Self-induced transparency when the resonance energy levels are degenerate

A. M. Basharov and A. I. Maĭmistov

Moscow Engineering Physical Institute
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In the framework of the generalized Maxwell-Bloch equations we consider the propagation of polarized ultra-short optical pulses in a two-level medium with energy levels which are degenerate with respect to the total angular momentum components. These equations allow a Lax representation only for the $1\leftrightarrow 0$ and $\frac{1}{2}\rightarrow\frac{1}{2}$ transitions and can be completely integrated by the inverse scattering method. We find soliton solutions—the analogs of the 2π pulses and we study the result of their collisions, in particular, the change in the polarizations of the colliding solitons.

INTRODUCTION

The self-induced transparency (SIT) effect[1] consists in the propagation of a powerful ultra-short pulse (USP) of light through a resonance medium without change of shape or energy losses. The group velocity of such a pulse (also called 2π- or SIT-pulse) is less than the phase velocity of light in the medium and depends on the length of the 2π-pulse: the shorter the 2π-pulse, the higher its velocity.[1] When two 2π-pulses propagate in the medium with different lengths the case is possible when the second of them overtakes the first and a "collision" takes place from which the 2π-pulses emerge without change in their shapes or velocities. This fundamental property of SIT pulses has been studied many times, both theoretically[2–10] and experimentally.[11,12] From the mathematical point of view it is the consequence of the complete integrability of the Maxwell-Bloch equations which describe the SIT effect in two-level media with nondegenerate resonance levels.[2–7] The SIT pulses correspond to the single-soliton solutions of these equations and the "collisional" properties reflect the "decay" of two-soliton solutions into single-soliton ones (see, e.g., Ref. 7).

In actual media degeneracy of the energy levels is natural and this manifests itself in special features of the propagation of polarized optical pulses in them. The effect of the degeneracy of the resonance levels on SIT has been considered before;[1,13–15] however, the polarization of the USP was always fixed and one was therefore not able to study polarization effects of SIT in all completeness. These effects manifest themselves especially clearly when the 2π-pulses with different polarizations collide and that has up to the present time not been studied. One of the reasons for this situation is that there are no exact solutions of the Maxwell-Bloch equations taking into account the degeneracy of the resonance levels.

In the present paper we find a Lax representation and we use the inverse scattering method to prove the integrability of the Maxwell-Bloch equations, describing SIT in the case of arbitrary polarizations of the light pulses in resonance with the quantum transitions

$$j_b=1\rightarrow j_a=0,\quad j_b=0\rightarrow j_a=1,\quad j_b=1/2\rightarrow j_a=1/2$$

between levels which are degenerate with respect to the components of the indicated angular momenta j_a and j_b. We allowed for arbitrary inhomogeneous broadening of the spectral lines of the resonance transition and detuning from resonance. We give in the first two sections all necessary information about the inverse scattering method applicable to the present problem. In sections 3 and 4 we study in the same way the polarization features of the collisions of SIT pulses (solitons) in the case of transitions with a change of angular momentum $1\leftrightarrow 0$ and $\frac{1}{2}\rightarrow\frac{1}{2}$. We establish that the nature of the collisions of USP depends on their polarization and on the kind of transition. In the case of the $1\leftrightarrow 0$ transitions there occurs a rotation of the polarization vectors of the solitons which collide with different linear polarizations. The polarization of circularly polarized solitons is not changed. In the case of the $\frac{1}{2}\rightarrow\frac{1}{2}$ transition there are no elliptically polarized solitons at all. Differently linearly polarized solitons retain their initial polarization after collisions. However, a collision of a linearly polarized soliton with a circularly polarized soliton leads to the appearance of three circularly polarized solitons. In this way it turns out that only circularly polarized solitons are stable.

All results obtained can be checked experimentally and can be in nonlinear spectroscopy. They are also of interest by themselves as in the case of the $1\leftrightarrow 0$ transitions they are a new example of a completely integrable system of nonlinear evolution equations, and in the case of the $\frac{1}{2}\rightarrow\frac{1}{2}$ transition they give a new physical interpretation of two independent sets of equations of which the integrability is known.

1. GENERALIZED MAXWELL-BLOCH EQUATIONS AND THEIR LAX REPRESENTATION

Let there propagate in the direction of the z-axis through a resonance medium, which is an ensemble of two-level atoms, a light pulse with an electric field strength of the form

$$\mathbf{E}=\vec{\mathscr{E}}\exp\left[i(kz-\omega t)\right]+\text{c.c.} \tag{1}$$

The carrier frequency $\omega = kc$ of the pulse (1) is close to the frequency $\omega_0=(E_b-E_a)/\hbar$ of the atomic transition $j_b\rightarrow j_a$ between the energy levels E_a and E_b which are degenerate with respect to the components m and μ of the total angular momenta j_a and j_b.

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The evolution of the pulse (1) in the resonance approximation is described by the generalized Maxwell-Bloch equations (see, e.g., Ref. 16) which in dimensionless form can be written as

$$\frac{\partial}{\partial \zeta}\varepsilon_q = i\sum_{\mu m}\langle R_{\mu m}\rangle J_{\mu m}{}^{q},$$

$$\left[\frac{\partial}{\partial \tau}+i(\eta-\Delta)\right]R_{\mu m} = i\sum_q \varepsilon_q\left(\sum_{m'} J^q_{\mu m'}R_{m'm} - \sum_{\mu'} R_{\mu\mu'}J^q_{\mu' m}\right),$$

$$\frac{\partial}{\partial \tau}R_{mm'} = i\sum_{q\mu}(\varepsilon_q^* J_{\mu m}{}^{q}R_{\mu m'} - \varepsilon_q R_{m\mu}J^q_{\mu m'}),$$

$$\frac{\partial}{\partial \tau}R_{\mu\mu'} = i\sum_{qm}(\varepsilon_q J_{\mu m}{}^{q}R_{m\mu'} - \varepsilon_q^* R_{\mu m}J^q_{\mu' m}). \quad (2)$$

In these equations the amplitude $\vec{\mathscr{E}}$ of the light pulse (1), the density matrices $\rho_{\mu\mu'}$ and $\rho_{mm'}$ characterizing the states of the atom in the upper and the lower resonance levels and also the optical coherence matrix $\rho_{\mu m}$ which describes the optical transitions between the resonance levels are connected with the quantities ε, $R_{\mu\mu'}$, $R_{mm'}$, and $R_{\mu m}$ through the relations

$$\vec{\mathscr{E}} = \hbar\varepsilon/t_0 d, \quad \rho_{\mu\mu'} = N_0 f(\eta)R_{\mu\mu'}, \quad \rho_{mm'} = N_0 f(\eta)R_{mm'},$$
$$R_{\mu m} = N_0 f(\eta)R_{\mu m}\exp[i(kz-\omega t)],$$

here d is the induced dipole moment of the $j_b \to j_a$ transition,[17]

$$N_0 = N_a(2j_a+1)^{-1} - N_b(2j_b+1)^{-1},$$

where N_b and N_a are the stationary populations of the upper and lower resonance levels when there are no external fields, $t_0 = (3\hbar/2\pi\omega|d|^2N_0)^{1/2}$ is a constant with the dimensions of time, $\eta = kvt_0$, $f(\eta) = T_0\pi^{-1/2}\exp[-(\eta T_0)^2]$ is the Maxwell distribution for the z-component v of the velocity of the resonance atoms which have u as their most probable velocity, $T_0 = 1/kut_0$,

$$\tau = (t - z/c)/t_0, \quad \zeta = z/ct_0,$$

$$\Delta = (\omega-\omega_0)t_0, \quad J_{\mu m}{}^{q} = (-1)^{j_b - m}\begin{pmatrix} j_a & 1 & j_b \\ -m & q & \mu \end{pmatrix}\sqrt{3}.$$

We indicate by the index q the spherical components of the vector ε (Ref. 17) and the angle brackets indicate averaging over the thermal motion of the resonance atoms:

$$\langle R_{\mu m}\rangle = \int_{-\infty}^{\infty} f(\eta)R_{\mu m}\,d\eta.$$

We have omitted in Eqs. (2) the relaxation terms as we consider here only USP the lengths of which are much smaller than all characteristic relaxation times in the medium.

The initial and boundary conditions for (1) have the form

$$\varepsilon|_{\tau=-\infty} = R_{\mu m}|_{\tau=-\infty} = 0,$$
$$R_{\mu\mu'}|_{\tau=-\infty} = N_b\delta_{\mu\mu'}/N_0(2j_b+1),$$
$$R_{mm'}|_{\tau=-\infty} = N_a\delta_{mm'}/N_0(2j_a+1),$$
$$\varepsilon|_{\zeta=0} = \varepsilon_0(\tau).$$

The point $\zeta = 0$ corresponds to the point where the excited pulse with an initial profile described by the function $\varepsilon_0(\tau)$ enters the resonance medium.

Equations (2) are a very complicated nonlinear set which have so far not been studied for the case of arbitrary angular momenta. It is, however, necessary to emphasize that even for the simplest resonance transitions $1\leftrightarrow 0$ and $\frac{1}{2}\to\frac{1}{2}$ Eqs. (2) do not reduce to the SIT equations considered earlier[1–15] if only the polarization vector of the light pulse

$$\varepsilon = \varepsilon \mathbf{l},$$

where $\mathbf{l}$ is independent of both τ and ζ, is not fixed.

In order to apply the inverse scattering method to the solution of the set of Eqs. (2) it is necessary to write them in the form of a condition of compatibility of a set of linear equations

$$\frac{\partial}{\partial \tau}Q = \hat{L}Q, \quad (3a)$$

$$\frac{\partial}{\partial \zeta}Q = \hat{A}Q, \quad (3b)$$

which can be expressed by the matrix equation

$$\partial\hat{L}/\partial\zeta = \partial\hat{A}/\partial\tau + \hat{A}\hat{L} - \hat{L}\hat{A}. \quad (4)$$

Let the matrices $\hat{L}$ and $\hat{A}$ (Lax pair) depend on the quantities ε, ε^*, $R_{\mu m}$, $R_{m\mu}$, $R_{\mu\mu'}$, and $R_{mm'}$ in the following way:

$$\hat{L} = X + \sum_q (Y_q\varepsilon_q + Z_q\varepsilon_q^*),$$

$$\hat{A} = \left\langle\left\{\sum_{\mu m}(U_{\mu m}R_{\mu m} + U_{m\mu}R_{m\mu}) + \sum_{mm'} U_{mm'}R_{mm'} + \sum_{\mu\mu'} U_{\mu\mu'}R_{\mu\mu'}\right\}\right\rangle, \quad (5)$$

where X, Y_q, Z_q, $U_{\mu m}$, $U_{mm'}$, and $U_{\mu\mu'}$ are square matrices of order $2(j_a + j_b + 1)$. In this case Q is a $2(j_a + j_b + 1)$ component column vector. Substituting (5) into (4) and using (2) to eliminate the derivatives with respect to τ and ζ we can find a set of algebraic equations for the matrices occurring in $\hat{L}$ and $\hat{A}$ from (5). This set turns out, however, to be soluble only for the cases $j_a = j_b = \frac{1}{2}$; $j_a = 1, j_b = 0$ and $j_a = 0, j_b = 1$. In that case

$$X_{bb'}=-i\lambda\delta_{bb'},\quad X_{aa'}=i\lambda\delta_{aa'},\quad (Y_q)_{ba}=(Z_q)_{ab}=iJ_{ba}{}^q,$$

$$(U_{\mu m})_{ba}=(U_{m\mu})_{ab}=\frac{i\delta_{\mu b}\delta_{ma}}{2\lambda-(\eta-\Delta)},$$

$$(U_{\mu\mu'})_{bb'}=\frac{i\delta_{\mu b}\delta_{\mu' b'}}{2\lambda-(\eta-\Delta)},\quad (U_{mm'})_{aa'}=\frac{i\delta_{ma}\delta_{m'a'}}{2\lambda-(\eta-\Delta)}. \tag{6}$$

We have written down here only the nonvanishing matrix elements. The indices b (b') and $a(a')$ vary, respectively, within the limits $-j_b\leqslant b\leqslant j_b$ and $-j_a\leqslant a\leqslant j_a$ while λ is an arbitrary parameter.

The original set (2) of Maxwell-Bloch equations can thus for the indicated cases of j_a and j_b be solved by the inverse scattering method with the spectral problem (3a) while the evolution of the scattering data is determined by Eqs. (3b). It is important that the polarizations of the solitons may be arbitrary and this makes it possible to study the SIT polarization effects. Even in the very particular cases these effects turn out to be nontrivial.

2. NECESSARY INFORMATION ABOUT THE INVERSE SCATTERING METHOD

We choose the quantization axis along the direction of propagation of the light wave (the ζ-axis). As the electromagnetic waves considered are transverse the interaction with a quantum system with such waves is accompanied by transitions with a change in the angular momentum component by ± 1. This enables us to simplify the Lax operators (5) and (6) for each of the $\frac{1}{2}\to\frac{1}{2}$ and $1\leftrightarrow 0$ transitions.

We consider first the transition $j_b=0\to j_a=1$ (one considers the transition $j_b=1\to j_a=0$ similarly) and introduce the notation

$$e_\pm=-i\varepsilon_{\pm 1},\quad n_b=R_{\mu\mu'}|_{\mu=\mu'=0},$$
$$n_{a\pm}=R_{mm'}|_{m=m'=\pm 1},\quad \nu_a=R_{mm'}|_{m=-m'=-1},$$
$$\nu_\pm=R_{\mu m}|_{\mu=0,\ m=\pm 1}.$$

In that case it follows from (5) that

$$\hat{L}=\begin{pmatrix} -i\lambda & e_- & e_+ \\ -e_-^* & i\lambda & 0 \\ -e_+^* & 0 & i\lambda \end{pmatrix}, \tag{7a}$$

$$\hat{A}=\left\langle \frac{i}{2\lambda-(\eta-\Delta)}\begin{pmatrix} n_b & \nu_- & \nu_+ \\ \nu_-^* & n_{a-} & \nu_a \\ \nu_+^* & \nu_a^* & n_{a+} \end{pmatrix}\right\rangle, \tag{7b}$$

where in Eqs. (3) with the matrices $\hat{L}$ and $\hat{A}$ in the form (7) the vector Q has three components.

The spectral problem (3a) with $\hat{L}$ in the form (7a) was studied in detail in Ref. 18 where the self-focusing of a polarized light beam in a Kerr medium was considered. For the following analysis of the SIT effect considered here it is necessary to give some information about the direct and the inverse scattering problem.[18]

The Jost functions for the spectral problem (3a) and (7a) $\Phi^{(i)}(\tau,\xi)$ and $\Psi^{(i)}(\tau,\xi)$ ($i=1$ to 3) are determined as the fundamental solutions of (3a) for real $\lambda=\xi$ with the following asymptotic behavior

$$\Phi^{(1)}\to g^{(1)}e^{-i\xi\tau},\quad \Phi^{(2)}\to g^{(2)}e^{i\xi\tau},\quad \Phi^{(3)}\to g^{(3)}e^{i\xi\tau},\quad \tau\to-\infty,$$
$$\Psi^{(1)}\to g^{(1)}e^{-i\xi\tau},\quad \Psi^{(2)}\to g^{(2)}e^{i\xi\tau},\quad \Psi^{(3)}\to g^{(3)}e^{i\xi\tau},\quad \tau\to\infty.$$

The quantities $\Phi^{(k)}$, $\Psi^{(k)}$, and $g^{(k)}$ are three-component column vectors and $g^{(k)}$ has an ith component $g_i^{(k)}$ equal to δ_{ki}. The scattering matrix S_{ij} is given in the basis of the Jost functions by

$$\Phi^{(i)}(\tau,\xi)=\sum_{j=1}^{3}S_{ij}(\xi)\Psi^{(j)}(\tau,\xi). \tag{8}$$

The eigenvalues of the spectral problem (3a) are found as the solutions of the equation $S_{11}(\lambda)=0$. If the $e_\pm$ decrease sufficiently rapidly as $|\tau|\to\infty$ then $S_{11}(\lambda)$, $\Phi^{(1)}$, $\Psi^{(2)}$, and $\Psi^{(3)}$ can be analytically continued in the upper half-plane of the complex λ-variable, and $\Phi^{(2)}$, $\Phi^{(3)}$, and $\Psi^{(1)}$ in the lower λ-half-plane.

Let λ_n be the zeroes of S_{11}, i.e., $S_{11}(\lambda_n)=0$, $n=1,\ldots,N$ ($\operatorname{Im}\lambda_n>0$); then

$$\Phi^{(1)}(\tau,\lambda_n)=C_{12}^{(n)}\Psi^{(2)}(\tau,\lambda_n)+C_{13}^{(n)}\Psi^{(3)}(\tau,\lambda_n). \tag{9}$$

The scattering data $\{\lambda_n\}$, $S_{ij}(\xi)$, $C_{12}^{(n)}$, and $C_{13}^{(n)}$ are sufficient to determine the "potentials" $e_\pm$ of the operator $\partial/\partial\tau-\hat{L}(\lambda;e_+,e_-)$. The solution of the inverse scattering problem is formulated on the basis of singular integral equations.[18] We shall use here the alternative approach based upon the Gel'fand-Levitan-Marchenko equations which when applied to the present problem have the form

$$-K^{(1)}(x,y)=\sum_{\beta=2}^{3}F_{1\beta}(x+y)g^{(\beta)}+\sum_{\beta=2}^{3}\int_x^\infty K^{(\beta)}(x,z)F_{1\beta}(z+y)\,dz,$$

$$K^{(\beta)}(x,y)=F_{1\beta}^*(x+y)g^{(1)}+\int_x^\infty K^{(1)}(x,z)F_{1\beta}^*(z+y)\,dz,\quad \beta=2,3.$$

Here

$$F_{1\beta}(y)=-i\sum_{n=1}^{N}\frac{C_{1\beta}^{(n)}}{S_{11}'(\lambda_n)}e^{i\lambda_n y}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_{1\beta}(\xi)}{S_{11}(\xi)}e^{i\xi y}\,d\xi,$$

$$S_{11}'(\lambda_n)=\left.\frac{dS_{11}}{d\lambda}\right|_{\lambda=\lambda_n}$$

and $K^{(1)}(x,y)$ and $K^{(\beta)}(x,y)$ are three-component column vectors. The potentials $e_\pm(\tau,\zeta)$ can be expressed in terms of the solution of these integral equations as $\tau'\to\tau+0$:

$$e_+(\tau)=-2\lim K_1^{(2)}(\tau,\tau')=2\lim K_2^{(1)*}(\tau,\tau'),$$
$$e_-(\tau)=-2\lim K_1^{(3)}(\tau,\tau')=2\lim K_3^{(1)*}(\tau,\tau').$$

If there is only one eigenvalue $\lambda_1 = i\sigma$ we have

$$F_{1\beta}(y) = -iC_\beta e^{-\sigma y}, \quad S_{11}(\lambda, \lambda_1) = (\lambda - i\sigma)/(\lambda + i\sigma), \tag{10}$$

$$S_{1\beta}(\lambda, \lambda_1) = S_{\beta 1}(\lambda, \lambda_1) = 0,$$

$$S_{\beta\beta'}(\lambda, \lambda_1) = \delta_{\beta\beta'} + \frac{2i\sigma}{\lambda - i\sigma}\frac{C_\beta^* C_{\beta'}}{|C_2|^2 + |C_3|^2}, \quad \beta, \beta' = 2, 3.$$

The solution of the Gel'fand-Levitan-Marchenko equations gives

$$e_-(\tau) = -2iC_2^*\chi(\tau), \quad e_+(\tau) = -2iC_3^*\chi(\tau),$$
$$\chi(\tau) = [e^{2\sigma\tau} + e^{-2\sigma\tau}(|C_2|^2 + |C_3|^2)/4\sigma^2]^{-1},$$
$$C_2^* = 2\sigma l_{-1}\exp(2\sigma\tau_0), \quad C_3^* = 2\sigma l_1 \exp(2\sigma\tau_0). \tag{11}$$

Expression (11) is the analog of the McCall-Hahn 2π-pulse

$$\boldsymbol{\varepsilon}(\tau) = 2\sigma \mathbf{l}\,\mathrm{sech}\,2\sigma(\tau - \tau_0) \tag{12}$$

with length $1/2\sigma$ and unit vector polarization **l**.

The evolution of the scattering data with ζ is given by the standard methods[19] from Eqs. (3b) and (7b):

$$S_{kj}(\xi, \zeta) = S_{kj}(\xi, \zeta = 0)\exp[(\gamma_j - \gamma_k)\zeta w(\xi)], \quad k, j = 1, 2, 3,$$
$$\gamma_1 = N_b/N_0, \quad \gamma_2 = \gamma_3 = N_a/3N_0, \tag{13a}$$

$$C_\beta(\zeta) = C_\beta(\zeta = 0)\exp[\zeta w(i\sigma)], \quad \beta = 2, 3, \tag{13b}$$

$$w(\xi) = \left\langle \frac{1}{2\xi - (\eta - \Delta)} \right\rangle. \tag{13c}$$

In the case $j_b = \frac{1}{2} \to j_a = \frac{1}{2}$ the matrices $\hat{L}$ and $\hat{A}$ have the form

$$\hat{L} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \tag{14}$$

where

$$L_\pm = \begin{pmatrix} -i\lambda & e_\pm \\ -e_\pm^* & i\lambda \end{pmatrix}, \quad A_\pm = \left\langle \frac{i}{2\lambda - (\eta - \Delta)} \begin{pmatrix} n_{b\pm} & \nu_\pm \\ \nu_\pm^* & n_{a\pm} \end{pmatrix} \right\rangle$$

$$e_\pm = \mp i e_{\pm 1}, \quad n_{b\pm} = R_{\mu\mu'}|_{\mu = \mu' = \mp 1/2},$$

$$n_{a\pm} = R_{mm'}|_{m = m' = \pm 1/2}, \quad \nu_\pm = R_{\mu m}|_{\mu = -m = \mp 1/2}. \tag{14a}$$

The spectral problem (3a) (like the second linear Eq. (3b) of the inverse scattering method) now splits into two independent problems which have been studied before:[19,20]

$$\frac{\partial}{\partial\tau}\varphi_\pm = L_\pm \varphi_\pm, \tag{14b}$$

where $\varphi_\pm$ are two-component column vectors. The Jost functions $\Phi_\pm^{(i)}(\tau, \xi)$ and $\Psi_\pm^{(i)}(\tau, \xi)$, $\lambda = \xi$ are determined for $\mathrm{Im}\,\lambda = 0$ as the fundamental solutions of (14b) with asymptotic behavior

$$\Phi_\pm^{(1)} \to g^{(1)}e^{-i\xi\tau}, \quad \Phi_\pm^{(2)} \to -g^{(2)}e^{i\xi\tau}, \quad \tau \to -\infty,$$
$$\Psi_\pm^{(1)} \to g^{(1)}e^{-i\xi\tau}, \quad \Psi_\pm^{(2)} \to -g^{(2)}e^{i\xi\tau}, \quad \tau \to +\infty,$$

where $g_k^{(i)} = \delta_{ik}$, $i, k = 1, 2$. The scattering matrix is the direct sum of $S_+ \otimes S_-$ 2×2 matrices such that

$$\Phi_\pm^{(i)}(\tau, \xi) = \sum_{i=1}^{2} S_{ik\pm}(\xi)\Psi_\pm^{(k)}(\tau, \xi), \quad k = 1, 2. \tag{8'}$$

The analytical properties of the Jost functions and of the matrices $S_{ik\pm}(\xi)$ are the same as in the normal Zakharov-Shabat spectral problem.[19,20] The discrete spectrum of the operator $\partial/\partial\tau - \hat{L}$ is determined by two sets of numbers λ_{n+} and λ_{n-} such that $S_{11\pm}(\lambda_{n\pm}) = 0$, while $\mathrm{Im}(\lambda_{n\pm}) > 0$. One can write the eigenfunctions of the discrete spectrum in the form

$$\Phi_\pm^{(1)}(\tau, \lambda_{n\pm}) = C_\pm^{(n)}\Psi_\pm^{(2)}(\tau, \lambda_{n\pm}). \tag{9'}$$

The construction of the "potentials" $e_\pm(\tau)$ from scattering data $\{\lambda_{n\pm}, n = 1, \ldots, N; C_\pm^{(n)}, S_{ij\pm}(\xi)\}$ is realized using the functions $K_\pm(\tau, \tau')$ with $\tau' \to \tau + 0$:

$$\boldsymbol{e_\pm(\tau) = -2\lim K_\pm(\tau, \tau')},$$

which satisfy the integral equation $(\tau' > \tau)$

$$K_\pm(\tau, \tau') + \int_\tau^\infty K_\pm(\tau, \eta_1)\int_\tau^\infty F_\pm(\eta_1 + \eta_2)F_\pm^*(\eta_2 + \tau')\,d\eta_1\,d\eta_2$$
$$= F_\pm^*(\tau + \tau'), \tag{15}$$

where

$$F_\pm(\eta) = -i\sum_{n=1}^{N}\frac{C_\pm^{(n)}\exp(i\tau\lambda_{n\pm})}{S'_{11\pm}(\lambda_{n\pm})} + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\,\frac{S_{12\pm}(\xi)}{S_{11\pm}(\xi)}e^{i\xi\tau}.$$

The dependence of the scattering data on the ζ-variable is given by the expressions

$$S_{11\pm}(\lambda, \zeta) = S_{11\pm}(\lambda, 0), \quad C_\pm^{(n)}(\zeta) = S_{12\pm}(\lambda_{n\pm}, \zeta),$$
$$S_{12\pm}(\lambda, \zeta) = S_{12\pm}(\lambda, 0)\exp[-2\zeta w(\lambda)],$$

where $w(\lambda)$ is given in (13c). Single-soliton solutions in this case of a resonance transition correspond to a situation where each of the branches of the discrete spectrum consists of a single point $\lambda_{1\pm} = i\sigma_\pm$:

$$e_\pm(\tau, \zeta) = -2i\sigma_\pm\,\mathrm{sech}\,\{2\sigma_\pm[\tau - \zeta\,\mathrm{Re}\,w(i\sigma_\pm)/\sigma_\pm - \tau_{0\pm}]\}, \tag{16}$$

where

$$\tau_{0\pm} = \frac{1}{2\sigma_\pm}\ln\left\{\frac{1}{2\sigma_\pm}\left|\frac{C_\pm^{(1)}}{S'_{11\pm}(i\sigma_\pm)}\right|\right\}.$$

3. COLLISIONS OF POLARIZED SOLITONS (1↔0 case)

Let at a time interval $\tau_2-\tau_1$ at the entrance $\zeta=0$ of a resonance medium two differently polarized solitons be incident of lengths $1/2\sigma_1$ and $1/2\sigma_2$ and with unit polarization vectors $\mathbf{l}^{(1)}$ and $\mathbf{l}^{(2)}$:

$$\varepsilon_0(\tau)=2\sigma_1\mathbf{l}^{(1)}\operatorname{sech}2\sigma_1(\tau-\tau_1)+2\sigma_2\mathbf{l}^{(2)}\operatorname{sech}2\sigma_2(\tau-\tau_2). \quad (17)$$

On the τ-axis the first pulse is positioned to the left of the second one when $\tau_2-\tau_1 \gg 1/2\sigma_n$, $n=1, 2$. If $\sigma_2>\sigma_1$ the propagation speed v_2 of the second soliton is larger than the speed v_1 of the first soliton,

$$v_n=c\left[1+\frac{1}{4\sigma_n}\operatorname{Re}w(i\sigma_n)\right]^{-1}.$$

(The speeds are here dimensional quantities.) As the solitons propagate in the medium the second soliton therefore overtakes the first one and, colliding with it, goes on so that as $\zeta\to\infty$ the second pulse turns out to be to the left of the first one on the τ-axis:

$$\varepsilon(\tau,\zeta)=2\sigma_1\mathbf{l}'^{(1)}e^{i\varphi_1}\operatorname{sech}2\sigma_1(\tau-\tau_1')$$
$$+2\sigma_2\mathbf{l}'^{(2)}e^{i\varphi_2}\operatorname{sech}2\sigma_2(\tau-\tau_2'). \quad (18)$$

Here $\mathbf{l}'^{(1)}$ and $\mathbf{l}'^{(2)}$ are the unit polarization vectors of the first and the second 2π pulses after the collision,

$$\tau_n'=\tau_n+\frac{\zeta}{4\sigma_n}\operatorname{Re}w(i\sigma_n)+\tau_{n0},\quad \varphi_n=\frac{\zeta}{2}\operatorname{Im}w(i\sigma_n),$$

the τ_{n0} are constants. Such a picture of the collision follows from the fact that the Maxwell-Bloch equations have two-soliton solutions with asymptotic forms which are the pulses (17) considered here. However, instead of a direct study of the two-soliton solution to determine the result of the collision of these pulses it is simpler to use the Zakharov-Shabat method.[21]

If $\zeta=0$, in the region $\tau\to-\infty$ the Jost functions $\Phi^{(1)}(\tau,i\sigma_1)$ and $\Phi^{(1)}(\tau,i\sigma_2)$ of the spectral problem (3a) and (7a) have, respectively, the form $g^{(1)}\exp(\sigma_1\tau)$ and $g^{(1)}(\sigma_2\tau)$. In the region between the solitons ($\tau_1 \ll \tau \ll \tau_2$) we have according to (8) and (9)

$$\Phi^{(1)}(\tau,i\sigma_1)=[C_{12}^{(1)}g^{(2)}+C_{13}^{(1)}g^{(3)}]\exp(-\sigma_1\tau),$$
$$\Phi^{(1)}(\tau,i\sigma_2)=S_{11}(i\sigma_2,i\sigma_1)g^{(1)}\exp(\sigma_2\tau).$$

Here $S_{11}(i\sigma_2,i\sigma_1)$ is given by Eq. (10) and the superscript of the coefficients C_{12} and C_{13} indicates the number of the soliton. When the Jost functions occurring on the right-hand sides of these expressions pass through the second soliton the quantities $g^{(2)}\exp(-\sigma_1\tau)$ and $g^{(3)}\exp(-\sigma_1\tau)$ are transformed according to (8) and $g^{(1)}\exp(\sigma_2\tau)$ according to (9). We therefore have as $\tau \gg \tau_2$

$$\Phi^{(1)}(\tau,i\sigma_1)=\sum_{\beta,\beta'=2}^{3}S_{\beta\beta'}(i\sigma_1,i\sigma_2)C_{1\beta}^{(1)}g^{(\beta')}\exp(-\sigma_1\tau),$$
$$\Phi^{(1)}(\tau,i\sigma_2)=S_{11}(i\sigma_2,i\sigma_1)\sum_{\beta=2}^{3}C_{1\beta}^{(2)}g^{(\beta)}\exp(-\sigma_2\tau). \quad (19)$$

If $\zeta\to\infty$, the order of the solitons is changed into the opposite one. In that case one can find similarly that as $\tau\to\infty$

$$\Phi^{(1)}(\tau,i\sigma_1)=S_{11}(i\sigma_1,i\sigma_2)\sum_{\beta=2}^{3}C_{1\beta}'^{(1)}g^{(\beta)}\exp(-\sigma_1\tau),$$
$$\Phi^{(1)}(\tau,i\sigma_2)=\sum_{\beta,\beta'=2}^{3}S_{\beta\beta'}'(i\sigma_2,i\sigma_1)C_{1\beta}'^{(2)}g^{(\beta')}\exp(-\sigma_2\tau). \quad (20)$$

The prime at the quantities $S'_{\beta\beta'}(i\sigma_2,i\sigma_1)$ and $C'^{(n)}_{1\beta}$ indicates that when they are calculated, using Eqs. (10) all quantities refer to the solitons after their collision.

Comparing (19) and (20) and using the explicit form of S_{jk} and $C^{(n)}_{1\beta}$ we find the following connection between the polarization vectors $\mathbf{l}^{(n)}$ and $\mathbf{l}'^{(n)}$:

$$\mathbf{l}'^{(1)}=b^{-1}\left\{-\mathbf{l}^{(1)}+\frac{2\sigma_2}{\sigma_2-\sigma_1}\mathbf{l}^{(2)}(\mathbf{l}^{(1)}\mathbf{l}^{(2)*})\right\},$$
$$\mathbf{l}'^{(2)}=b^{-1}\left\{\mathbf{l}^{(2)}+\frac{2\sigma_1}{\sigma_2-\sigma_1}\mathbf{l}^{(1)}(\mathbf{l}^{(1)*}\mathbf{l}^{(2)})\right\},$$
$$b^2=1+\frac{4\sigma_1\sigma_2}{(\sigma_2-\sigma_1)^2}|\mathbf{l}^{(1)}\mathbf{l}^{(2)*}|^2, \quad (21)$$

where

$$\mathbf{l}'^{(1)}\mathbf{l}'^{(2)*}=\mathbf{l}^{(1)}\mathbf{l}^{(2)*},$$
$$\tau_{10}=\frac{1}{2\sigma_1}\ln\frac{\sigma_1+\sigma_2}{\sigma_2-\sigma_1}b,\quad \tau_{20}=-\frac{1}{2\sigma_2}\ln\frac{\sigma_1+\sigma_2}{\sigma_2-\sigma_1}b. \quad (22)$$

If before the collision the pulses (17) were linearly polarized, according to Eq. (21) they will also be linearly polarized after the collision. The polarization vectors of the first and the second solitons are in this case rotated over different angles χ_1 and χ_2:

$$\mathbf{l}^{(1)}\mathbf{l}'^{(1)}=\cos\chi_1=-(1+B_{12}\cos^2\chi)(1-B_{12}B_{21}\cos^2\chi)^{-1/2},$$
$$\mathbf{l}^{(2)}\mathbf{l}'^{(2)}=\cos\chi_2=(1+B_{21}\cos^2\chi)(1-B_{12}B_{21}\cos^2\chi)^{-1/2},$$
$$B_{12}=2\sigma_2/(\sigma_1-\sigma_2),\quad B_{21}=2\sigma_1/(\sigma_2-\sigma_1),\quad \cos\chi=\mathbf{l}^{(1)}\mathbf{l}^{(2)}. \quad (23)$$

Nonetheless as a result of such a rotation the angle between the polarization vectors of the solitons is unchanged and equal to χ. It follows from (23) that there is no rotation of the polarization plane if $\chi=0$ or $\chi=\pi/2$, i.e., if the pulses (17) are polarized colinearly or at right angles. Circularly polarized solitons also collide without a change in their polarization.

As a result of the collision of a linearly polarized soliton with a circularly polarized one two solitons with elliptical polarization are formed. The parameters of the polarization ellipses follow from (19) and are determined by the lengths of the pulses and the magnitude of the induced dipole moment of the 1→0 or 0→1 transition.

For any polarization in the 1↔0 transitions the pulses (17) behave as repelling particles: the overtaken soliton gets a positive increase in its coordinate $\tau_{10}>0$, while the overtaking soliton has a negative change $\tau_{20}<0$.

The behavior of the polarizations found here is independent of the degree of inhomogeneous broadening of the spectral line and of the detuning. The latter affect only the propagation speed of the soliton (as long as the resonance approximation is valid).

4. COLLISION OF POLARIZED SOLITONS IN THE $\frac{1}{2}\to\frac{1}{2}$ CASE

It is a peculiar feature of the $\frac{1}{2}\to\frac{1}{2}$ case that left- and right-hand circularly polarized waves propagate independently, which is reflected in the reducibility of the operators (5). This fact determines both the polarization of the solitons and their polarization properties when they collide.

It follows from Eq. (16) that only when $\sigma_+ = \sigma_- = \sigma$ the electromagnetic pulse corresponding to the soliton (16) will propagate as a single unit with velocity

$$v=c[1+\mathrm{Re}\, w(i\sigma)/4\sigma]^{-1},$$

and with a length of $1/2\sigma$ in that case. If the initial light pulse were linearly polarized the equality of σ_+ and σ_- is guaranteed (in the general case $\lambda_{n+} = \lambda_{n-}$; $n = 1, \ldots, N$). If, however, the incoming pulse is elliptically polarized, we have $\sigma_+ \neq \sigma_-$ and the situations when $\sigma_+ \neq 0, \sigma_- = 0$ or the other way round, are possible. It is clear from (16) that the propagation velocities of the solitons $e_+(\tau,\zeta)$ and $e_-(\tau,\zeta)$ are different (like their lengths). The elliptically polarized initial USP thus splits up into a pair of (left- and right-handedly) circularly polarized solitons with lengths $1/2\sigma_\pm$ and propagation speeds $v_\pm = c[1 + \mathrm{Re}\, w(i\sigma_\pm)/4\sigma_\pm]^{-1}$. When one of the eigenvalues $\sigma_\pm$ is equal to zero there appears only one circularly polarized soliton. In order that such an evolution of an elliptically polarized incoming pulse occurs it is necessary that the area under the envelopes $e_+(\tau)$ or $e_-(\tau)$ be less than π. This result follows simply from the McCall-Hahn area theorem[1] for SIT in the case of nondegenerate resonance levels. Finally, a circularly polarized outgoing USP evolves into a circularly polarized soliton (or N_1 solitons and N_2 breathers in the general case, where $N = N_1 + 2N_2$).

In the case of a collision of differently linearly polarized solitons ($\sigma_{i+} = \sigma_{i-}$; $i = 1,2$; $\sigma_1 \pm \neq \sigma_2 \pm$) one can easily show by using a method analogous to the one given in the preceding section that the polarization of the solitons remains unchanged while the solitons themselves undergo additional coordinate shifts τ_{10} and τ_{20} given by Eqs. (20) with $b = 1, \sigma_1 = \sigma_{1\pm}, \sigma_2 = \sigma_{2\pm}$. A similar result holds when solitons with identical circular polarizations collide $\sigma_{1\pm} \neq \sigma_{2\pm}$, $\sigma_{1\mp} = \sigma_{1\mp} = 0$. The collision of left- and right-hand polarized solitons is not accompanied at all with any effects.

An interesting picture will be observed when a linearly polarized soliton collides with a circularly polarized soliton ($\sigma_{2+} = \sigma_{2-}$ and $\sigma_{1+} = 0, \sigma_{1-} \neq 0$ or $\sigma_{1-} = 0, \sigma_{1+} \neq 0$). Let $\sigma_2 > \sigma_1$ and let the linearly polarized soliton overtake the circularly polarized soliton

$$\varepsilon_0(\tau)=2\sigma_1\mathbf{l}_{\mathrm{circ}}\,\mathrm{sech}\, 2\sigma_1(\tau-\tau_1)+2^{1/2}\sigma_2\mathbf{l}_x\,\mathrm{sech}\, 2\sigma_2(\tau-\tau_2), \qquad (24)$$

where $\mathbf{l}_{\mathrm{circ}} = \mathbf{l}_+ = -(\mathbf{l}_x + i\mathbf{l}_y)/\sqrt{2}$ for right- and $\mathbf{l}_{\mathrm{circ}} = \mathbf{l}_- = (\mathbf{l}_x - i\mathbf{l}_y)/\sqrt{2}$ for left-hand circular polarizations, while $\mathbf{l}_x$ and $\mathbf{l}_y$ are unit vectors along Cartesian axes. To fix the ideas we put $\mathbf{l}_{\mathrm{circ}} = \mathbf{l}_+$. Performing the procedure of taking the Jost functions $\Phi^{(i)}_\pm(\tau,\lambda_{n\pm})$ through the soliton potentials (24) we can obtain expressions similar to (19) and (20). It follows from these expressions that the first soliton after the collision, retaining its shape, is shifted along the τ-axis by an amount

$$\Delta\tau_1=\zeta\,\mathrm{Re}\, w(i\sigma_{1+})/4\sigma_{1+}-\sigma_{1+}^{-1}\ln[(\sigma_{2+}-\sigma_{1+})/(\sigma_{1+}+\sigma_{2+})].$$

The left-hand polarized part of the second soliton is shifted by an amount

$$\Delta\tau_2=\zeta\,\mathrm{Re}\, w(i\sigma_{2-})/4\sigma_{2-}$$

along the τ-axis whereas the right-hand polarized part of the same soliton is shifted by an amount

$$\Delta\tau_3=\zeta\,\mathrm{Re}\, w(i\sigma_{2+})/4\sigma_{2+}+(\sigma_{2+})^{-1}\ln[(\sigma_{2+}-\sigma_{1+})/(\sigma_{1+}+\sigma_{2+})].$$

The differently circularly polarized components of a linearly polarized soliton are thus separated. As the result of a collision of a circularly polarized soliton with a linearly polarized one three circularly polarized solitons have thus been formed (the total number of points of the discrete spectrum of the operator $\partial/\partial\tau - \hat{L}(\lambda,e_+,e_-)$ has, of course, not been changed):

$$\begin{aligned}\boldsymbol{\varepsilon}(\tau,\zeta)=&2\sigma_1\mathbf{l}_+\exp(i\varphi_1)\,\mathrm{sech}\,[2\sigma_1(\tau-\tau_1-\Delta\tau_1)]\\ &+2\sigma_2\mathbf{l}_-\exp(i\varphi_2)\\ &\times\mathrm{sech}\,[2\sigma_2(\tau-\tau_2-\Delta\tau_2)]\\ &+2\sigma_2\mathbf{l}_+\exp(i\varphi_2)\,\mathrm{sech}[2\sigma_2(\tau-\tau_2-\Delta\tau_3)],\end{aligned} \qquad (25)$$

where

$$\sigma_2=\sigma_{2+}=\sigma_{2-},\quad \sigma_1=\sigma_{1+},\quad \varphi_k=\zeta\,\mathrm{Im}\, w(i\sigma_k)/2,\quad k=1,2.$$

Two (of the three) solitons formed move with the same velocity and they are farther separated the smaller the difference in velocities of the colliding solitons. One can easily understand these results if we consider a linearly polarized pulse as a combination of independent left- and right-hand circularly polarized waves. When a linearly polarized soliton interacts with a soliton with circular (e.g., right-hand) polarization the right-hand polarized component of the first soliton interacts with the second soliton like repelling particles while the left-hand polarized component evolves in the usual way. Due to this the left-hand polarized components of the first soliton are spatially separated after the collision and as the result of the collision the three solitons (25) are formed which are circularly polarized.

A similar result ensues when a circularly polarized soliton overtakes and interacts with a linearly polarized soliton.

CONCLUSION

In contrast to the well-known solitons of the Korteweg–de Vries, the sine-Gordon, and the nonlinear Schrödinger equations the soliton solutions of the generalized Maxwell-Bloch equations found here have an additional parameter—the polarization. Collision of two solitons now leads not only to a phase shift but also (in general) to a rotation of the polarization vectors. A similar situation is encountered in the problem of the self-focusing of a polarized light beam.[18] Boomerons[22] also are polarized solitons but so far they have not found a physical interpretation. Solitons in SIT theory in a three-level nondegenerate medium[8–10] called simultons in Ref. 8 can be interpreted as polarized solitons[10] if we take the relative amplitudes of the pulses which form the simulton as the components of the polarization vector. The solitons studied in the present paper complete this small list. We have not considered here the problem of finding the infinite series of constants of the motion although it would not be too complicated to do this on the basis of the Lax representation which we found.

The fact that for arbitrary j_a and j_b we did not succeed in finding a Lax representation serves, apparently, as an indication that the Eqs. (2) in the general case are not integrable. Although one can find sometimes by standard methods (see, e.g., Ref. 1 for $j_b = 1 \rightarrow j_a = 1$ or Ref. 13 for $j_b = 2 \rightarrow j_a = 2$) stationary solitary wave collisions of such waves, especially of differently polarized ones, lead, evidently, to their disintegration. It must be of great interest to study such a process by numerical means.

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