This asymptotic law characterizes also the scales of the varying regimes, definable for a k-turn cycle ($k=2^n$) by the quantity $\xi_n=-f^k(1)-1$. At large n we have $\xi_{n+1}\sim\xi_n^{\varkappa}$, i.e.,

$$\xi_n\approx CB^{(\varkappa^n)}, \tag{3}$$

where the coefficient C is determined by the units in which ξ is measured, and the constant B ($B<1$) depends on ν.

We note that the fraction of the parameter intervals corresponding to the existence of nontrivial attracting regimes decreases with increasing serial number of the bifurcations:

$$\left(\frac{\mu_n^*-\mu_n'}{\mu_{n+1}'-\mu_n'}\right)\sim\left(\frac{\mu_{n-1}^*-\mu_{n-1}'}{\mu_n'-\mu_{n-1}'}\right)^{\varkappa}, \tag{4}$$

where μ_n^* is the nth bifurcation value of the third type. These asymptotic rules can be obtained from (1) by using an approximate analysis. We represent the higher degree of the mapping (1) as the same mapping with different a and μ: at $f(x)<-1$ we have

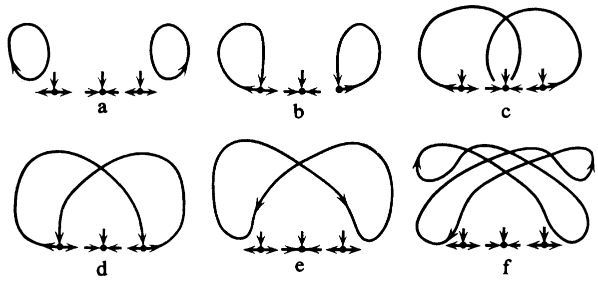

FIG. 2.

$$f(f(x))\approx\mu-a(\mu-1)^{1-\nu}+a^2(1-\nu)(\mu-1)^{-\nu}(x-1)^{1-\nu}, \tag{5}$$

i.e., $a_1=a_0^2(1-\nu)(\mu_0-1)^{-\nu}$ and $\xi_1=\mu_1-1=a_0\xi_0^{1-\nu}-\xi_0-2$. Continuing this process, we get

$$a_{i+1}=(1-\nu)a_i^2\xi_i^{-\nu}, \tag{6}$$

$$\xi_{i+1}=a_i\xi_i^{1-\nu}-\xi_i-2. \tag{7}$$

Putting $\xi_{i+2}\ll\xi_{i+1}\ll\xi_i$, we get

$$a_{i+1}\xi_{i+1}^{1-\nu}-\xi_{i+1}\approx a_i\xi_i^{1-\nu}-\xi_i\approx 2. \tag{8}$$

Hence

$$a_i^2\xi_i^{-\nu}(1-\nu)\xi_{i+1}^{1-\nu}\approx 2, \tag{9}$$

i.e.,

$$\xi_{i+1}\approx\left(\frac{2}{1-\nu}\frac{\xi_i^{\nu}}{a_i^2}\right)^{1/(1-\nu)}\approx(2-2\nu)^{1/(\nu-1)}\xi_i^{(2-\nu)/(1-\nu)}. \tag{10}$$

Thus, for the limiting value μ_∞ the mappings of the point $x=1$ form after 2^j turns a sequence $\{\xi_j\}$ that converges to zero in the opposite direction and generate corresponding heteroclinic trajectories. In this case

$$\Delta_n=\mu_\infty-\mu_n\sim\xi_n\bigg/\frac{\partial\xi_n}{\partial\mu};$$

the ratio of the increments increases therefore in inverse proportion of ξ_{n+1}:

$$\rho_n=\frac{\Delta_n}{\Delta_{n+1}}\approx\frac{\xi_n}{\xi_{n+1}}\frac{\partial\xi_{n+1}}{\partial\xi_n}\sim\frac{1}{\xi_{n+1}}. \tag{11}$$

Thus, $\rho_{n+1}\sim\rho_n^{(2-\nu)/(1-\nu)}$, i.e., we again obtain a power law with the same exponent as in (10):

(12)

A bifurcation sequence of this type is realized in a system of three ordinary differential equations which describes in the one-mode approximation the averaged regimes of convection of a viscous incompressible liquid in an oscillating gravitational field. An increase of the modulation amplitudes stabilizes the equilibrium, as a result of which we get in phase space a node (equilibrium) neighboring on a pair of saddles (unstable convective banks).[6] Numerical integration, using appropriate values of the parameters, reveals the described alternation of the regimes. Estimates of $\varkappa(\nu)$, using either the differences of the bifurcation values of the parameter [through relation(2)] or the geometric characteristics of limit cycles [through relation (3)] at various fixed ν agree with one another as well as with results of a numerical investigation of the mapping (1). It turns out here that the approximate formula (12) is only a slight overestimate. The details of the calculations will be published elsewhere.

This picture can apparently be explained by assuming that the renormalization transformation of the doubling type has a stationary point. The transformation must be singular at this point; the quantity $1/\varkappa$ determines the degree of the first term of the expansion.

[1]M. Feigenbaum, Journ. Stat. Phys. **19**, 25 (1978).
[2]A. Arneodo, P. Coullet, and C. Treser, Phys. Lett. **81A**, 197 (1981).

[3]D. V. Lyubimov and M. A. Zaks, Physica (Utrecht) **9D**, 52 (1983).
[4]L. P. Shil'nikov, in: J. Marsden and M. McCracken, McCracken, Bifurcation Creation of a Cycle and Its Applications [Russ. transl.], Mir, 1980, pp. 317–335.
[5]J. Coste and N. Peraud, Physica (Utrecht), **5D**, 415 (1982).
[6]M. A. Zaks, D. V. Lyubimov, and V. I. Chernatynskii, Izv. AN SSSR **19**, 312 (1982).

Translated by J. G. Adashko