Theoretical description of elastic reflection of particles (photons) incident at grazing angles without the use of the diffusion approximation

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The process of elastic reflection of an external beam striking the boundary of matter at a grazing angle is considered. An integral equation for the direct determination of the reflection function is derived on the basis of the Boltzmann transport equation and then solved. A simple analytic expression describing the angular spectrum of the reflected radiation is found for the various laws of single scattering. It is shown that for any law the most probable angle in the reflection spectrum is equal to the grazing angle. Analysis of the solution obtained shows that, if the cross section for single scattering decreases with increasing scattering angle ϑ more slowly than ϑ^{-4}, then the small-angle reflection process cannot be described in the diffusion approximation. If, on the other hand, the single-scattering cross section decreases with increasing ϑ faster than the Rutherford cross section, then the diffusion approximation is applicable, and the angular spectrum of the reflected radiation can be computed with the aid of the Firsov formula.

1. INTRODUCTION

A large number of different experiments have been performed in the last few years to study the laws governing the reflection of ions,[1–3] electrons,[4–6] and protons[7] from thick material layers in the case when the angle ζ_0 between the velocity of the incident particles and the surface of the scatterer is sufficiently small and the scattering of the particles by the atoms of the medium is highly anisotropic: $\vartheta_{\text{eff}} \ll 1$ (ϑ_{eff} is the effective single-scattering angle). The value of the grazing angle ζ_0 was varied within wide limits from one degree to several score degrees; the particle energy, from several keV to several MeV. In each of these experiments, without exception, there was found in the angular spectrum of the reflected radiation a sharp anisotropy that bears no resemblance to the well-known cosine law: there was observed a strongly pronounced peak at reflection angles close to the grazing angle (the so-called law of specular reflection). The smaller the energy lost by the particles inside the material, i.e., the greater the contribution made by the purely elastic scattering to the reflected beam, is, the more distinctly the law of specular reflection manifests itself.

The violation of the cosine law indicates that the majority of the particles are reflected back from the material before the angular distribution becomes isotropic. At the same time, the occurrence of a sharp peak in the reflected-radiation spectrum cannot be explained within the framework of the single- or low-multiplicity-scattering model.[8] This means that, in the case in question, the reflected beam is produced largely as a result of essentially multiple scattering.

The computation of the spectra of reflected radiation is an importtant particular case of the general problem of transport theory underlying which is the Boltzmann transport equation. The same equation describes the process of photon propagation in turbid media (dust, fog, aerosol), when the waves scattered by the individual optical inhomogeneities of the medium are completely incoherent, so that their phases are not connnected by stable relations.[9,10] Therefore, below we shall, by the term "reflected radiation" mean any kind of radiation, be it a beam of reflected charged particles or a beam of reflected photons.

Since the Boltzmann transport equation was formulated many attempts have been made to solve it for different kinds of incident radiation and different scattering-medium geometries. Thus far, it has been possible to carry out the fullest investigation for the process of radiation passage through relatively thin layers of matter, when the reflection of the particles (photons) can be entirely ignored. But the situation is highly complicated in the case of the solution of the albedo problems of transport theory, when the particle (light) flux crosses one and the same boundary of the material twice: on entering, and on going out of, the medium. Because of the great difficulties encountered in the solution of such problems, it was rarely possible to obtain a closed analytic expression for the computation of the reflected-radiation spectra. And in those rare cases where it was possible to do this, the results obtained became classical: both the results themselves and the method used to obtain them found numerous applications in different areas of physics and mathematics. An example of this is the classical problem of the computation of the spectrum of neutrons reflected from a semi-infinite scatterer, when the scattering of the neutrons by the nuclei is assumed to be isotropic. Even the solution of this important problem, which is, at the same time, one of the simplest albedo problems of transport theory, required the development of a fundamentally new method—the Wiener-Hopf method—of solving a certain class of integral equations. This method can also be used in those cases when the scattering by the individual centers is nearly isotropic, and we can limit ourselves in the expansion of the scattering cross section in a series in terms of the Legendre polynomials to the consideration of two or three terms of the series.

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But if the wavelength of the incident radiation is much smaller than the dimensions of the scattering centers (i.e., if $\lambda\!\!\!^{-} \ll a$), then the scattering is highly anisotropic (in the case of light it is also essential that the relative refractive index of the large-sized scattering particles be close to unity: $n_{\rm rel} \approx 1$). As the degree of anisotropy of the single scattering increases (and it is precisely this case that is of interest to us), the reflection-spectrum calculations become so complicated that it was for a long time thought that the analytic solution of such problems was practically impossible without the use of a computer. But in 1966 Firsov demonstrated for the first time that, in precisely the case of grazing angles of incidence of external radiation on the surface of a scatterer, it is possible to compute the reflection spectra by analytic methods entirely different from the Wiener-Hopf method.[11] Indeed, analysis of the experimental data shows that, in the case of grazing angles of incidence, the majority of the particles (photons) are reflected from the material at small angles to its surface ($|\zeta| \ll 1$), so that the total multiple-scattering angle $\theta_{\rm scat} \sim \zeta_0 + |\zeta|$ is also small. Thus, fortunately, we can use the small-angle approximation to theoretically describe the reflection process.

We should note at once the great difference that exists between the small-angle reflection theory and the small-angle transmission theory. In the theory of small-angle transmission of particles, the reflection is entirely neglected, and therefore the angular distribution of the radiation at the boundary of the material is assumed to be known in the entire angle range. As to the reflection theory, in this case the angular spectrum of only the incident flux is assumed to be known at the boundary of the material: the angular distribution of the radiation reflected from the medium should be determined in the course of the solution of the problem. But if the reflection occurs in the case of small-angle scattering, then it is possible to compute the reflection spectrum analytically without the use of any numerical methods. The last circumstance is especially important, since the analytic expression for the reflection spectrum explicitly contains the radiation-scattering center interaction parameters. For example, we can, without difficulty, follow how the reflected-radiation spectrum depends on the specific law governing the scattering occurring in one collision. It can also be used to check numerical computations in a broader range of problems of the theory of reflection for arbitrary angles of incidence.

To date the theoretical description of the process of small-angle reflection of charged particles and ions of high and intermediate energies, when the cross section for scattering by the atoms of the medium is highly anisotropic, has been carried out largely within the framework of the diffusion approximation in terms of the angle variables.[11–19] The same approximation has been used to compute the angular spectra of photons.[20] Computations of the angular spectra of reflected radiation in the diffusion approximation are distinguished by relative simplicity, but the region of applicability of the approximation is narrow. It is well known that the diffusion approximation fairly adequately describes the process of elastic scattering only in those cases when the scattered-radiation spectrum is formed largely as a result of a large number of collisions in each of which deflection through small angles occurs.[21]

For example, this approximation furnishes a good description of the scattering of fast heavy particles, i.e., particles whose mass m is much greater than the electron mass m_e ($m \gg m_e$) and whose de Broglie wavelength $\lambda\!\!\!^{-}$ is much smaller than the dimensions r_n of the atomic nuclei of the medium ($\lambda\!\!\!^{-} \ll r_n$). Such particles are practically scattered in one collision into the narrow angle range $0 \leqslant \vartheta \lesssim (\lambda\!\!\!^{-}/r_n) \ll 1$ (Ref. 22). Therefore, if the glancing angle $\zeta_0 \gg \lambda\!\!\!^{-}/r_n$, then we can assert with confidence that only those particles can be reflected from the material which have actually undergone essentially multiple scattering inside the medium, more specifically, only those particles which have interacted with the atoms of the material not less than $\zeta_0/(\lambda\!\!\!^{-}/r_n) \gg 1$ times, and have, in each of these collisions, been scattered through a small (in comparison with ζ_0) angle.

The situation can be quite different for particles of lower energy or smaller mass (ions of intermediate energies, electrons) and also for photons, since the limitation on the maximum value of the single-scattering angle disappears. It becomes possible for the particles (photons) to be deflected in individual collisions through angles comparable in magnitude to the total observed scattering angle: $\vartheta \sim \zeta_0 + |\zeta|$. The relative contribution from such collisions to the reflected flux depends both on the relation between the total scattering angle $\zeta_0 + |\zeta|$ and the angle $\vartheta_{\rm eff}$ and on the specific angular dependence of the single-scattering cross section.

If $\zeta_0 + |\zeta| \sim \vartheta_{\rm eff}$, then an appreciable contribution to the reflected flux will be made by the low-multiplicity collision processes, so that the inapplicability of the diffusion approximation is evident. But even in the case when $\zeta_0 + |\zeta| \gg \vartheta_{\rm eff}$ and the reflected flux is formed largely as the result of essentially multiple scattering, the possibility of describing the reflection process in the diffusion approximation is far from being apparent: everything depends on how rapidly the single-scattering cross section decreases as the angle ϑ increases.[23]

In the present paper we investigate the problem of small-angle reflection of charged particles (photons) in the general case without the use of the diffusion approximation. Formally, this means that, to compute the reflected radiation spectrum, we use the Boltzmann transport equation with a collision integral in the general form, and not in the differential form. We obtain a simple analytic expression for the spectrum of elastically reflected charged particles, ions, or photons, when the probability for single scattering through an angle $\vartheta \ll 1$ is given by the following two-parameter expression:

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega} \equiv I(\vartheta) = \frac{1}{\pi}\frac{\nu\vartheta_{\rm eff}^{2\nu}}{[\vartheta_{\rm eff}^2 + \vartheta^2]^{1+\nu}}. \tag{1.1}$$

Here $I(\vartheta)$ is the scattering indicatrix and ν is a parameter determining the rapidity of decrease of the scattering probability as the angle ϑ increases. The second parameter $\vartheta_{\rm eff}$ determines the degree of anisotropy of the scattering. The value of $\vartheta_{\rm eff}$ depends on the law governing the interaction of the particles being scattered in the case of large impact parameters. The quantity $I(\vartheta)$ is normalized by the condition

$$2\pi\int_0^\infty I(\vartheta)\vartheta\, d\vartheta=1.$$

The expression (1.1) can be used to describe the scattering process in a large number of important cases involving different types of interaction between the radiation and the scattering centers. For example, if we set $\nu = 1$ in (1.1), we obtain an expression for the probability for small-angle scattering of fast charged particles (i.e., electrons).[21] In this case $\vartheta_{\text{eff}} \sim \lambda\!\!\!^{-}/r_{\text{at}}$, where r_{at} is the mean radius of the atoms of the material of the scatterer. If we set $\vartheta_{\text{eff}} = 0$ in the denominator of (1.1), then we arrive at the well-known expression for the probability for small-angle scattering in an inverse-power potential field[1] $U(r) \propto r^{-\beta}$, where $\beta = 1/\nu$:

$$d\sigma_\beta(\vartheta) \sim \vartheta^{-2(1+1/\beta)}\vartheta d\vartheta.$$

The case $\nu = 1/2$ is of special interest in connection with both the study of the scattering of ions of intermediate energies ($U(r) \propto r^{-2}$ is the Firsov potential[1,24]) and the study of light scattering in turbid media (the Heiney-Greenstein indicatrix[9]). In the latter case $\vartheta_{\text{eff}} \sim \lambda\!\!\!^{-}_0/a$. The expression (1.1) with $\nu = 1/3$ is used to describe the scattering of protons with an energy of several keV.[24] Finally, the case $\nu = 5/6$ corresponds to the indicatrix for photon scattering in a turbulent continuous medium (the Kolmogorov-Obukhov scattering law[10]).

2. EQUATION FOR THE REFLECTION FUNCTION

Let a broad beam of particles (photons) be incident on the plane surface of a homogeneous semi-infinite material layer at a small angle ζ_0 to the surface: $\zeta_0 \ll 1$. Let the xy plane coincide with the surface of the medium, and let the z axis be directed into the interior of the material. The direction of propagation of the radiation is determined by the angles ζ and φ, where ζ is the angle between the velocity and the boundary of the material and φ is the azimuthal angle. Since $\zeta = \pi/2 - \theta$ (θ is the angle between the direction of propagation of the radiation and the z axis), below we shall call ζ the polar angle. To the particles moving into the interior of the medium correspond the values of $\zeta > 0$, while to the particles moving toward the boundary correspond negative values of ζ: $\zeta = -|\zeta| < 0$. We shall assume that the velocity of the incident particles is parallel to the xz plane, so that the initial azimuth $\varphi_0 = 0$.

In the case of grazing angles, when the deflection from the initial direction of motion is small, i.e., for

$$\cos\theta=\sin\zeta\approx\zeta, \quad (\zeta-\zeta_0)^2+\varphi^2\ll 1, \tag{2.1}$$

the transport equation for the radiation flux density $N(\tau, \zeta, \varphi)$ has the form

$$\zeta\frac{\partial N}{\partial\tau}=\iint_{-\infty}^{\infty} d\varphi'\, d\zeta'\, I(\vartheta'')\{N(\tau;\zeta',\varphi')-N(\tau,\zeta,\varphi)\}. \tag{2.2}$$

Here $\tau = z/l$ is the depth, measured in units of the mean free path l; $I(\vartheta'')$ is the probability for scattering in one collision from the state (ζ', φ') into the state (ζ, φ). For grazing angles of motion

$$(\vartheta'')^2\approx(\zeta'-\zeta)^2+(\varphi'-\varphi)^2. \tag{2.3}$$

If the intensity of the incident flux is equal to unity, then the boundary condition at the surface of the material has the form

$$N(\tau=0;\ \zeta>0,\ \varphi)=\delta(\zeta-\zeta_0)\delta(\varphi). \tag{2.4}$$

The angular distribution of the reflected radiation is determined by the reflection function $S(|\zeta|, \varphi; \zeta_0)$, which is connected with the flux density at the boundary of the medium by the relation

$$S(|\zeta|,\varphi;\zeta_0)=|\zeta|N(\tau=0;\ \zeta=-|\zeta|;\ \varphi). \tag{2.5}$$

For such a definition $Sd|\zeta|d\varphi$ is the number of particles flying back out of the material from a unit area of its surface in unit time in the angle ranges from $|\zeta|$ to $|\zeta| + d|\zeta|$ and φ to $\varphi + d\varphi$. Thus, $S(|\zeta|, \varphi; \zeta_0)$ can be directly measured in backscattering experiments. Therefore, the computation of this quantity is the primary problem of the theory of reflection.

There exist different methods of deriving the equation for the reflection function from the more general Boltzmann transport equation, which determines not only the angular, but also the spatial distribution of the radiation both on the surface and inside the scattering medium. For example, using the method of "invariant inclusion," we can derive for S in the case of reflection from a scatterer of infinite thickness a closed equation, which, however, turns out to be nonlinear.[21] The solution to this equation has been found for the case of isotropic s scattering, and is expressed in terms of the Chandrasekhar functions.[9]

In the opposite case, when the scattering is highly anisotropic (i.e., when $\vartheta_{\text{eff}} \ll 1$), which is the case of interest to us here, the invariant-inclusion method turns out to be ineffective. Therefore, to solve the problem in question we use in the present paper another approach, based on the expansion of the spatial-angular distribution of the radiation in terms of the angular eigenfunctions of the homogeneous Boltzmann equation. The method of eigenfunctions allows us obtain quite easily an equation for the direct determination of the reflection function. It is important to emphasize that the thus obtained equation for S is linear.

Since the transport equation (2.2) allows the separation of the variables, its general solution, bounded with respect to the depth, can be represented in the form

$$N(\tau,\zeta,\varphi)=\int_0^\infty \lambda C(\lambda)\exp(-\lambda^3\tau)\,\Phi_\lambda(\zeta,\varphi)\,d\lambda. \tag{2.6}$$

Here the $\Phi_\lambda(\zeta, \varphi)$ are the angular eigenfunctions of the problem under consideration and the λ are the eigenvalues. The equation for the angular functions has the form

$$\zeta\lambda^3\Phi_\lambda(\zeta,\varphi)=\int_{-\infty}^{\infty} d\zeta'\, d\varphi'\, I[(\zeta'-\zeta)^2+(\varphi'-\varphi)^2] \times\{\Phi_\lambda(\zeta,\varphi)-\Phi_\lambda(\zeta',\varphi')\}. \tag{2.7}$$

In order to obtain the equation for the direct determination of the reflection function S, let us note that, at any depth $\tau \geqslant 0$, the flux of the particles moving toward the boundary (i.e., for which the angles $\zeta = -|\zeta| < 0$) can be represented in the form of a linear functional of the flux of the radiation propagating into the interior of the material. Since the scattering medium occupies the entire $0 \leqslant \tau < \infty$ region, and is homogeneous, the kernel of this functional does not depend on the depth τ, and is precisely the reflection function:

$$|\zeta| N(\tau; -|\zeta|; \varphi) = \int_{-\infty}^{\infty} d\varphi'' \int_{0}^{\infty} d\zeta'' S(|\zeta|; \varphi-\varphi''; \zeta'') \times N(\tau; \zeta'', \varphi''). \quad (2.8)$$

Now, substituting (2.6) into (2.8), and taking account of the fact that the relation (2.8) should be fulfilled for any $\tau \geqslant 0$, we obtain the following equation for S:

$$|\zeta| \Phi_\lambda(-|\zeta|; \varphi) = \int_{-\infty}^{\infty} d\varphi'' \int_{0}^{\infty} d\zeta'' S(|\zeta|; \varphi-\varphi''; \zeta'') \Phi_\lambda(\xi''; \varphi''). \quad (2.9)$$

Equation (2.9) should be satisfied for all the eigenvalues λ in the interval $0 \leqslant \lambda < \infty$.

Thus, the computation of the angular spectrum of the reflected radiation reduces to the following: First, we must determine from Eq. (2.7) the explicit form of the angular functions $\Phi_\lambda(\zeta, \varphi)$, and then, knowing these functions, solve the linear integral equation (2.9) and compute the reflection function.

3. COMPUTATION OF THE ANGULAR FUNCTIONS

Since Eq. (2.7) is an integral equation with a difference kernel in the variables ζ and φ, it can be easily solved with the aid of a double Fourier transformation. Multiplying (2.7) by $\exp(-i\omega\zeta - iq\varphi) d\zeta d\varphi$, and integrating over infinite ranges of ζ and φ, we obtain for the double Fourier transform $\overline{\Phi}_\lambda(\omega, q)$ the following equation:

$$\lambda^3 \partial \overline{\Phi}_\lambda(\omega, q)/\partial\omega = -i[1-I(\omega, q)] \overline{\Phi}_\lambda(\omega, q). \quad (3.1)$$

Here $I(\omega, q)$ is the Fourier transform of the scattering indicatrix:

$$I(\omega, q) = \int_{-\infty}^{\infty} d\zeta\, d\varphi\, I(\zeta^2+\varphi^2) e^{-i(\omega\zeta+q\varphi)} = 2\pi \int_0^{\infty} I(\vartheta) J_0(\vartheta(\omega^2+q^2)^{1/2}) \vartheta\, d\vartheta, \quad (3.2)$$

where J_0 is the Bessel function of order zero.

For any eigenvalues $-\infty < \lambda < \infty$, Eq. (3.1) possesses a solution bounded in ω and q, and having the form

$$\overline{\Phi}_\lambda(\omega, q) = A(\lambda) \exp\left\{-\frac{i}{\lambda^3} \int_0^{\omega} [1-I(\omega', q)] d\omega'\right\}, \quad (3.3)$$

where $A(\lambda)$ is a normalized constant.

Now, carrying out a inverse Fourier transformation and taking account of the fact that, according to (3.2), $I(\omega, q)$ is an even function of the variables ω and q, we obtain the following expression for the angular eigenfunctions Φ_λ of the problem under consideration:

$$\Phi_\lambda(\zeta, \varphi) = \frac{\sqrt{3}}{2\pi^2|\lambda|} \int_0^{\infty} dq\, d\omega \cos(q\varphi) \cos\left\{\omega\zeta - \frac{1}{\lambda^3} \int_0^{\omega} G(\omega', q) d\omega'\right\}, \quad (3.4)$$

where

$$G(\omega, q) = 2\pi \int_0^{\infty} \vartheta\, d\vartheta\, I(\vartheta^2) [1-J_0(\vartheta(\omega^2+q^2)^{1/2})]. \quad (3.5)$$

The normalization constant $A(\lambda)$ has been chosen to be equal to $\sqrt{3}/|\lambda|$. It is not difficult to verify that the following relations obtain:

$$\int_{-\infty}^{\infty} d\varphi\, d\zeta\, \Phi_\lambda(\zeta, \varphi) \Phi_{\lambda'}(\zeta, \varphi) = \frac{1}{\lambda'} \delta(\lambda-\lambda'),$$
$$\int_{-\infty}^{\infty} \lambda\, d\lambda\, \Phi_\lambda(\zeta, \varphi) \Phi_\lambda(\zeta', \varphi') = \frac{1}{\zeta'} \delta(\zeta-\zeta') \delta(\varphi'-\varphi). \quad (3.6)$$

Thus, the expression (2.6) is the expansion of the solution to the transport equation (2.2) for grazing motion angles in terms of the complete orthonormalized system of functions, in which we have retained only those modes ($\lambda \geqslant 0$) which guarantee the boundedness of the solution with respect to the depth τ: $N(\tau \to \infty, \zeta, \varphi) = 0$.

It should be emphasized that (3.4) gives the explicit form of the eigenfunctions for an arbitrary single-scattering law. Therefore, by substituting (3.4) into (2.9), we can obtain for the reflection function a closed integral equation that is also valid for any form of the scattering indicatrix $I(\vartheta)$.

Below we shall consider the case when the scattering indicatrix is given by the expression (1.1). Then, substituting (1.1) into (3.5), we obtain

$$G_\nu(\omega, q) = 1 - \frac{2^{1-\nu}}{\Gamma(\nu)} [\vartheta_{\text{eff}} (\omega^2+q^2)^{1/2}]^\nu K_\nu(\vartheta_{\text{eff}} (\omega^2+q^2)^{1/2}). \quad (3.7)$$

Here $\Gamma(\nu)$ is the Euler gamma function and K_ν is the Macdonald function. For $\nu = 1/2$ the expression (3.7) is significantly simpler. Using the explicit form of $K_{1/2}(x)$, we have

$$G_{1/2}(\omega, q) = 1 - \exp[-\vartheta_{\text{eff}} (\omega^2+q^2)^{1/2}]. \quad (3.8)$$

It can be seen from (3.7) and (3.8) that, as $\omega^2 + q^2 \to \infty$, the quantity $G_\nu \to 1$. Therefore, the functions $\Phi_\lambda(\zeta, \varphi)$ have a delta-function singularity, i.e., they pertain to the class of singular functions. This property of the eigenfunctions occurs in other problems of transport theory, e.g., in neutron physics.[25]

4. ANGULAR SPECTRUM OF THE REFLECTED RADIATION

Now, knowing the form of the eigenfunctions, we can write down in its explicit form the equation for the reflection function. Since the function S entering into (2.8) depends on the angle difference $\varphi' - \varphi$, it is convenient to use the Fourier transformation in the azimuth. Multiplying (2.8) by $\cos(k\varphi/\vartheta_{\text{eff}}) d\varphi$, and integrating with respect to φ over the interval $0 \leqslant \varphi < \infty$ with allowance for the expression (3.4), we obtain an equation for the Fourier transform of the reflection function $S_k(|\zeta|; \zeta_0)$:

$$\int_0^\infty d\zeta_0 S_k(|\zeta|;\zeta_0)\int_0^\infty d\omega'\cos\Bigl\{\omega'\frac{\zeta_0}{\vartheta_{\rm eff}}-\frac{1}{\lambda^3}\int_0^{\omega'}d\omega\Bigl[1-\frac{2^{1-\nu}}{\Gamma(\nu)}(\omega^2+k^2)^{\nu/2}K_\nu((\omega^2+k^2)^{1/2})\Bigr]\Bigr\}$$

$$=|\zeta|\int_0^\infty d\omega'\cos\Bigl\{\omega'\frac{|\zeta|}{\vartheta_{\rm eff}}+\frac{1}{\lambda^3}\int_0^{\omega'}d\omega\Bigl[1-\frac{2^{1-\nu}}{\Gamma(\nu)}(\omega^2+k^2)^{\nu/2}K_\nu\times((\omega^2+k^2)^{1/2})\Bigr]\Bigr\},\qquad(4.1)$$

where we have set $\lambda^3\vartheta_{\rm eff}\equiv\lambda^3$.

If we are able to solve Eq. (4.1), then the distribution of the reflected radiation over the angle variable $|\zeta|$ and φ can be computed from the formula

$$S(|\zeta|,\varphi,\zeta_0)=\frac{1}{2\pi\vartheta_{\rm eff}}\int_{-\infty}^{\infty}dk\,S_k(|\zeta|;\zeta_0)\cos\Bigl(k\frac{\varphi}{\vartheta_{\rm eff}}\Bigr).\qquad(4.2)$$

Unfortunately, the kernel of Eq. (4.1) has such a complicated form that it has to date not been possible to obtain even an approximate solution to this equation. In other words, thus far there has not been found for the reflection spectrum an expression that is simultaneously analytic in both the polar and the azimuthal angles. (Within the framework of the diffusion approximation, this problem is solved in Refs. 13 and 14.) Below we shall limit ourselves to the computation of the distribution of the reflected radiation over only the polar angle $|\zeta|$, without reference to the azimuthal angle. Setting $k=0$ in (4.1), we obtain the following equation for $S(|\zeta|;\zeta_0)$:

$$\int_0^\infty d\zeta_0 S(|\zeta|;\zeta_0)\times\int_0^\infty d\omega'\cos\Bigl\{\omega'\frac{\zeta_0}{\vartheta_{\rm eff}}-\frac{1}{\lambda^3}\int_0^{\omega'}d\omega\Bigl[1-\frac{2^{1-\nu}}{\Gamma(\nu)}\omega^\nu K_\nu(\omega)\Bigr]\Bigr\}$$

$$=|\zeta|\int_0^\infty d\omega'\cos\Bigl\{\omega'\frac{|\zeta|}{\vartheta_{\rm eff}}+\frac{1}{\lambda^3}\int_0^{\omega'}d\omega\Bigl[1-\frac{2^{1-\nu}}{\Gamma(\nu)}\omega^\nu K_\nu(\omega)\Bigr]\Bigr\}.\qquad(4.3)$$

Equation (4.3) assumes its simplet form in the important $\nu=1/2$ case:

$$\int_0^\infty d\zeta_0 S_{1/2}(|\zeta|;\zeta_0)\int_0^\infty d\omega'\cos\Bigl\{\omega'\frac{\zeta_0}{\vartheta_{\rm eff}}+\frac{1}{\lambda^3}[1-\omega'-e^{-\omega'}]\Bigr\}$$

$$=|\zeta|\int_0^\infty d\omega'\cos\Bigl\{\omega'\frac{|\zeta|}{\vartheta_{\rm eff}}-\frac{1}{\lambda^3}[1-\omega'-e^{-\omega'}]\Bigr\}.\qquad(4.4)$$

If we were able to solve Eq. (4.3) (or even the simpler equation (4.4)) exactly, we would obtain an expression for the distribution of the reflected radiation over the polar angle $|\zeta|$ for any relation connecting the angles ζ_0, $|\zeta|$, and $\vartheta_{\rm eff}$, provided all these angles were small. We then could investigate the angular dependence of the reflected spectrum for any relation connecting the contributions made by the single-low-multiplicity-, and high-multiplicity-scattering processes to the reflected radiation.

But in many cases the anisotropy of the single scattering is fairly high, so that the following inequality is satisfied even for small glancing angles ζ_0:

$$1>\zeta_0\gg\vartheta_{\rm eff}.\qquad(4.5)$$

As noted above, when the condition (4.5) is fulfilled, the reflected flux is formed largely as the result of essentially multiple scattering. In this case, in the most interesting spectral region ($|\zeta|\sim\zeta_0$), where the flux has its maximum, we have $\zeta_0/\vartheta_{\rm eff}\gg1$; $|\zeta|/\vartheta_{\rm eff}\gg1$. Therefore, the dominant contribution to both ω' integrals will be made by the region $\omega'\ll1$, where rapid oscillations do not occur. This circumstance allows us to replace the functions $K_\nu(\omega)$ entering into the internal ω integrals by their values in the region of small ω when evaluating these integrals. To do this, let us use the well-known formula connecting the function $K_\nu(\omega)$ (where ν is not equal to a whole number) with the Bessel functions of imaginary argument:

$$K_\nu(\omega)={}^1/_2\Gamma(\nu)\Gamma(1-\nu)[I_{-\nu}(\omega)-I_\nu(\omega)].$$

Then, using the representation for $I_{\pm\nu}(\omega)$ in form of a series, we write

$$1-\frac{2^{1-\nu}}{\Gamma(\nu)}\omega^\nu K_\nu(\omega)$$

$$=\frac{\Gamma(1-\nu)}{2^\nu}\Bigl\{\sum_{k=0}^{\infty}\frac{(\omega/2)^{2(k+\nu)}}{k!\Gamma(k+\nu+1)}-\sum_{n=1}^{\infty}\frac{(\omega/2)^{2n}}{n!\Gamma(n-\nu+1)}\Bigr\}.\qquad(4.6)$$

In the region $\omega\ll1$ the form of the expression (4.6) depends essentially on whether the value of the paameter ν is greater or smaller than unity. Therefore, let us consider the two cases separately.

a) The parameter ν greater than unity

In this case, in the small-ω region, the dominant term in (4.6) is the first ($n=1$) term of the second series. Therefore,

$$1-\frac{2^{1-\nu}}{\Gamma(\nu)}\omega^\nu K_\nu(\omega)\approx\frac{\omega^2}{(\nu-1)2^{2+\nu}}\quad(\omega\ll1,\ \nu>1).\qquad(4.7)$$

Thus, the expressions standing in the square brackets in (4.3) are proportional to ω^2, irrespective of the numerical value of ν. This property obtains in the case of integral ν values as well.

Now, using the approximation (4.7), we can evaluate the ω' integrals in the kernel and the right-hand side of Eq. (4.3):

$$\int_0^\infty d\omega'\cos\Bigl\{\omega'\frac{\zeta}{\vartheta_{\rm eff}}\mp\frac{\omega'^3}{3\lambda^3(\nu-1)2^{2+\nu}}\Bigr\}=\frac{\pi}{\sqrt3}\vartheta_{\rm eff}\tilde\lambda\,{\rm Ai}^*(\pm\tilde\lambda\zeta),\qquad(4.8)$$

where

$$\tilde\lambda=\lambda[(\nu-1)2^{2+\nu}]^{1/3}\vartheta_{\rm eff}^{-1},$$

and the $\mathrm{Ai}^*(\pm\tilde\lambda\zeta)$ are the orthonormalized Airy functions.[13,14] Substituting (4.8) into (4.3), we obtain

$$\int_0^\infty d\zeta_0 S(|\zeta|;\zeta_0){\rm Ai}^*(\tilde\lambda\zeta_0)=|\zeta|{\rm Ai}^*(-\tilde\lambda|\zeta|)\quad(0\leq\tilde\lambda<\infty).\qquad(4.9)$$

We can see that Eq. (4.9) does not contain ν at all. Therefore, it will have the same solution for all values of $\nu>1$. Equation

(4.9) was first obtained in 1966 by Firsov,[11] who, to describe the process of small-angle reflection, employed the diffusion approximation from the very beginning. The solution to this equation is given by the Firsov formula

$$S(|\zeta|;\zeta_0;\nu>1)=S_\Phi(|\zeta|;\zeta_0)=\frac{3}{2\pi}\left[\left(\frac{|\zeta|}{\zeta_0}\right)^{3/2}+\left(\frac{\zeta_0}{|\zeta|}\right)^{3/2}\right]^{-1}. \quad (4.10)$$

Thus, if the single-scattering cross section decreases with increasing ϑ faster than the Rutherford cross section, then, to compute the multiply scattered component of the reflected radiation, we can, in the first approximation, use the diffusion approximation.

b) The parameter ν smaller than unity

In this case, in the region $\omega \ll 1$, the dominant term in the expansion (4.8) is the first ($k = 0$) term of the first series:

$$1-\frac{2^{1-\nu}}{\Gamma(\nu)}\omega^\nu K_\nu(\omega)\approx\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{\omega^2}{8}\right)^\nu \quad (\omega\ll 1,\ \nu<1). \quad (4.11)$$

In contrast to the preceding case, now the expressions standing in the square brackets in (4.3) are proportional to $\omega^{2\nu}$, i.e., they depend essentially on the value of the parameter ν. Therefore, the solution to Eq. (4.3) will also have different forms for different ν values: $S(|\zeta|;\zeta_0)=S_\nu(|\zeta|;\zeta_0)$.

Substituting (4.11) into (4.3), we write the equation for the reflection function in the following form:

$$\int_0^\infty d\zeta_0\, S_\nu(|\zeta|;\zeta_0)D_\nu^-\left(\frac{\zeta_0}{\varepsilon}\right)=|\zeta|D_\nu^+\left(\frac{|\zeta|}{\varepsilon}\right), \quad (4.12)$$

where

$$D_\nu^\pm\left(\frac{\zeta}{\varepsilon}\right)=\int_0^\infty d\omega\cos\left(\omega\frac{\zeta}{\varepsilon}\pm\omega^{1+2\nu}\right). \quad (4.13)$$

Here ε is a new independent variable connected with λ by the relation

$$\varepsilon=\vartheta_{\rm eff}\,[2^{-3\nu}\Gamma(1-\nu)/\lambda^3(1+2\nu)\Gamma(1+\nu)]^{1/(1+2\nu)}.$$

Like λ, the variable ε varies in the range from 0 to ∞.

Although the kernel D_ν^- of Eq. (4.12) is, as usual, expressed in terms of an integral that cannot be taken, it, unlike the kernel of the original equation (4.3), depends on the ratio ζ_0/ε, i.e., it is a homogeneous function of ζ_0 and ε. This circumstance radically simplifies the problem, since it allows us to use the Mellin transformation [26] to solve Eq. (4.12) (see the Appendix). As a result, we find

$$S_\nu(|\zeta|;\zeta_0)=\frac{1+2\nu}{\pi(1+\nu)}\sin\frac{\pi\nu}{1+\nu}\left[2\cos\frac{\pi\nu}{1+\nu}+\left(\frac{|\zeta|}{\zeta_0}\right)^{\frac{1+2\nu}{1+\nu}}+\left(\frac{\zeta_0}{|\zeta|}\right)^{\frac{1+2\nu}{1+\nu}}\right]^{-1}. \quad (4.14)$$

The expression (4.14) gives the distribution of the elastically reflected radiation over the polar angle $|\zeta|$ (without reference to the azimuthal angle φ) in the case when the single-scattering cross section decreases with increasing ϑ slower than the Rutherford cross section, i.e., slower than ϑ^{-4}.

The expression (4.14) is an approximate solution to Eq. (4.3) when it is assumed that the condition (4.5) is fulfilled. It is not without interest, however, to note that at th same time this expression is not an approximation, but the exact solution to the small-angle reflection problem if we set $\nu_{\rm eff}=0$ in the denominator of (1.1) right from the start, i.e., if we assume that $d\sigma_\nu(\vartheta)\propto\vartheta^{-2(1+\nu)}\vartheta d\vartheta$. Then the transport equation for the flux density $N(\tau,\zeta)$, integrated over φ, has the form

$$\zeta\frac{\partial N(\tau,\zeta)}{\partial\tau}=\int_{-\infty}^{\infty}\frac{N(\tau,\zeta')-N(\tau,\zeta)}{|\zeta'-\zeta|^{1+2\nu}}d\zeta'. \quad (4.15)$$

Although in this case the differential scattering cross section has a singularity at $\vartheta\to 0$, in the transport equation (4.15) this singularity is a nonessential one, since as the scattering angle decreases (i.e., as $|\zeta'-\zeta|\to 0$), the expression $N(\tau,\zeta_1)-N(\tau,\zeta)$ also vanishes.

Here we encounter a situation similar to the one that arises in the computation of the energy spectrum of relativistic electrons, when the dominant mechanism of energy loss is the bremsstrahlung process. The bremsstrahlung cross section has a singularity in the low γ-quantum energy region (the "infrared catastrophe"). Nevertheless, this singularity turns out to be suppressed in the transport equation,[18,22] and does not have a strong effect on the energy spectrum of the particles.

The above-indicated circumstance indicates that the magnitude of the Boltzmann collision integral is not as sensitive to the law governing scattering through very small angles as the single-scattering cross section itself. Therefore, it can be expected that the solution (4.14) obtained by us will also describe the angular spectrum of the reflected radiation sufficiently well in the case when the strong inequality (4.5) is replaced by the weaker condition $\zeta_0\gtrsim\vartheta_{\rm eff}$.

Finally, let us turn to the case $\nu=1$. In this case the formula (4.6) is not applicable, and we cannot therefore use any of the approximations (4.7) and (4.11). For $\nu=1$ we have in the region of small ω the relation

$$1-\omega K_1(\omega)\approx{}^1/_4\omega^2[\ln(\omega^2/4)+2C-1], \quad (4.16)$$

where $C=0.577$ is the Euler constant. Thus, in this case, when $\omega\ll 1$, there arises in (4.16) a logarithmic factor besides a power one. Because of this, it is not possible to simplify the basic equation (4.3) to such an extent that it becomes soluble. The $\nu=1$ case is investigated in Ref. 27 in the study of small-angle reflection of fast electrons. In that investigation the following approximation is used to simplify the equation for the reflection function (Eq. (8) of the present paper): $\omega^3\ln\omega^2\approx\omega^{3-\alpha}$, where $0<\alpha<1$ is an approximation parameter of the problem. This essentially reduces the equation for the reflection function in Ref. 27 to Eq. (4.12) of the present paper if we set $\nu=1-\alpha/2$ in it. Thus, it is, in the light of the investigation carried out above, clear that the approimate method used in Ref. 27 to solve the equation for the reflection function actually implies the replacement of the screened Coulomb potential $U(r)\propto r^{-1}\exp(-r/r_{\rm at})$ by an inverse-power potential of the form $U(r) \propto r^{-2/(2-\alpha)}$.

5. DISCUSSION OF THE RESULTS OBTAINED. COMPARISON WITH THE RESULTS OF THE CALCULATIONS OF OTHER AUTHORS

Let us, in proceeding to analyze the results obtained, note the following general properties of the reflection function.

1. The expression (4.14) is invariant under the interchange $|\zeta| \leftrightarrow \zeta_0$, which is a consequence of the well-known reciprocity theorem[25] in the one-velocity problems of transport theory.

2. We can easily convince ourselves by direct verification that the reflection function (4.14), like (4.10), satisfies the following normalization condition for any values of the parameter ν:

$$\int_0^\infty d|\zeta| S_\nu(|\zeta|;\zeta_0) = \zeta_0. \tag{5.1}$$

Since the quantity ζ_0 (for unit incident-flux intensity) is equal to the number of particles entering the medium in unit time through a unit area, the condition (5.1) expresses the law of conservation of the particle number in the case of purely elastic scattering. Therefore, the probability for backscattering into the range of reduced polar-angle values from $|\psi|$ to $|\psi| + d|\psi|$, where $|\psi| = |\zeta|/\zeta_0$, is given by the expression

$$\frac{dw_{\mathrm{ref}}(|\psi|)}{d|\psi|} = \frac{1+2\nu}{\pi(1+\nu)} \sin\frac{\pi\nu}{1+\nu}\left[2\cos\frac{\pi\nu}{1+\nu} + |\psi|^{\frac{1+2\nu}{1+\nu}} + |\psi|^{-\frac{1+2\nu}{1+\nu}}\right]^{-1}. \tag{5.2}$$

Figure 1 shows plots of the function $dw_{\mathrm{ref}} d|\psi|$ for different values of the parameter ν.

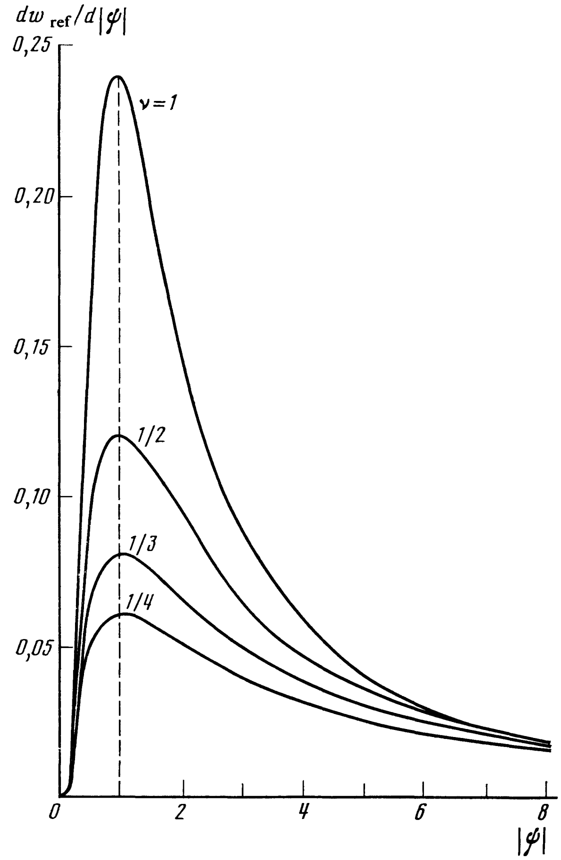

FIG. 1. Dependence of the differential reflection coefficient, computed from the formula (5.2), on the reduced angle of emission $|\psi| = |\zeta|/\zeta_0$ (without reference to the azimuthal angle) for different values of the parameter ν.

3. The results obtained are valid not only for a homogeneous, but also for a stratified, medium, when the density of the scattering centers depends on the depth: $n_0 = n(z)$. This becomes clear when we take account of the fact that the basic transport equation (2.2) has the same form for both a homogeneous and an inhomogeneous medium if we introduce in place of the depth z the "optical" depth

$$\tau = \sigma_{\mathrm{elas}} \int_0^z n(z')\,dz',$$

where σ_{elas} is the total cross section for elastic scattering.

4. The reflection function (4.14), like the differential backscattering coefficient (5.2), has a strongly pronounced peak at $|\zeta| = \zeta_0$ (i.e., $|\psi| = 1$). Thus, in the case of elastic scattering through small angles, the specular reflection law turns out to be valid not only in the diffusion approximation (4.10), but also in a more general case, i.e., it is obeyed irrespective of the specific form of the angular dependence (1.1) of the single scattering for any ν value. It is precisely because of this circumstance that the specular reflection law is observed in virtually all backscattering experiments performed for the case of grazing angles of incidence of fast particles (electrons, protons), as well as for the case of reflection of ions of intermediate energies.

5. In contrast to the diffusion approximation (4.10), which contains no information at all about the scattering properties of the medium (since it depends only on the angles $|\zeta|$ and ζ_0), the obtained solution (4.14) depends essentially on the numerical value of the parameter ν, which determines the rapidity of decrease of the single-scattering cross section in the region of relatively large angles $\vartheta \gg \vartheta_{\mathrm{eff}}$. As can be seen from Fig. 1, as ν decreases, there occurs a broadening of the reflection spectrum in the region of relatively large angles. This is explained by the fact that in the case $\nu < 1$, when the single-scattering cross section decreases with increasing ϑ slower than the Rutherford cross section, the ponderable contribution to the reflected flux of multiply scattered particles is made by those collisions in which there occurs scattering through an angle comparable to the total scattering angle $\vartheta \sim \zeta_0 + |\zeta|$. This contribution is the greater, the more the "tail" of the scattering indicatrix $I(\vartheta)$ is stretched out, i.e., the higher the probability for single scattering through relatively large angles.

The broadening of the spectrum is accompanied by the decrease of the value of the reflection coefficient at the peak (at $|\psi| = 1$):

$$\left(\frac{dw_{\mathrm{ref}}}{d|\psi|}\right)_{\max} = \frac{1}{2\pi}\frac{1+2\nu}{1+\nu}\,\mathrm{tg}\,\frac{\pi\nu}{2(1+\nu)}. \tag{5.3}$$

Thus, when $\nu = 1/2$, the peak value of the reflection coefficient is 0.12, while when $\nu = 1/3$ it is 0.08. (The correspond-

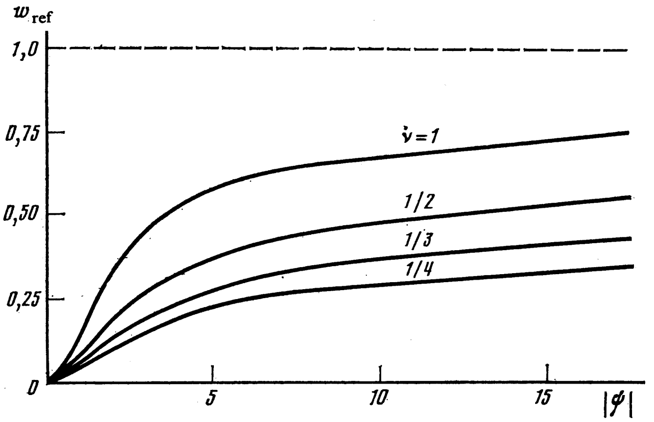

FIG. 2. Probability, integrated over the azimuthal angle, of elastic scattering into the interval of angles smaller than $|\psi| = |\zeta|/\zeta_0$ for different values of the parameter ν.

ing value in the diffusion approximation is $3/4\pi \approx 0.24$.) This circumstance allows, for example, the determination from (5.3), after the reflection coefficient for the angle $|\zeta| = \zeta_0$ has been determined in experiment, of the value of the parameter ν, i.e., the determination of, for example, the potential function for the interaction of ions with the atoms of the medium in the case of relatively small impact parameters, when $U(r) \propto r^{-1/\nu}$.

6. Using the expression (5.2), we can compute the probability that the particle will be reflected from the material in the finite angle range from zero to $|\psi|$:

$$w_{\text{ref}}(|\psi|) = \int_0^{|\psi|} dw_{\text{ref}}(|\psi'|). \tag{5.4}$$

Figure 2 shows plots of this probability function for different values of ν. It can be seen that, as ν decreases, the reflection probability (5.4) decreases, since the fraction of the particles reflected into the interval of angles greater than $|\psi|$ increases because of the increase of the probability for single scattering through relatively large angles.

7. If we set $\nu = 1/2$ in the general expression (4.14), we obtain

$$S_{1/2}(|\zeta|;\zeta_0) = \frac{2}{\pi\sqrt{3}}\left[1 + \left(\frac{|\zeta|}{\zeta_0}\right)^{4/3} + \left(\frac{\zeta_0}{|\zeta|}\right)^{4/3}\right]^{-1}. \tag{5.5}$$

The expression (5.5) describes the angular spectrum of the reflected radiation in the case when the interaction between the ions and the atoms of the medium is described by an inverse-square law potential, or the spectrum of the reflected photons when the law governing the single scattering process is modeled by the Heiney-Greenstein indicatrix. The result (5.5) was first obtained by Firsov[28] in 1970.

8. Although the expression (4.14) was obtained for the $\nu < 1$ case, we obtain the Firsov formula (4.10) again when we set $\nu = 1$ in it. Thus, the solution (4.14) found for $\nu < 1$ joins onto the results obtained in the diffusion-approximation calculation for the case when $\nu > 1$. Consequently, the expression (4.14) for the reflection function generalizes all the results obtained in previously published calculations for the angular spectra of elastically scattered radiation (without reference to the azimuthal angle) in the case of grazing angles of incidence of the external flux on the surface of the material.

In conclusion, we note that all these calculations of the angular spectra of the reflected radiation were carried out under the assumption that the surface of the medium is absolutely smooth. In reality, any material surface exhibits some roughness. If the characteristic dimensions of the irregularity profile of the surface are $\sim h$, then the roughness of the surface can be ignored in the case when the reflected radiation is produced in the region of depths $z \gtrsim z_0 \gg h$, where z_0 is that minimum depth at which the particles are deflected through angles comparable to ζ_0, and, consequently, can get out of the medium: $\langle\theta^2\rangle z_0 \sim \zeta_0^2$. Since at a depth of z_0, the particles traverse on the average a path of length $\sim z_0/\zeta_0$, $\langle\theta^2\rangle z_0 \sim z/\zeta_0 l_{\text{tr}}$, where l_{tr} is the transport mean free path characterizing the elastic scattering. Thus, the roughness of the surface can be ignored if $\zeta_0^3 \gg h/l_{\text{tr}}$.

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APPENDIX

Solution of equation (4.12)

Let us multiply both sides of Eq. (4.12) by $\varepsilon^{s-1}\,d\varepsilon$ and integrate over the range from zero to infinity. Then, by using the convolution theorem and the inversion formula for the Mellin transformation, we can represent the solution to Eq. (4.12) in the following form:

$$S_\nu(|\zeta|;\zeta_0) = \frac{1}{2\pi i}\int_{-i\infty+\delta}^{i\infty+\delta} ds\left(\frac{|\zeta|}{\zeta_0}\right)^{1+s}\frac{D_\nu^+(s)}{D_\nu^-(s)}. \tag{A.1}$$

Here

$$D_\nu^\pm(s) = \int_0^\infty D_\nu^\pm(u)\,u^{-(1+s)}\,du. \tag{A.2}$$

The functions $D_\nu^+(u)$ are given by the relation (4.13). The integration in (A.1) is performed in the complex plane along any straight line parallel to the imaginary axis in the analyticity band of the function D_ν^+/D_ν^-. Let us compute the function $D_\nu^+(s)$. Substituting (4.13) into (A.2), and changing the integration order, we obtain

$$D_\nu^+(s) = \int_0^\infty d\omega\left\{\cos\omega^{1+2\nu}\int_0^\infty du\,u^{-(1+s)}\cos\omega u - \sin\omega^{1+2\nu}\right.$$

$$\left.\times\int_0^\infty du\,u^{-(1+s)}\sin\omega u\right\}.$$

Since[26]

$$\int_0^\infty du\,u^{-(1+s)}\cos\omega u = -\frac{\pi}{2}\frac{\omega^s}{\Gamma(1+s)\sin(\pi s/2)},$$

$$-1 < \operatorname{Re} s < 0,$$

and

$$\int_0^\infty du\,u^{-(1+s)}\sin\omega u = \frac{\pi}{2}\frac{\omega^s}{\Gamma(1+s)\cos(\pi s/2)}, \quad -1 < \operatorname{Re} s < 1,$$

we obtain

$$D_\nu^+(s)=-\frac{\pi}{2\Gamma(1+s)}\left\{\left(\sin\frac{\pi s}{2}\right)^{-1}\int_0^\infty d\omega\,\omega^s\cos\omega^{1+2\nu}+\left(\cos\frac{\pi s}{2}\right)^{-1}\int_0^\infty d\omega\,\omega^s\sin\omega^{1+2\nu}\right\}. \quad (A.3)$$

The two ω integrals entering into this expression reduce to the u integrals written out above when we set $\omega^{1+2\nu}=\tilde{\omega}$, and are easily evaluated:

$$\int_0^\infty d\omega\,\omega^s\cos\omega^{1+2\nu}=\frac{\pi}{2}\operatorname{cosec}\left[\frac{\pi(1+s)}{2(1+2\nu)}\right]\Big/(1+2\nu)\Gamma\left(1-\frac{1+s}{1+2\nu}\right),$$

$$2\nu>\operatorname{Re}s>-1,$$

$$\int_0^\infty d\omega\,\omega^s\sin\omega^{1+2\nu}=\frac{\pi}{2}\sec\left[\frac{\pi(1+s)}{2(1+2\nu)}\right]\Big/(1+2\nu)\Gamma\left(1-\frac{1+s}{1+2\nu}\right),$$

$$2\nu>\operatorname{Re}s>-2(1+\nu).$$

Substituting the above-found values of the integrals into (A.3), and using the formula

$$\Gamma\left(\frac{1+s}{1+2\nu}\right)\Gamma\left(1-\frac{1+s}{1+2\nu}\right)=\pi\operatorname{cosec}\frac{\pi(1+s)}{2(1+2\nu)},$$

we obtain the expression for the function $D_\nu^+(s)$ in the following form:

$$D_\nu^+(s)=\frac{1}{1+2\nu}\Gamma\left(\frac{1+s}{1+2\nu}\right)\Gamma(-s)\cos\frac{\pi(1-2\nu s)}{2(1+2\nu)}. \quad (A.4)$$

Similarly, we can compute the quantity $D_\nu^-(s)$. As a result, we obtain

$$D_\nu^-(s)=\frac{1}{1+2\nu}\Gamma\left(\frac{1+s}{1+2\nu}\right)\Gamma(-s)\cos\frac{\pi[1+2(1+\nu)s]}{2(1+2\nu)}. \quad (A.5)$$

Now, by substituting (A.4) and (A.5) into the solution (A.1), we can represent the epression for the reflection function in the form

$$S_\nu(|\zeta|;\zeta_0)=\frac{1}{2\pi i}\int_{-i\infty+\delta}^{i\infty+\delta}ds\times\left(\frac{|\zeta|}{\zeta_0}\right)^{1+s}\cos\frac{\pi(1-2\nu s)}{2(1+2\nu)}\Big/\cos\frac{\pi[1+2(1+\nu)s]}{2(1+2\nu)}. \quad (A.6)$$

The integration in (A.6) is performed in the band $-1<\operatorname{Re}s\leqslant 0$. Making in (A.6) the change of integration variable

$$[1+2(1+\nu)s](1+2\nu)^{-1}=\tilde{s},$$

and making the integration contour for the new variable $\tilde{s}$ coincident with the imaginary axis (i.e., setting $\tilde{s}=i\omega$), we obtain

$$S_\nu(|\zeta|;\zeta_0)=\frac{1+2\nu}{4\pi(1+\nu)}\gamma\int_{-\infty}^{\infty}d\omega\frac{\exp(i\omega\ln\gamma)}{\operatorname{ch}(\pi\omega/2)}\times\left[\cos\frac{\pi}{2(1+\nu)}\operatorname{ch}\frac{\pi\nu\omega}{2(1+\nu)}+i\sin\frac{\pi}{2(1+\nu)}\operatorname{sh}\frac{\pi\nu\omega}{2(1+\nu)}\right]. \quad (A.7)$$

Here we have set

$$\gamma=(|\zeta|/\zeta_0)^{(1+2\nu)/2(1+\nu)}.$$

The ω integral entering into (A.7) can be represented in the form of two tabular integrals:

$$\int_{-\infty}^{\infty}d\omega\exp(i\omega\ln\gamma)\operatorname{ch}\frac{\pi\nu\omega}{2(1+\nu)}\Big/\operatorname{ch}\frac{\pi\omega}{2}=4\left(\gamma+\frac{1}{\gamma}\right)\cos\frac{\pi\nu}{2(1+\nu)}\Big/\left[\gamma^2+\frac{1}{\gamma^2}+2\cos\frac{\pi\nu}{1+\nu}\right],$$

$$i\int_{-\infty}^{\infty}d\omega\exp(i\omega\ln\gamma)\operatorname{sh}\frac{\pi\nu\omega}{2(1+\nu)}\Big/\operatorname{ch}\frac{\pi\omega}{2}=-4\left(\gamma-\frac{1}{\gamma}\right)\sin\frac{\pi\nu}{2(1+\nu)}\Big/\left[\gamma^2+\frac{1}{\gamma^2}+2\cos\frac{\pi\nu}{1+\nu}\right].$$

Substituting the values of the evaluated integrals into (A.7), and reducing such terms, we finally obtain the expression (4.14) for the reflection function.

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