Radiative effects in stochastic acceleration and universal spectra of fast cosmic-ray particles

V. N. Tsytovich

Institute of General Physics, USSR Academy of Sciences
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In the framework of a theory that takes into account radiative effects in stochastic acceleration of particles it is shown that a power-law distribution of the fast (tail) particles is formed independently of the type of random field, the turbulence spectrum, the nature of the correlations (strong or weak turbulence), and the charge and spin of the accelerated particles. The basic characteristics of the theoretically obtained spectrum (the power-law dependence $1/\varepsilon^\gamma$ on the energy with $\gamma = 3$ for $\varepsilon \gg mc^2$; the flattening of the spectrum of all ions (nuclei) for $\varepsilon < mc^2$, i.e., at energies less than 1 GeV per nucleon; the relative concentration of the electrons and protons, and also of the protons and the heavy nuclei; and the relative concentration of thermal and fast particles) are very close to the observed characteristics of the electron and ion cosmic-ray spectrum. The theory predicts the presence of a relatively large number of low-energy electrons, so that the total energy transferred in acceleration processes to cosmic-ray electrons can be greater than the energy transferred to the cosmic-ray ions. This changes the estimates obtained from observations of the energy released in known sources of cosmic rays.

§1. INTRODUCTION. QUALITATIVE PICTURE

The radiative effects associated with the change in the electromagnetic mass of a particle accelerated in random fields by wave-particle resonances were discussed in the framework of the classical approach in Ref. 1 and in the framework of the relativistic quantum kinetic approach for spin $\frac{1}{2}$ particles in Refs. 2 and 3. In both the classical approach (for a physically reasonable cutoff of divergent integrals) as well as in the quantum approach the relative magnitude of the radiative effects is formally of order $q^2/\hbar c$, where q is the charge of the particle. However, as will be seen from what follows, this estimate is somewhat formal in nature, and it also corresponds to the estimate of the relative fraction of integrated characteristics such as, for example, the change in the mean energy of the accelerated particles, etc.

With regard to the energy distribution of the particles, it can be shown that radiative effects become decisive in the generation of the small number of very energetic particles. Their spectrum has a number of universal characteristics, which will be the subject of the present paper.

By "formal" allowance for radiative effects, we mean formal expansion of the interaction with respect to $q^2/\hbar c$. Whereas in the first approximation the interaction of the particles with the random resonance field E_{k_1} (in what follows $k_1 = \{\mathbf{k}_1, \omega_1\}$ denotes the 4-momentum vector of the resonance field, and $\mathbf{k}$ is the momentum of the virtual field) is proportional to $q^2|E_{k_1}|^2$, in the following approximation in the square of the charge it is proportional to $q^4|E_{k_1}|^2$.

In a classical description, an additional q^2 can occur only in dimensionless combinations containing classical quantities. Since we are considering the effects associated with the change in the electromagnetic mass of the particles due to the presence of the resonance fields, the additional q^2 must occur in the expression equal to the ratio of the energy of the self-field of the particle to its total energy $\varepsilon_p = (p^2c^2 + m^2c^4)^{1/2}$. The electrostatic energy of a point charge is equal to q^2/r as $r \to 0$ or $q^2 \int d\,\mathbf{k}/2\pi^2 k^2$. It was shown in Ref. 1 that only the transverse part of the self-energy contributes to the radiative corrections to the stochastic acceleration (the retardation of the virtual photon is important; cf. Ref. 4). Therefore, instead of the Green's Function $1/2\pi^2k^2$ of the longitudinal field, the expressions obtained in Ref. 1 contained the Green's function of the transverse field: $1/[k^2 - (\mathbf{k}\cdot\mathbf{v})^2/c^2]$, where $\mathbf{v}$ is the particle velocity. Integration of such expressions over the angles of the virtual photon $\mathbf{k}$ gives divergent integrals of the same type $\Lambda(v) \int d\,\mathbf{k}/k^2$ as in the case of the electrostatic energy, though the factors $\Lambda(v)$ that depend on the particle velocity (expressions for $\Lambda(v)$ are given in Ref. 1). For $\varepsilon \gg mc^2$, when $v \to c$, these factors are in general certain constants.

In other words, in the classical approach the additional terms with extra q^2 for $\varepsilon \gg mc^2$ contain q^2 in the combination ($\varepsilon_p \approx cp$)

$$\frac{q^2}{cp} \int \frac{d\mathbf{k}}{2\pi^2 k^2}. \tag{1}$$

This result is given here to illustrate qualitatively and nonrigorously how the universal power-law spectra are obtained in the quantum treatment. Before doing this, we note that if in the integral (1) we cut off k at $k_{\max} \sim p/\hbar$, then (1) gives $q^2/\hbar c$.

In the quantum treatment, naturally, divergent integrals do not arise, but this is achieved by the renormalization subtractional procedure. The subtractional procedure must give expressions that in the limit $k \to \infty$ are also asymptotically power functions $1/k^\nu$ with integral ν [in (1), $\nu = 2$]. It is obvious that convergence occurs for $\nu \geqslant 4$. This, of course, is only an indirect indication that ν may be equal to 4. In any case, this value of ν is the smallest possible value that can "survive" asymptotically, unless, of course, its coefficient

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vanishes for certain symmetry reasons. In fact, this is not the case, as will be demonstrated in the quantitative calculation made below. In the present qualitative analysis, we shall assume that the coefficient does not vanish.

From the quantum point of view, the self-energy of the particles is produced by the emission and absorption of a virtual photon (momentum $\hbar\mathbf{k}$). Then a particle with momentum $\mathbf{p}$ goes over as a result of emission of, for example, a photon $\hbar\mathbf{k}$ into the state $\mathbf{p}-\hbar\mathbf{k}$. If $\hbar\mathbf{k}$ is sufficiently large (and the divergence of the integral (1) gives grounds for assuming that very large $\mathbf{k}$ can be important in this process), so that $\hbar k \gg p$, the recoil momentum of the particle after emission of the $\mathbf{k}$ photon will be large and $|\mathbf{p}-\hbar\mathbf{k}|\gg p$, i.e., a slow particle is transformed into a fast particle and a fast photon. This "dissociation" occurs only temporarily, and subsequently the virtual photon is absorbed by the particle, which again becomes slow.

It is also posssible to have another process, in which a fast particle, emitting a photon with large momentum, becomes slow "for a time." Because of the renormalization, this process is absent (is included in the particle mass) for free particles (not accelerated by external factors). But in the presence of the external fields leading to acceleration processes, this process is included (as an effect additional with respect to the vacuum), and one can say that in the presence of accelerating fields the emission of virtual photons causes the particles to "jitter," a fast particle becoming a slow one for short periods and vice versa.

We shall be interested in stochastic acceleration, assuming that the accelerating fields are random, and we shall consider an ensemble of accelerated particles. Let $\Phi_\mathbf{p}$ be the mean probability of finding a particle with momentum $\mathbf{p}$. The averaging is over a mixed state, i.e., it includes the quantum-mechanical probability of finding one particle in the state with momentum $\mathbf{p}$. The states that arise on the emission of a virtual photon are also of this kind. Because of the indistinguishability of the particles, it is meaningless to try and distinguish those that at a given time have gone over virtually to a state with a different value $\mathbf{p}'$ of the momentum from those that at the given time have the same momentum $\mathbf{p}'$. It is only meaningful to speak of the averaged probability over the mixed state. In fact, $\Phi_\mathbf{p}$ is a component of the averaged density matrix (see below).

We assume that when radiative effects are ignored the rate of stochastic acceleration is described by the operator $\hat{I}_\mathbf{p}$, i.e.,

$$d\Phi_\mathbf{p}/dt=\hat{I}_\mathbf{p}\Phi_\mathbf{p}. \tag{2}$$

Let us find how frequently we shall "see" the appearance of fast particles. Let $\mathbf{p}$ be the momentum of a fast particle. The momentum of the slow particle will be $\mathbf{p}-\hbar\mathbf{k}$ (where $\hbar\mathbf{k}\approx\mathbf{p}$, so that $|\mathbf{p}-\hbar\mathbf{k}|\ll p$). We write down Eq. (2) for the slow particles:

$$d\Phi_{\mathbf{p}-\hbar\mathbf{k}}/dt=\hat{I}_{\mathbf{p}-\hbar\mathbf{k}}\Phi_{\mathbf{p}-\hbar\mathbf{k}}. \tag{3}$$

The factor (1) can be regarded as the probability of different $\mathbf{k}$. Replacing $1/k^2$ in (1) by the renormalized expression, denoted by $G(k^2)$, we obtain from (3) for the fast particles

$$\frac{d\Phi_\mathbf{p}}{dt}=\frac{q^2}{pc}\int G(k^2)\hat{I}_{\mathbf{p}-\hbar\mathbf{k}}\Phi_{\mathbf{p}-\hbar\mathbf{k}}\hbar^3 d\mathbf{k}. \tag{4}$$

Replacing $\mathbf{p}-\hbar\mathbf{k}$ in (4) by $\mathbf{p}'$, we obtain

$$\frac{d\Phi_\mathbf{p}}{dt}=\frac{q^2}{pc}\int G((\mathbf{p}-\mathbf{p}')^2)\hat{I}_{\mathbf{p}'}\Phi_{\mathbf{p}'}d\mathbf{p}'. \tag{5}$$

We now consider $p\gg p'$, using the asymptotic behavior for $G(k^2)$ that we have already discussed above. We set

$$G((\mathbf{p}-\mathbf{p}')^2)\to\frac{p'^2}{\hbar p^4}G_0. \tag{6}$$

Here, the factor $p'^2/\hbar$ is added for dimensional reasons to make G_0 in (6) dimensionless. Indeed, for this choice the factor $q^2p'^2/\hbar cp^5$ has dimensions $1/p^3$. So does $d\mathbf{p}'$. The dependence (6) on p as $1/p^4$ follows from the asymptotic behavior

$$G(k^2)\to\frac{1}{k^4}=\frac{\hbar^4}{(\mathbf{p}-\mathbf{p}')^4}\approx\frac{\hbar^4}{p^4}.$$

Thus, we obtain the rate of generation of fast particles:

$$\frac{d\Phi_\mathbf{p}^{fast}}{dt}=\frac{q^2}{\hbar c}\frac{G_0}{p^5}\int p'^2\hat{I}_{\mathbf{p}'}\Phi_{\mathbf{p}'}d\mathbf{p}'. \tag{7}$$

This expression gives a universal energy distribution. If, for example, we integrate (7) over the complete interval of time during which the acceleration takes place, making the assumption that $\Phi_{\mathbf{p}'}$ describes only the low-energy particles, whose number is much greater than the number of high-energy particles, we obtain from (7)

$$\Phi_\mathbf{p}^{fast}\propto 1/p^5. \tag{8}$$

The energy distribution $\Phi_\varepsilon(\varepsilon\approx cp)$ differs from $\Phi_\mathbf{p}$ in (8) by the phase space $4\pi p^2$, i.e.,

$$\Phi_\varepsilon^{fast}\propto 1/\varepsilon^3. \tag{9}$$

In these arguments, we have not particularized the operator $\hat{I}_\mathbf{p}$—it may correspond either to Fermi acceleration or to the quasilinear acceleration more common in plasmas. In the latter case, $\hat{I}_\mathbf{p}\propto q^2|E_{k_1}|^2$ (as already discussed above). The operator $\hat{I}_\mathbf{p}$ may also correspond to scattering processes (for example, for induced scattering, $\hat{I}_\mathbf{p}\propto q^4|E_{k_1}|^4$).

In the present paper, we shall construct a quantitative theory that confirms these qualitative arguments, using the equations for the Wigner density matrix of particles of arbitrary momenta (nonrelativistic and relativistic) described by Klein-Gordon equations, i.e., having vanishing spin. This treatment will show that the main results of the qualitative treatment do not depend on the spin of the particles [for spin $\frac{1}{2}$, the spectrum (9) was obtained analytically in Ref. 3) and, in addition, the case of spin 0 is somewhat simpler as regards the final analytic result and makes it somewhat easier to find the integrated characteristics of the radiative effects in stochastic acceleration. For simplicity, we shall discuss only the radiative effects for quasilinear acceleration by electrostatic fields ($\hat{I}_\mathbf{p}\propto q^2|E_{k_1}|^2$) in the absence of external magnetic fields. The resonance condition for the fields E_{k_1} then reduces to the Cherenkov condition

$$\omega_1=\mathbf{k}_1\mathbf{v}. \tag{10}$$

We shall assume that the fields E_{k_1} are classical, i.e., the quantum effects for them are negligibly small. This means that $\hbar k_1 \ll p, \hbar\omega_1 \ll \varepsilon_p$, the resonance condition $\varepsilon_p - \varepsilon_{p-\hbar k} = \hbar\omega_1$ reduces to (10), and the pair production resonance $\hbar\omega_1 = \varepsilon_p + \varepsilon_{p+\hbar k_1}$ is not realized. Then the operator $\hat{I}_p$ has the form

$$\hat{I}_p = \hat{I}_p^{ql} = q^2\pi \int dk_1 \frac{|E_{k_1}|^2}{k_1^2}\left(\mathbf{k}_1\frac{\partial}{\partial \mathbf{p}}\right)\delta(\omega_1 - \mathbf{k}_1\mathbf{v})\left(\mathbf{k}_1\frac{\partial}{\partial \mathbf{p}}\right). \quad (11)$$

In what follows, we shall use the system of units with $\hbar = c = 1$. The superscript ql in the operator (11) indicates that the acceleration process is quasilinear.

§2. BASIC EQUATIONS. RADIATIVE COLLISION INTEGRAL

In accordance with Ref. 5, a particle of spin 0 can be described by a scalar wave function ψ and the fourth component ψ' of the 4-vector $\{\psi', i\psi'\}$. Although the two functions ψ and ψ' are related to each other, the relativistic quantum density matrix can be conveniently constructed by writing the basic equations

$$(p_\mu - qA_\mu)\psi_\mu = -m\psi, \qquad (p_\mu - qA_\mu)\psi = m\psi_\mu$$

in the form of two equations for ψ and ψ' in the momentum representation:

$$\psi(\mathbf{r},t) = \int \frac{\psi_\mathbf{p}(t)e^{i\mathbf{pr}}}{(2\pi)^{3/2}}d\mathbf{p}, \quad \psi'(\mathbf{r},t) = \int \frac{\psi_\mathbf{p}'(t)e^{i\mathbf{pr}}d\mathbf{p}}{(2\pi)^{3/2}},$$

$$\varphi(\mathbf{r},t) = \int \varphi_\mathbf{k}(t)e^{i\mathbf{kr}}d\mathbf{k},$$

$$A(\mathbf{r},t) = \int \mathbf{A}_\mathbf{k}(t)e^{i\mathbf{kr}}d\mathbf{k}, \quad A_\mu = \{\mathbf{A}, i\varphi\}.$$

The initial equations are

$$i\frac{\partial\hat{\psi}_\mathbf{p}(t)}{\partial t} = m\hat{\psi}_\mathbf{p}'(t) + q\int \hat{\varphi}_{\mathbf{k}_1}(t)\hat{\psi}_{\mathbf{p}-\mathbf{k}_1}(t)d\mathbf{k}_1, \quad (12)$$

$$i\frac{\partial\hat{\psi}_\mathbf{p}'(t)}{\partial t} = \frac{\varepsilon_p^2}{m}\hat{\psi}_\mathbf{p}(t) + q\int \hat{\varphi}_{\mathbf{k}_1}(t)\hat{\psi}'_{\mathbf{p}-\mathbf{k}_1}(t)d\mathbf{k}_1$$

$$-\frac{2q}{m}\int\left(\left(\mathbf{p}-\frac{\mathbf{k}_1}{2}\right),\hat{\mathbf{A}}_{\mathbf{k}_1}(t)\right)\hat{\psi}_{\mathbf{p}-\mathbf{k}_1}(t)d\mathbf{k}_1 + \frac{q^2}{m}\int(\hat{\mathbf{A}}_{\mathbf{k}_1}(t)$$

$$\times\hat{\mathbf{A}}_{\mathbf{k}_2}(t))\hat{\psi}_{\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2}d\mathbf{k}_1d\mathbf{k}_2, \quad (13)$$

and also

$$i\frac{\partial\hat{\psi}_\mathbf{p}^+(t)}{\partial t} = -m\hat{\psi}_\mathbf{p}'^+(t) - q\int \hat{\psi}^+_{\mathbf{p}+\mathbf{k}_1}(t)\hat{\varphi}_{\mathbf{k}_1}(t)d\mathbf{k}_1, \quad (14)$$

$$i\frac{\partial\hat{\psi}_\mathbf{p}'^+(t)}{\partial t} = -\frac{\varepsilon_p^2}{m}\hat{\psi}_\mathbf{p}^+(t) - q\int \hat{\psi}'^+_{\mathbf{p}+\mathbf{k}_1}(t)\hat{\varphi}_{\mathbf{k}_1}(t)d\mathbf{k}_1$$

$$+\frac{2q}{m}\int\hat{\psi}^+_{\mathbf{p}+\mathbf{k}_1}(t)\left(\mathbf{p}+\frac{\mathbf{k}_1}{2},\hat{\mathbf{A}}_{\mathbf{k}_1}(t)\right)$$

$$d\mathbf{k}_1 - \frac{q^2}{m}\int\hat{\psi}^+_{\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2}(t)\left(\hat{\mathbf{A}}_{\mathbf{k}_1}(t)\hat{\mathbf{A}}_{\mathbf{k}_2}(t)\right)d\mathbf{k}_1d\mathbf{k}_2. \quad (15)$$

At the same time $\varepsilon_p^2 = p^2 + m^2$, and the potentials $\hat{\varphi}_{\mathbf{k}_1}(t)$ and $\hat{\mathbf{A}}_{\mathbf{k}_1}(T)$ are determined from Maxwell's equations, whose right-hand sides contain the charge and current densities

$$\hat{\rho}_\mathbf{k}(t) = \frac{qm}{2(2\pi)^3}\int[\hat{\psi}^+_{\mathbf{p}'-\mathbf{k}}(t)\hat{\psi}_{\mathbf{p}'}'(t) + \hat{\psi}_{\mathbf{p}'}'(t)\hat{\psi}^+_{\mathbf{p}'-\mathbf{k}}(t)$$

$$+\hat{\psi}'^+_{\mathbf{p}'-\mathbf{k}}(t)\hat{\psi}_{\mathbf{p}'}(t) + \hat{\psi}_{\mathbf{p}'}(t)\hat{\psi}'^+_{\mathbf{p}'-\mathbf{k}}(t)]d\mathbf{p}',$$

$$\mathbf{j}_\mathbf{k}(t) = \frac{2q}{(2\pi)^3}\int\left(\mathbf{p}'-\frac{\mathbf{k}}{2}\right)\hat{\psi}^+_{\mathbf{p}'-\mathbf{k}}(t)\hat{\psi}_{\mathbf{p}'}(t)d\mathbf{p}'$$

$$-\frac{q^2}{(2\pi)^3}$$

$$\int[\hat{\psi}^+_{\mathbf{p}'-\mathbf{k}+\mathbf{k}_1}(t)\hat{\psi}_{\mathbf{p}'}(t)\hat{\mathbf{A}}_{\mathbf{k}_1}(t) + \hat{\mathbf{A}}_{\mathbf{k}_1}(t)\hat{\psi}^+_{\mathbf{p}'-\mathbf{k}+\mathbf{k}_1}(t)\psi_{\mathbf{p}'}(t)]d\mathbf{p}'.$$

The charges and currents are introduced in such a way that their vacuum expectation values are zero.

The introduction of two functions, ψ and ψ', is dictated by the presence of particles with two possible signs of the energy (particles and antiparticles), but for both free particles the operators ψ and ψ' contain both terms with positive and negative energy.

It can be seen from the commutation relations

$$\hat{\psi}_\mathbf{p}(t)\hat{\psi}'^+_{\mathbf{p}'}(t) - \hat{\psi}'^+_{\mathbf{p}'}(t)\hat{\psi}_\mathbf{p}(t)$$

$$= \hat{\psi}_\mathbf{p}'(t)\hat{\psi}^+_{\mathbf{p}'}(t) - \hat{\psi}_{\mathbf{p}'}^+(t)\hat{\psi}_\mathbf{p}'(t) = \frac{1}{m}\delta(\mathbf{p}-\mathbf{p}'), \quad (16)$$

$$\hat{\psi}_\mathbf{p}(t)\hat{\psi}^+_{\mathbf{p}'}(t) - \hat{\psi}_{\mathbf{p}'}^+(t)\hat{\psi}_\mathbf{p}(t) = \hat{\psi}_\mathbf{p}'(t)\hat{\psi}'^+_{\mathbf{p}'}(t) - \hat{\psi}'^+_{\mathbf{p}'}(t)\hat{\psi}_\mathbf{p}'(t) = 0 \quad (17)$$

that in fact the "conjugate" of the operator $\hat{\psi}_\mathbf{p}(t)$ is $\hat{\psi}_\mathbf{p}'^+(t)$, and not $\hat{\psi}_\mathbf{p}^+(t)$, and the "conjugate" of the operator $\hat{\psi}_\mathbf{p}'(t)$ is $\hat{\psi}_\mathbf{p}^+(t)$.

It is convenient to introduce the sign $\lambda = \pm 1$ of the energy of the free particles (superscript (0) of the operator $\hat{\psi}$):

$$\hat{\psi}_\mathbf{p}^{(0)}(t) = \frac{1}{(2\varepsilon_\mathbf{p})^{1/2}}\sum_\lambda a_\mathbf{p}^\lambda \exp(-i\lambda\varepsilon_\mathbf{p}t),$$

$$\hat{\psi}_\mathbf{p}'^{(0)}(t) = \left(\frac{\varepsilon_\mathbf{p}}{2m^{2}}\right)^{1/2}\sum_\lambda \lambda a_\mathbf{p}^\lambda \exp(-i\lambda\varepsilon_\mathbf{p}t),$$

$a_\mathbf{p} = a_\mathbf{p}^+, a_\mathbf{p}^{-1} = b_{-\mathbf{p}}^+$; $a_\mathbf{p}^+$ and $a_\mathbf{p}$ and $b_\mathbf{p}^+$ and $b_\mathbf{p}$ are, respectively, the operators of creation and annihilation of particles and antiparticles.

We introduce density matrix operators in the Wigner representation $\hat{f}^{\lambda,\lambda'}_{\mathbf{p},\mathbf{k}}(t)$ and in the representation of the signs of the energy of the free particles ($\lambda, \lambda' = \pm 1$):

$$\hat{f}^{\lambda,\lambda'}_{\mathbf{p},\mathbf{k}} = \frac{1}{4}\left(\lambda'\hat{f}_{\mathbf{p},\mathbf{k}} + \lambda\frac{\varepsilon_{\mathbf{p}-\mathbf{k}/2}}{\varepsilon_{\mathbf{p}+\mathbf{k}/2}}\hat{f}'_{\mathbf{p},\mathbf{k}} + \frac{\varepsilon_{\mathbf{p}-\mathbf{k}/2}}{m}\hat{\xi}_{\mathbf{p},\mathbf{k}} + \frac{m\lambda\lambda'}{\varepsilon_{\mathbf{p}+\mathbf{k}/2}}\hat{\xi}'_{\mathbf{p},\mathbf{k}}\right), \quad (18)$$

where

$$\hat{f}_{\mathbf{p},\mathbf{k}}(t) = m(\hat{\psi}'^+_{\mathbf{p}-\mathbf{k}/2}(t)\hat{\psi}_{\mathbf{p}+\mathbf{k}/2}(t) + \hat{\psi}_{\mathbf{p}+\mathbf{k}/2}(t)\hat{\psi}'^+_{\mathbf{p}-\mathbf{k}/2}(t)),$$

$$\hat{f}'_{\mathbf{p},\mathbf{k}}(t) = m(\hat{\psi}^+_{\mathbf{p}-\mathbf{k}/2}(t)\hat{\psi}'_{\mathbf{p}+\mathbf{k}/2}(t) + \hat{\psi}'_{\mathbf{p}+\mathbf{k}/2}(t)\hat{\psi}^+_{\mathbf{p}-\mathbf{k}/2}(t)), \quad (19)$$

$$\hat{\xi}_{\mathbf{p},\mathbf{k}}(t) = m(\hat{\psi}^+_{\mathbf{p}-\mathbf{k}/2}(t)\hat{\psi}_{\mathbf{p}+\mathbf{k}/2}(t) + \hat{\psi}_{\mathbf{p}+\mathbf{k}/2}(t)\hat{\psi}^+_{\mathbf{p}-\mathbf{k}/2}(t)) - \frac{m}{\varepsilon_\mathbf{p}}\delta(\mathbf{k}),$$

$$\hat{\xi}'_{\mathbf{p},\mathbf{k}}(t) = m(\hat{\psi}'^+_{\mathbf{p}-\mathbf{k}/2}(t)\hat{\psi}'_{\mathbf{p}+\mathbf{k}/2}(t) + \hat{\psi}'_{\mathbf{p}+\mathbf{k}/2}(t)\hat{\psi}'^+_{\mathbf{p}-\mathbf{k}/2}(t)) - \frac{\varepsilon_\mathbf{p}}{m}\delta(\mathbf{k}).$$

For free particles, $\langle a_\mathbf{p} + a_{\mathbf{p}'}\rangle = \Phi_\mathbf{p}\delta(\mathbf{p}-\mathbf{p}')$. In the absence of antiparticles, $\langle b_\mathbf{p} + b_{\mathbf{p}'}\rangle = 0$. The particle concentration n is related to $\mathbf{\Phi}_\mathbf{p}$ by

$$n = \int\frac{\Phi_\mathbf{p}d\mathbf{p}}{(2\pi)^3}; \quad (20)$$

we have

$$\langle f_{\mathbf{p},\mathbf{k}}^{1,1(0)}\rangle=\Phi_{\mathbf{p}}\delta(\mathbf{k}),\quad \langle f_{\mathbf{p},\mathbf{k}}^{1,-1(0)}\rangle=\langle f_{\mathbf{p},\mathbf{k}}^{-1,1(0)}\rangle=\langle f_{\mathbf{p},\mathbf{k}}^{-1,-1(0)}\rangle=0.$$

The averaging symbol corresponds to averaging over the vacuum and over the statistical ensemble.

In the general case, one can introduce the averaged density matrix

$$\Phi_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'}=\langle\hat f_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'}\rangle$$

and the operator of fluctuations

$$\delta\hat f_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'}=\hat f_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'}-\langle\hat f_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'}\rangle.$$

In the approximation of free particles,

$$\delta\hat f_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'(0)}(t)=\delta\hat f_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'(0)}(0)\exp\{-i(\lambda\varepsilon_{\mathbf{p}+\mathbf{k}/2}-\lambda'\varepsilon_{\mathbf{p}-\mathbf{k}/2})t\},$$

and the mean values of the fluctuations do not vanish only for the products

$$\begin{aligned}&\langle\delta f_{\mathbf{p},\mathbf{k}}^{1,1(0)}(0)\,\delta f_{\mathbf{p}',\mathbf{k}'}^{1,1(0)}(0)\rangle=\Phi_{\mathbf{p}-\mathbf{k}/2}\delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{k}+\mathbf{k}'),\\ &\langle\delta f_{\mathbf{p},\mathbf{k}}^{1,-1(0)}(0)\,\delta f_{\mathbf{p}',\mathbf{k}'}^{-1,1(0)}(0)\rangle=\Phi_{\mathbf{p}+\mathbf{k}/2}\delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{k}+\mathbf{k}').\end{aligned}\tag{21}$$

The equations for $f_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'}$ can be obtained from (12)–(15). They can be most readily obtained for the second terms of (19) containing the operators $\hat\psi$ and $\hat\psi'$ in an order which ensures that the operators $\hat{\mathbf{A}}$ and $\hat\varphi$ occur only in front of or after the combinations containing the operators ψ. Further, using (16) and (17) we can show that the first terms of (19) satisfy the same equations. The equation then obtained has a form very similar to the classical kinetic equation:

$$\begin{aligned}&i\frac{\partial}{\partial t}\hat f_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'}(t)-(\lambda\varepsilon_{\mathbf{p}+\mathbf{k}/2}-\lambda'\varepsilon_{\mathbf{p}-\mathbf{k}/2})\hat f_{\mathbf{p},\mathbf{k}}^{\lambda,\lambda'}(t)=\frac{q}{2}\sum_{\lambda''}\int d\mathbf{k}_1\\ &\times\Big\{\hat\varphi_{\mathbf{k}_1}(t)\left(1+\lambda\lambda''\frac{\varepsilon_{\mathbf{p}-\mathbf{k}_1+\mathbf{k}/2}}{\varepsilon_{\mathbf{p}+\mathbf{k}/2}}\right)\hat f_{\mathbf{p}-\mathbf{k}_1/2,\mathbf{k}-\mathbf{k}_1}^{\lambda'',\lambda'}(t)-\hat f_{\mathbf{p}+\mathbf{k}_1/2,\mathbf{k}-\mathbf{k}_1}^{\lambda,\lambda''}(t)\\ &\times\left(\lambda'\lambda''+\frac{\varepsilon_{\mathbf{p}-\mathbf{k}/2}}{\varepsilon_{\mathbf{p}+\mathbf{k}_1-\mathbf{k}/2}}\right)\hat\varphi_{\mathbf{k}_1}(t)\Big\}\\ &-q\sum_{\lambda''}\int d\mathbf{k}_1\Big\{\frac{\lambda}{\varepsilon_{\mathbf{p}+\mathbf{k}/2}}\left(\left(\mathbf{p}+\frac{\mathbf{k}}{2}-\frac{\mathbf{k}_1}{2}\right),\right.\\ &\left.\hat{\mathbf{A}}_{\mathbf{k}_1}(t)\right)\hat f_{\mathbf{p}-\mathbf{k}_1/2,\mathbf{k}-\mathbf{k}_1}^{\lambda'',\lambda'}(t)\\ &-\frac{\lambda'}{\varepsilon_{\mathbf{p}+\mathbf{k}_1-\mathbf{k}/2}}\hat f_{\mathbf{p}+\mathbf{k}_1/2,\mathbf{k}-\mathbf{k}_1}^{\lambda,\lambda''}(t)\left(\left(\mathbf{p}-\frac{\mathbf{k}}{2}+\frac{\mathbf{k}_1}{2}\right),\hat{\mathbf{A}}_{\mathbf{k}_1}(t)\right)\Big\}\\ &+q^2\sum_{\lambda''}\int d\mathbf{k}_1\,d\mathbf{k}_2\Big\{\frac{\lambda}{\varepsilon_{\mathbf{p}+\mathbf{k}/2}}(\hat{\mathbf{A}}_{\mathbf{k}_1}(t)\hat{\mathbf{A}}_{\mathbf{k}_2}(t))\hat f_{\mathbf{p}-\mathbf{k}_1/2-\mathbf{k}_2/2,\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}^{\lambda'',\lambda'}(t)\\ &-\hat f_{\mathbf{p}+\mathbf{k}_1/2+\mathbf{k}_2/2,\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}^{\lambda,\lambda''}(t)\frac{\lambda}{\varepsilon_{\mathbf{p}+\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}/2}}(\hat{\mathbf{A}}_{\mathbf{k}_1}(t)\mathbf{A}_{\mathbf{k}_2}(t))\Big\}.\end{aligned}\tag{22}$$

For $\lambda=\lambda'=1$ and $k\ll p$, the left-hand side of (22) has the standard classical form $i\partial f/\partial t+(\mathbf{k}\cdot\mathbf{v})f$.

From (22) we can obtain by averaging an equation for $\Phi^{\lambda\lambda'}$, and by subtracting the averaged equation from (22) we can obtain an equation for the fluctuations $\delta f^{\lambda\lambda'}$. The relations given above are sufficient to obtain the radiative collision integral.

We use the Coulomb gauge. We assume that there is a random external classical electrostatic field:

$$\langle\varphi_{k_1}^{R},\varphi_{k_1'}^{R}\rangle=\frac{|E_{k_1}|^2}{k_1^2}\delta(k_1+k_1').\tag{23}$$

Besides this, there is the self-field φ^q, $\mathbf{A}^q$ of the particles and the vacuum electromagnetic field $\mathbf{A}^v$ (the field of the zero-point fluctuations).

To obtain the quasilinear equation, it is sufficient to take into account in δf only the terms linear in φ^R and ignore the terms associated with the self-fields and vacuum fields, while on the right-hand side it is necessary to use the asymptotic behaviour (as $t\to\infty$)

$$\frac{\sin(\omega_1-\varepsilon_{\mathbf{p}+\mathbf{k}_1}+\varepsilon_{\mathbf{p}})t}{\omega_1-\varepsilon_{\mathbf{p}+\mathbf{k}_1}+\varepsilon_{\mathbf{p}}}\to\pi\delta(\omega_1-\varepsilon_{\mathbf{p}+\mathbf{k}_1}+\varepsilon_{\mathbf{p}}),\tag{24}$$

which is valid for $t\gg\max\{1/\omega_1,1/\mathbf{k}_1\mathbf{v}\}$; finally, it is necessary to expand the result in k_1/p, assuming $k_1/p\ll1$. Then for

$$\Phi_{\mathbf{p}}(\mathbf{r},t)=\int\Phi_{\mathbf{p},\mathbf{k}}^{1,1}(t)e^{i\mathbf{k}\mathbf{r}}dk$$

we obtain

$$d\Phi_{\mathbf{p}}/dt=\hat I_{\mathbf{p}}^{ql}\Phi_{\mathbf{p}},$$

where $\hat I_{\mathbf{p}}^{ql}$ is described by Eq. (11). In the same approximation we obtain expressions for the nondiagonal components of the density matrix,

$$\Phi_{\mathbf{p}}^{1,-1}=-\Phi_{\mathbf{p}}^{-1,1}=\frac{q^2\pi}{4\varepsilon_{\mathbf{p}}^2}\int\frac{\omega_1|E_{k_1}|^2}{k_1^2}\delta(\omega_1-\mathbf{k}_1\mathbf{v})\left(\mathbf{k}_1\frac{\partial\Phi_{\mathbf{p}}}{\partial\mathbf{p}}\right)dk_1,\tag{25}$$

which contain $\hbar$ in the numerator [in (25), $\hbar=1$] and, therefore, describe quantum effects. They must be taken into account in the calculation of the terms of the following order in q^2, i.e., the radiative effects. To describe the latter, it is necessary, as we have noted, to take into account the self-fields and the vacuum fields and to renormalize. These calculations are very lengthy, but they actually correspond to a simple procedure of expansion with respect to q^2 and averaging by means of (21) and (23), and also to the standard renormalization procedure. We shall give here only the final result (for details of the calculations, see Ref. 5, and also Refs. 2 and 3):

$$d\Phi_{\mathbf{p}}/dt=\hat I_{\mathbf{p}}^{ql}\Phi_{\mathbf{p}}+\hat I_{\mathbf{p}}^{rad}\Phi_{\mathbf{p}},\tag{26}$$

$$\begin{aligned}\hat I_{\mathbf{p}}^{rad}\Phi_{\mathbf{p}}=\pi q^2\int\frac{d\mathbf{p}'}{(2\pi)^3}\{&G_{\mathbf{p},\mathbf{p}'}\hat I_{\mathbf{p}'}^{ql}\Phi_{\mathbf{p}'}\\ &-G_{\mathbf{p}',\mathbf{p}}\hat I_{\mathbf{p}}\Phi_{\mathbf{p}}+\hat I_{\mathbf{p}}^{ql}G_{\mathbf{p}',\mathbf{p}}\Phi_{\mathbf{p}}-\hat I_{\mathbf{p}}G_{\mathbf{p},\mathbf{p}'}\Phi_{\mathbf{p}'}\},\end{aligned}\tag{27}$$

where

$$G_{\mathbf{p},\mathbf{p}'}=\frac{8[\mathbf{p},\times\mathbf{p}']^2(|\mathbf{p}-\mathbf{p}'|+\varepsilon_{\mathbf{p}})}{\varepsilon_{\mathbf{p}}|\mathbf{p}-\mathbf{p}'|^3[(|\mathbf{p}-\mathbf{p}'|+\varepsilon_{\mathbf{p}})^2-\varepsilon_{\mathbf{p}'}^2]^2}.\tag{28}$$

The radiative collision integral satisfies the particle number conservation law

$$\int\hat I_{\mathbf{p}}^{rad}\Phi_{\mathbf{p}}\frac{d\mathbf{p}}{(2\pi)^3}=0.$$

Its characteristic feature is the integral [and not differential, as in (11)] nature of the interaction of particles of different momenta. For the fast ("tail") particles with momenta greater than those that quasilinear diffusion has "reached," all the terms containing $\Phi_{\mathbf{p}}$ can be ignored, it being assumed that the number of low-energy particles greatly exceeds the number of fast particles. In addition, the last term of (27) is also usually small compared with the first ($p \gg p'$). We have

$$\frac{d\Phi_{\mathbf{p}}^{fast}}{dt} = \pi q^2 \int \frac{d\mathbf{p}'}{(2\pi)^3} \frac{[\mathbf{p},\times\mathbf{p}']^2}{p^7} \hat{I}_{\mathbf{p}'}^{ql} \Phi_{\mathbf{p}'}. \tag{29}$$

Thus, the really rigorous analytic treatment confirms the validity of the relation (7) obtained from the semiqualitative arguments. Moreover, for the isotropic case

$$\overline{[\mathbf{p},\mathbf{p}']^2/p^2p'^2} = \frac{2}{3}, \qquad G_0 = 1/12\pi^2$$

(the bar denotes averaging over the angles).

For nonrelativistic fast particles $p \gg p'$, $p \ll m$,

$$G_{\mathbf{p},\mathbf{p}'} \approx 2[\mathbf{p},\mathbf{p}']^2/m^2p^5,$$

$$\frac{d\Phi_{\mathbf{p}}^{fast}}{dt} \approx \frac{q^2}{6p^3m^2\pi^2} \int p'^2 \hat{I}_{\mathbf{p}'}^{ql} \Phi_{\mathbf{p}'} \, dp' \tag{30}$$

and, therefore, the distribution with respect to the kinetic energies $\varepsilon = p^2/2m$ is

$$\Phi_\varepsilon = \frac{pm}{2\pi^2} \Phi_{\mathbf{p}} \propto \frac{m}{p^2} \propto \frac{1}{\varepsilon}.$$

In the general case we have for $p' \ll m$ and for arbitrary relationship between p and m

$$G_{\mathbf{p},\mathbf{p}'} \approx 2[\mathbf{p},\times\mathbf{p}']^2/p^5\varepsilon_{\mathbf{p}}(p+\varepsilon_{\mathbf{p}}),$$

$$\frac{d\Phi_\varepsilon^{fast}}{dt} \approx \frac{2q^2}{3\pi\varepsilon(\varepsilon+2m)} \int \frac{p'^2 \hat{I}_{\mathbf{p}'}^{ql} \Phi_{\mathbf{p}'} \, dp'}{(2\pi)^3[\varepsilon + (\varepsilon(\varepsilon+2m))^{1/2} + m]}, \tag{31}$$

where $\varepsilon = (p^2 + m^2)^{1/2} - m$. This distribution describes the universal variation of the spectrum at $\varepsilon \approx$ m, i.e., for nuclei at an energy of 1 GeV per nucleon.

§3. MEAN CHANGE IN THE ENERGY OF PARTICLES DUE TO RADIATIVE INTERACTIONS

We determine the total energy of the particles by the relation

$$E = \int \varepsilon_{\mathbf{p}} \Phi_{\mathbf{p}} \frac{d\mathbf{p}}{(2\pi)^3}.$$

By virtue of conservation of the particle number, the change in the mean energy is proportional to the change in the total energy. The change in the energy due to the quasilinear acceleration is

$$\frac{dE^{ql}}{dt} = \int \varepsilon_{\mathbf{p}} \hat{I}_{\mathbf{p}}^{ql} \Phi_{\mathbf{p}} \frac{d\mathbf{p}}{(2\pi)^3}$$

$$= -\pi q^2 \int \frac{\omega_1 |E_{\mathbf{k}_1}|^2 d\mathbf{k}_1}{k_1^2} \delta(\omega_1 - \mathbf{k}_1\mathbf{v}) \left(\mathbf{k}_1 \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right) \frac{d\mathbf{p}}{(2\pi)^3}. \tag{32}$$

This expression will be compared with the radiative effects:

$$\frac{dE^{rad}}{dt} = \pi q^2 \int \frac{d\mathbf{p}\, d\mathbf{p}'}{(2\pi)^6}$$

$$\{(\varepsilon_{\mathbf{p}'} - \varepsilon_{\mathbf{p}}) G_{\mathbf{p}',\mathbf{p}'} \hat{I}_{\mathbf{p}}^{ql} \Phi_{\mathbf{p}} + (\varepsilon_{\mathbf{p}} \hat{I}_{\mathbf{p}}^{ql} - \varepsilon_{\mathbf{p}'} \hat{I}_{\mathbf{p}'}^{ql}) G_{\mathbf{p}',\mathbf{p}} \Phi_{\mathbf{p}}\}. \tag{33}$$

We have here replaced the integration ($\mathbf{p}' \rightleftarrows \mathbf{p}$) in all terms containing $\Phi_{\mathbf{p}'}$ to ensure that only $\Phi_{\mathbf{p}}$ occurs. The operators $\hat{I}_{\mathbf{p}}^{ql}$ and $\hat{I}_{\mathbf{p}}^{ql}$ contain two derivatives with respect to the momenta. We use the first of them, integrating by parts in (38). We obtain

$$\frac{dE^{rad}}{dt} = \pi^2 q^4 \int \frac{d\mathbf{p}\, d\mathbf{p}'}{(2\pi)^6} d\mathbf{k}_1 \frac{|E_{\mathbf{k}_1}|^2}{k_1^2} \left\{ (\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}'}) \left(\mathbf{k}_1 \frac{\partial G_{\mathbf{p}',\mathbf{p}}}{\partial \mathbf{p}}\right) \right.$$

$$\times \delta(\omega_1 - \mathbf{k}_1\mathbf{v}) \left(\mathbf{k}_1 \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right) - (\mathbf{k}_1\mathbf{v}) \left(\mathbf{k}_1 \frac{\partial G_{\mathbf{p}',\mathbf{p}}}{\partial \mathbf{p}}\right) \delta(\omega_1 - \mathbf{k}_1\mathbf{v}) \Phi_{\mathbf{p}}$$

$$\left. + (\mathbf{k}_1\mathbf{v}') \delta(\omega_1 - \mathbf{k}_1\mathbf{v}') \left(\mathbf{k}_1 \frac{\partial G_{\mathbf{p}',\mathbf{p}}}{\partial \mathbf{p}}\right) \Phi_{\mathbf{p}} \right\}. \tag{34}$$

The relation (34) already differs in structure qualitatively from (32), since it contains not only the derivative of the distribution function (which can decrease fairly rapidly in the case of quasilinear relaxation) but also the function $\Phi_{\mathbf{p}}$ itself (a further integration by parts leads already to derivatives of $\delta(\omega_1 - \mathbf{k}_1 \cdot \mathbf{v})$, indicating that the derivative of $\Phi_{\mathbf{p}}$ still occurs in the result). Such terms could occur if $\hat{I}_{\mathbf{p}}^{rad}$ were a Fokker-Planck operator and contained a friction force. However, $\hat{I}_{\mathbf{p}}^{rad}$ describes a nonlocal coupling between particles of different momenta and is not a Fokker-Planck operator (the quasilinear operator $\hat{I}_{\mathbf{p}}^{ql}$ does not contain a friction force). Therefore, the last two terms of (34) describe some effective mean friction force.

The relation (34) is convenient for calculating the change in the energy for the nonrelativistic part of the particle distribution $\Phi_{\mathbf{p}}$, i.e., for $p \ll m$. At the same time, it is possible to use the expression (31) averaged over the angles, bearing in mind that characteristically $p' \sim m$, i.e., $p' \gg p$:

$$\overline{G_{\mathbf{p}',\mathbf{p}}} = 4p^2/3p'^3\varepsilon_{\mathbf{p}'}(\mathrm{p}' + \varepsilon_{\mathbf{p}'}). \tag{35}$$

For $p \ll m$, the second and third terms of (34) are p^2/m^2 times smaller than the first. With allowance for (35) and only the first term of (34), we have

$$\frac{dE^{rad}}{dt} = -\frac{8\pi^2 q^4}{3} \int \frac{|E_{\mathbf{k}_1}|^2 m(\varepsilon_{\mathbf{p}'} - m)\delta(\omega_1 - \mathbf{k}_1\mathbf{v})}{k_1^2 \mathrm{p}'^3 \varepsilon_{\mathbf{p}'}(\mathrm{p}' + \varepsilon_{\mathbf{p}'})}$$

$$\left(\mathbf{k}_1 \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right) d\mathbf{k}_1 \omega_1 \, d\mathbf{p}\, d\mathbf{p}'/(2\pi)^6. \tag{36}$$

Here, in the first approximation in $p \ll m$ we have set $\varepsilon_{\mathbf{p}'} - \varepsilon_{\mathbf{p}} \approx \varepsilon_{\mathbf{p}'} - m, \mathbf{k}_1 \cdot \mathbf{p} \approx m\omega$. The same expression can be obtained from the first term of (33) if we set $\varepsilon_{\mathbf{p}} \approx m$. Thus, the difference between the first terms of (33) and (34) has the relative order p^2/m^2, i.e., for $p \ll m$ it is small. On the transformation of (33) into (34) some of the first term is transferred to the second and third terms of (34). They both have the order p^2/m^2 relative to the first term.

Integrating in (36) over $\mathbf{p}'$, we obtain

$$\frac{dE^{rad}}{dt} = -\frac{8}{3}\left(\ln 2 - \frac{1}{2}\right) q^4 \int \frac{|E_{\mathbf{k}_1}|^2}{k_1^2} \omega_1 \delta(\omega_1 - \mathbf{k}_1\mathbf{v})$$

$$\left(\mathbf{k}_1 \frac{\partial \Phi_{\mathbf{p}}}{\partial \mathbf{p}}\right) \frac{d\mathbf{p}\, d\mathbf{k}_1}{(2\pi)^3}. \tag{37}$$

Comparison of the radiative effects described by (37) with the quasilinear effects described by (32) shows that the result differs by a numerical factor ($\hbar\neq 1, c\neq 1$)

$$\frac{8}{3\pi}\left(\ln 2-\frac{1}{2}\right)\frac{q^2}{\hbar c}\approx 0{,}164\frac{q^2}{\hbar c}\approx 1{,}2\cdot 10^{-3}Z^2. \qquad (38)$$

The last number is written down for $q = Ze$ (where e is the electron charge), $e^2/\hbar c \approx 1/137$.

It is appropriate to recall that the quasilinear change in the energy is an inversion of Landau damping. Therefore the radiative corrections γ^{rad} to the Landau damping γ^L are

$$\gamma^L+\gamma^{rad}=\gamma^L\left[1+\frac{8}{3\pi}\left(\ln 2-\frac{1}{2}\right)\frac{q^2}{\hbar c}\right]. \qquad (39)$$

The second and third terms of (34), like the corrections p^2/m^2 to the first term (which have the same order), diverge logarithmically at small p' if the approximate expression (35), which is valid for $p'\gg p$, is used. If we are interested in these corrections with logarithmic accuracy, we can truncate the integration over p' at $p'\approx p$.

It should here be pointed out that $G_{\mathbf{p},\mathbf{p}'}$ has a singularity as $\mathbf{p}'\to\mathbf{p}$, and therefore the integration at $p'\sim p$ must in general take into account all the terms of Eq. (27), which does not have a singularity as $\mathbf{p}'\to\mathbf{p}$.

The correctness of this last remark can be seen by replacing $\mathbf{p}'$ by $\mathbf{p}+\mathbf{k}$ and using for $k\ll p$ the expansion

$$G_{\mathbf{p},\mathbf{p}+\mathbf{k}}\approx\left(1+\frac{1}{2}\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)G_{\mathbf{p}}^{(0)}(\mathbf{k}),$$

$$G_{\mathbf{p}+\mathbf{k},\mathbf{p}}\approx\left(1+\frac{1}{2}\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)G_{\mathbf{p}}^{(0)}(-\mathbf{k});$$

$$G_{\mathbf{p}}^{(0)}(\mathbf{k})=\frac{2[\mathbf{k},\times\mathbf{p}]^2}{k^3\varepsilon_p^2(k-\mathbf{k}\mathbf{p}/\varepsilon_p)^2}.$$

Equation (27) then reduces to a form corresponding to the classical result of Ref. 1, which does not contain a divergence as $\mathbf{k}\to 0$ ($\mathbf{p}'\to\mathbf{p}$):

$$I_{\mathbf{p}}^{rad}\Phi_{\mathbf{p}}=-\frac{q^4}{8\pi}\int\frac{|E_{k_1}|^2}{k_1^2}d\mathbf{k}_1\,d\mathbf{k}\left\{\left(\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega_1-\mathbf{k}_1\mathbf{v})\right.$$
$$\times\left(\mathbf{k}_1\frac{\partial G_{\mathbf{p}}^{(0)}(\mathbf{k})}{\partial\mathbf{p}}\right)\left(\mathbf{k}_1\frac{\partial\Phi_{\mathbf{p}}}{\partial\mathbf{p}}\right)$$
$$+\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega_1-\mathbf{k}_1\mathbf{v})\left(\mathbf{k}_1\frac{\partial G_{\mathbf{p}}^{(0)}(\mathbf{k})}{\partial\mathbf{p}}\right)$$
$$\times\left(\mathbf{k}\frac{\partial\Phi_{\mathbf{p}}}{\partial\mathbf{p}}\right)-\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\left(\mathbf{k}\frac{\partial\delta(\omega_1-\mathbf{k}_1\mathbf{v})}{\partial\mathbf{p}}\right)G_{\mathbf{p}}^{(0)}(\mathbf{k})\left(\mathbf{k}_1\frac{\partial\Phi_{\mathbf{p}}}{\partial\mathbf{p}}\right)$$
$$\left.+\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\delta(\omega_1-\mathbf{k}_1\mathbf{v})\Phi_{\mathbf{p}}\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)\left(\mathbf{k}_1\frac{\partial}{\partial\mathbf{p}}\right)G_{\mathbf{p}}^{(0)}(\mathbf{k})\right\}.$$

This result together with (39) shows, among other things, that Eq. (27), and also the expressions (33) and (34) for the mean change in the energy do not contain either infrared ($k\to 0$) or ultraviolet ($k\to\infty$ or $p'\to\infty$) divergences.

Note also that (33) is effectively identical to the expression for the change in the mean energy of the fast particles, which are described in the limiting cases $p\gg m$ and $p\ll m$ by (29) and (30), and in the general case by (31). Thus, for the considered case of nonrelativistic background particles the radiative corrections are entirely due to the generation of fast particles. The analysis shows that a relatively small transfer of turbulence energy is associated with the generation of fast particles; the main expenditure of turbulence energy is on the quasilinear heating of the main background particles.

Equations (26), (27), and (32)–(34) describe the general case of arbitrary distributions. We have here restricted ourselves to the most real case, when the number of background (almost thermal, and in a number of cases for low-frequency oscillations simply thermal) particles greatly exceeds the number of fast particles, i.e., the case having direct bearing on the problem of fast (cosmic ray) particles in cosmic plasmas.

§4. COMPARISON OF THEORETICAL SPECTRA WITH OBSERVED COSMIC RAY SPECTRA

For the comparison, we shall consider the most realistic picture of inhomogeneous unsteady turbulence with intermittent regions of turbulence in space and in time[3] (the model of so-called spike turbulence). It is important that in each bounded region or during the time of each intermittent burst the same spectra (31) are generated. Therefore, the particles produced in different spatial regions and at different times are simply superimposed on one another (if, of course, the losses in the intervals are small).

We discuss the question of the extent to which the resulting spectra are universal. In the general case, Eqs. (29)–(31) give only the source of fast particles, but there may also be losses. For $\varepsilon\gg m$, for example, we obtain

$$\frac{\partial\Phi_\varepsilon}{\partial t}+\mathbf{v}\frac{\partial\Phi_\varepsilon}{\partial\mathbf{r}}=\frac{Q}{\varepsilon^3}+\frac{\partial}{\partial\varepsilon}P(\varepsilon)\Phi_\varepsilon, \qquad (40)$$

where $P(\varepsilon)$ is the power of the loss of the fast particles, and the term $\mathbf{v}\partial\Phi_\varepsilon/\partial\mathbf{r}$ describes the convective transport of the particles out of the acceleration region. We discuss two questions: 1) when losses are important in the forming of the spectrum, 2) the extent to which the sources are universal.

The answer to the first question differs from the traditional answer for the mechanism we are discussing. The source formation time is determined by the attainment of the asymptotic behavior (24) after the turbulence has been "switched on," i.e., by a term of order $1/\omega_1$. The higher the characteristic frequency of the turbulence, the smaller the part played by the losses. However, in cosmic plasmas the losses in the acceleration process are usually not very important even for very small ω_1. This is also indicated by the estimates for the known loss mechanism.[3,7] In any case, it is easy to satisfy the conditions of small losses for the generation of even the particles of the highest energies.[3] When accelerated high-energy particles pass through a certain thickness of substance in the source, cascade showers must naturally arise, and this may limit the contribution to the high-energy particles from sufficiently dense turbulent regions of space. However, the question of the particles of the highest energies in cosmic rays is now posed in a quite different framework, since their generation by close sources is not ruled out.

The second question concerns the universality of the source Q/ε^3 giving a spectrum $1/\varepsilon^\gamma$ with $\gamma=3$. It was already noted above that the integral (36) depends on the total integral of the correlation function of the fields, i.e., a spectrum with $\gamma = 3$ is obtained for any correlation function with width greater than or less than the characteristic pulsation frequency or, in other words, in both the case of weak and strong turbulence. It does not depend on the type of the turbulent pulsations. Although (36) is written down for the case of electrostatic oscillations, generalization of the result to any oscillations of another type (not electrostatic) is trivial, since (36) simply contains a different operator $\hat{I}^{ql}$ by the operator describing the process of induced scattering. Similarly, an external magnetic field has little effect on the result for $\hbar\omega_H \ll mc^2$. Finally, comparison of (29) and (30) with the results of Ref. 3 for spin $\frac{1}{2}$ particles shows that the basic characteristics of the spectrum do not depend on the spin of the particles (only the numerical coefficient of order unity is somewhat changed, and then only for $\varepsilon \gg mc^2$).

The emergence of the particles from the source can slightly change the spectrum through the excitation of Alfvén waves and the diffusion process. However, for cosmic-ray electrons the excitation of Alfvén waves is difficult. Therefore, their spectrum must be close to $\gamma = 3$, whereas for protons and nuclei $2 < \gamma < 3$ possibly (see Ref. 8). These theoretical ideas agree well with the observations. First, the spectrum of cosmic-ray electrons is close to $1/\varepsilon^3$. For the relativistic cosmic-ray nuclei the spectra are almost independent of Z, the atomic number of the nucleus, in accordance with the theoretical results.

Thus, we have found an explanation for the two most mysterious properties of the observed cosmic-ray spectrum—the constancy of γ in a very wide range of energies and the Z-independence of γ.

Further, one must expect changes in the spectrum of the nuclei at the energies at which the escape of the nuclei from the sources is significant. For the escape of cosmic-ray nuclei from the Galaxy, the energy is 10^{15}–10^{17} eV, at which certain changes in the spectrum are in fact observed.

Another characteristic property of the theoretical spectrum is that for all nuclei a characteristic change in the spectrum [see (31) and (36)] occurs at $\varepsilon \sim mc^2$, i.e., at the same energy per nucleon (1 GeV per nucleon). This is also in agreement with the observations.

However, these are not all the possibilities of theoretical explanations of the observations. One of the most important characteristics is the chemical composition of cosmic rays, i.e., in (40) the distribution of Q with respect to Z. The fragmentation of nuclei and the change in the chemical composition as the cosmic rays travel from the sources is a well-studied problem.[7] This however cannot explain the relatively large (large by almost two orders of magnitude) number of nuclei in cosmic rays with $Z > 10$. According to the latest data,[9] the abundance of nuclei in cosmic rays for $Z > 10$ is in general close to the abundance of the elements in cosmic space (with some not very significant deviations for some isotopes). But the ratio of the number of protons to the number of nuclei with $Z > 10$ is about 10^{-2}.

This can be partly understood by means of (36) if it is assumed that the acceleration takes place in a plasma of sufficiently high temperatures, so that nuclei with relatively small Z are fully ionized. Then

$$q = Ze; \qquad p'^2 \hat{I}_p^{ql} \sim \frac{1}{v_T} \approx \left(\frac{m}{T}\right)^{1/2} \propto A^{1/2} \propto Z^{1/2},$$

$$\int_\varepsilon^\infty \Phi_\varepsilon \, d\varepsilon \propto \frac{q^4 Z^{1/2}}{A^2} \propto Z^{5/2},$$

and for $Z \sim 10$ this gives $\sim 3 \times 10^2$. We note that the theory is not valid for $Z^2 > \hbar c/q^2$ and in accordance with (39) it is also invalid at least for $Z^2 > 10^3$ or $Z > 30$.

An important characteristic is the relative number of electrons and protons in the cosmic rays. The observations indicate that the number of cosmic-ray electrons with energy greater than 1 GeV is approximately two orders of magnitude less than the number of cosmic-ray protons. It is necessary to consider what theory gives for particles of different masses but $Z = 1$. In accordance with (30), the number of particles with $\varepsilon \sim mc^2$ is $1/mv_T \propto 1/m^{3/2}$, since $\int p'^2 \hat{I}_{p'}^{ql} \Phi_{p'} dp' \propto 1/v_T \propto m^{1/2}$, $\delta(\omega_1 - \mathbf{k}\cdot\mathbf{v}) \sim 1/k_1 v \sim 1/k_1 v_1$.

For spin $\frac{1}{2}$ particles (electrons and protons) it is, naturally, necessary to use the results of Ref. 3. However, as we have noted, the corresponding expressions differ from (30) only by numerical factors of order unity, and therefore for estimates we can here use the expression (30). If we compare the number of electrons with $\varepsilon \approx m_e c^2$ with the number of protons with $\varepsilon \approx m_p c^2$, there will be more electrons by the factor $(m_p/m_e)^{3/2}$. But in accordance with (29) the number of electrons with $\varepsilon \gg m_p c^2$ is $(m_e/m_p)^2$ times less than the number of electrons with $\varepsilon \gtrsim m_e c^2$, i.e., the theory indicates that the number of electrons with $\varepsilon \gtrsim m_p c^2 \sim 1$ GeV must be $(m_e/m_p)^{1/2} \approx 3 \times 10^{-2}$ times less than the number of protons, and this is somewhat greater (by 1.5 times) than the observed number. It is however clear that our estimate cannot claim high accuracy. In addition, it is well known that for electrons radiative losses of various kind are more important.

These qualitative and quantitative agreements between the theory and observations make it possible to reexamine the general question of the origin of the cosmic rays and, specifically, on the basis of theory find the parameters of the cosmic plasma of the sources (density, temperature, etc.) that can ensure the observed characteristics of the cosmic rays at the Earth, due allowance being made, of course, for the effects associated with their propagation. For high-energy particles this gives us a requirement on the density; for the chemical composition, a requirement on the degree of ionization (i.e., the temperature) and so forth. This program is also of interest for particles of maximal energies in cosmic rays.

Also of interest is the question of new effects that can be predicted by the theory. The first and most important of them is that the main component of the cosmic rays in the interstellar medium may be electrons and not protons and nuclei. The maximum in the electron spectrum, at ~ 1 MeV, cannot be observed near the Earth due to modulations by the solar wind. However, these electrons may be the most significant as regards the number of particles in the interstellar cosmic medium outside the solar system. True, this conclusion is obtained without allowance for the energy loss of the electrons on ionization, emission, and so forth, i.e., in the regions of their generation (acceleration). With allowance for the losses, the energy in the electrons may be of the same order or even less than the energy of the protons.

Another conclusion is that the nonrelativistic component of the cosmic rays, both electrons and ions, may be the main fraction of the cosmic rays (they are sometimes called subcosmic rays). Indeed, for a distribution $\sim 1/\varepsilon$ for $\varepsilon \ll mc^2$ the total number of subcosmic rays (i.e., particles with $\varepsilon < mc^2$) will be in order of magnitude $\ln(c^2/v_T^2)$ times greater than the total number of cosmic rays (i.e., particles with $\varepsilon > mc^2$) of the given species (i.e., with the same q and m). For the electrons, this estimate is not always true. If, for example, ionization losses are important, $P(\varepsilon) \propto \varepsilon^{-1/2}$, then for the source Q/ε the spectrum ε^{-1} is replaced by $\varepsilon^{1/2}$, i.e., $\ln(c^2/v_T^2)$, which gives an estimate of the relative numbers of nonrelativistic and relativistic electrons, is changed by $\ln(c^2/v_{\min}^2)$, where $v_{\min}$ is the electron velocity below which ionization losses are important (it is assumed that $v_{\min} \ll c$). The large number of electrons leads, for example, to strong heating of the interstellar medium.[10]

The energy in the cosmic-ray electrons must be $(m_p/m_e)^{1/2}$ times greater than in the ions if, of course, the ionization losses for relativistic electrons do not have an effect for $\varepsilon > m_e c^2$. Let $\varepsilon_{cr} > m_e c^2$ be the value of the electron energy below which the ionization losses are important; then for $\varepsilon < \varepsilon_{cr}$ the spectrum becomes ε^{-2}. Extrapolating this electron spctrum to $\varepsilon \sim m_e c^2$, we obtain a ratio of the energy of the cosmic-ray electrons to the energy of the ions that is smaller than $(m_p/m_e)^{1/2}$. If $\varepsilon_{cr} \sim 100$ keV, then the electron energy will be of approximately the same order as that of the ions. But this is a consequence of the losses (in the given example, on ionization and heating of the medium). The sources are such that they will transfer to the cosmic-ray electrons $(m_p/m_e)^{1/2}$ times more energy. This energy then goes on heating of the medium through ionization losses. The formation of the electron spectrum can also be affected by reabsorption, which leads to the creation of a turbulent plasma reactor.[11]

Thus, the energy of the cosmic-ray electrons (at the time of their generation) is approximately $(m_p/m_e)^{1/2}$ times greater than the energy of the ions. In turn, the energy transferred to the cosmic-ray electrons is 1.2×10^{-4} times less than the energy transferred to the background (thermal) electrons, i.e., the energy expended on heating the electrons in the sources (turbulent regions).

With regard to the well-known general energy estimates[7] leading to the conclusions that the cosmic rays are generated in the envelopes of supernovae or active stars, they remain as before. In these regions there is merely generated a more intense turbulence, leading to a greater contribution of the corresponding region to the generation of the cosmic rays. Essentially new is the result that low-activity regions can also make a contribution, since the spectrum is universal, and the superposition of all the existing sources determines only the total concentration of the cosmic rays. Many weak sources can give the same effect as a small number of strong ones. A more careful analysis of the number of weak sources is therefore needed. Shock waves can also be sources of cosmic rays, since turbulence develops at their shock fronts, but Fermi acceleration of the first kind at a shock front cannot ensure the common power-law spectrum in the observed wide range of energies and therefore cannot be the main mechanism of generation of cosmic rays. For sufficiently low (above all nonrelativistic) energies, quasilinear acceleration is predominant, and this gives distributions of exponential type: $f_\varepsilon \propto \varepsilon^{-\alpha} \exp(-\varepsilon^{\beta})$. This explains why exponential rather than power-law spectra are observed in weak flares on the Sun.

The effects of the generation of the fast electrons are naturally important only only for cosmic but also laboratory plasmas. They must be a necessary concomitant of resonance microwave heating of plasmas in facilities with magnetic confinement. If they are not confined in the system, then their flux to the walls must be accompanied by the appearance of impurities. The energy loss in the fast-electron channel may be comparable with the energy lost by bremsstrahlung.

Thus, we formulate the main conclusions.

The first is that the very simple expression (31) provides an explanation for five independent observations: 1) the universality of the spectrum and the fact that it does not depend on the particle species or the nature of the turbulence; 2) the power-law nature of the spectrum with exponent close to the observed value; 3) the change in the spectrum of all nuclei at energies of 1 GeV per nucleon; 4) the chemical composition of the cosmic rays and the predominant abundance of heavy nuclei with $Z \geqslant 10$; 5) the deviation of the number of cosmic-ray electrons from the number of ions at energies greater than 1 GeV.

The second is the prediction that the main fraction of the energy in the sources is transferred to cosmic-ray electrons (about 40 times more energy is transferred to the electrons than to the ions), which then in general is expended partly on heating and ionizing the medium. The energy transferred to the cosmic-ray electrons is approximately 10^3 times less than the energy transferred to heat and ionization. These estimates can give more reliable ideas about the energy balance of the cosmic sources.

The third is the prediction that the cosmic rays of the highest energies can be generated in fairly near and, possibly, galactic sources. The generation of cosmic rays is a general physical phenomenon, and sources with different power must change only the number of particles but not their spectrum. In this sense, all solar flares must be accompanied by the generation of fast ions and give radiation in γ lines.

The fourth conclusion is the possibility of a laboratory test of the considered mechanism of generation of a universal spectrum of fast particles. This is most readily done for electrons. The efficiency of such a quantum accelerator, as it may be called, decreases rapidly with increasing energy. Thus, for the plasma-beam intereaction the number of relativistic electrons with energy greater than $\varepsilon_0 \gg mc^2$ estimated by means of (29) is

$$\frac{n_{fast}}{n_b} \approx 10^{-3} \frac{v_b^2}{c^2} \frac{\tau}{\tau_{ql}} \left(\frac{mc^2}{\varepsilon_0}\right)^2, \quad n_{fast} = \int_{\varepsilon_0}^{\infty} \Phi_\varepsilon^{fast} d\varepsilon,$$

where τ_{ql} is the quasilinear time, v_b is the velocity, and n_b is the density of the beam. For $v_0^2/c^2 \simeq 0.1$, $(mc^2/\varepsilon_0)^2 \sim 0.1$, and $\tau \sim \tau_{ql} \sim \omega_{pe} n_b / n_0$ we have $n_{\text{fast}}/n_b \sim 10^{-5}$, and already for $\varepsilon_0 \sim 0.01$ GeV when $(mc^2/\varepsilon_0)^2 \sim 10^{-4}$ we have $n_{\text{fast}}/n_b \sim 10^{-8} \tau/\tau_{ql}$. The maximal $\tau \sim l/c$ is determined by the flight time (if there is no confinement). Therefore, observation of the spectrum of relativistic electrons in a restricted energy interval in a sufficiently large volume of turbulent plasma is an experimental task that is not without hope.

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