Riemann invariants and the propagation of nonlinear waves in superfluid helium

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A system of equations, written in terms of Riemann invariants, and describing the propagation of intense acoustic waves in superfluid helium, are derived. A qualitative investigation of these equations shows that a high-intensity entropy pulse decays into a "precursor"—a pressure wave propagating with the velocity of first sound—and a "mixture" of entropy and density perturbations moving with the velocity of second sound.

At low amplitudes the coupling between density and entropy perturbations in superfluid helium, i.e., between first and second sounds, is determined by the expansion coefficient $\beta_T = -(\partial\rho/\partial T)/\rho$, and is accordingly weak. Therefore, we can, with a high degree of accuracy, consider these wave modes to be independent. At high amplitudes there arises between them an interaction due to the nonlinear terms in the equations of motion. A number of investigations have recently been carried out in which certain effects connected with this interaction are described.[1–4] In the cited papers the investigation is carried out on the basis of the Hamiltonian formalism for He II hydrodynamics.[1] But this method, which has proved itself to be very effective in the case of waves in a plasma, or, for example, in the case of waves in the ocean, is not suitable for helium. The point is that sound-velocity dispersion practically does not occur in He II, and the small parameter connected with this dispersion vanishes.

In this respect a more general approach is the one based on the standard Landau-Khalatnikov equations of motion.[5] One of the first theoretical investigations in which the nonlinear effects associated with sound propagation is contained in Khalatnikov's paper.[6] In this paper the nonlinear corrections to the sound velocities are found, which makes it possible to explain the heat-pulse distortion observed by Osborne.[7] The method used in Ref. 6 does not reveal the wave-mode interaction effect. In Refs. 8 and 9 a Burgers-type equation is derived for the evolution of nonlinear waves with allowance for the viscosity. These investigations also miss the effects connected with the nonlinear interaction between first and second sound. If in Ref. 8 the reason for this is the proximity to the λ transition (the intermode coupling is $\sim\rho_s/\rho$), in Ref. 9 the cause is the unnatural omission of the terms that give rise to the interaction. Putterman and Garret,[10] applying the method of successive approximations to the nonlinear wave equations, found the effect whereby pressure waves are generated under conditions of second-sound pumping. But this method, in the form developed by Putterman and Garret,[10] does not allow us to describe the nonlinear distortion of the waves, and, consequently, does not adequately describe the evolution of pulses.

In the present paper we derive for intense first- and second-sound waves a system of evolutional equations that take account of both the interaction processes and the processes of nonlinear twisting of the wave front. The system is written in the form of equations for Riemann invariants.[11] On the one hand, this significantly simplifies the analytic and numerical investigation of specific problems, and, on the other, it is a convenient starting point for further generalizations of the type in which the viscosity is allowed for. In the second section of the paper, using the equations obtained, we qualitatively solve the problem involving the nonlinear decay of an entropy wave and the appearance of a "precursor"—a pressure wave propagating with the velocity of first sound.

1. DERIVATION OF THE EQUATIONS OF MOTION

Let us choose the following quantities as the variables describing the flow of superfluid helium: the density perturbation ρ', the entropy perturbation σ', the mean-mass velocity $\mathbf{v}(\rho\mathbf{v}=\mathbf{j})$, and the relative velocity $\mathbf{w}=\mathbf{v}_n-\mathbf{v}_s$. Up to terms of second order in these quantities (below we shall limit ourselves to this approximation) the equations of one-dimensional motion will be the following:

$$\frac{\partial\varphi_i}{\partial t}+\sum_j A_{ij}\frac{\partial\varphi_j}{\partial x}=0,\quad i,j=1,2,3,4. \tag{1}$$

Here φ is a column vector formed by the quantities $\varphi_1=\rho'$, $\varphi_2=v$, $\varphi_3=\sigma'$, and $\varphi_4=w$, where $v=v_x$ and $w=w_x$. The elements of the 4×4 matrix A_{ij} contain the φ_k powers not higher than the first, and have the following form:

$$A_{13}=A_{14}=A_{23}=A_{32}=A_{41}=0,\qquad A_{12}=\rho+\rho',$$

$$A_{11}=A_{22}=v,\quad A_{21}=\frac{1}{\rho}\frac{\partial p}{\partial\rho}-\frac{\rho'}{\rho^2}\frac{\partial p}{\partial\rho}+\frac{\rho'}{\rho}\frac{\partial^2 p}{\partial\rho^2},$$

$$A_{24}=\frac{2\rho_s\rho_n}{\rho^2}w-\rho w\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho},\qquad A_{31}=\frac{\sigma}{\rho}\frac{\partial\rho_s}{\partial\rho}w,$$

$$A_{33}=v+\frac{w}{\rho}\frac{\partial}{\partial\sigma}\rho_s\sigma,\quad A_{34}=\sigma\frac{\rho_s}{\rho}+\sigma\rho'\frac{\partial}{\partial\rho}\frac{\rho_s}{\rho}+\sigma'\frac{\partial}{\partial\sigma}\frac{\rho_s\sigma}{\rho},$$

$$A_{42}=\left[1-\frac{\rho^2}{\rho_n}\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}\right]w,$$

$$A_{44}=\left[\frac{3\rho_s}{\rho}-\frac{\rho\sigma}{\rho_n}\left(1+\frac{\rho_s}{\rho}\right)\frac{\partial}{\partial\sigma}\frac{\rho_n}{\rho}\right]w+v, \tag{2}$$

$$A_{43}=\frac{\rho}{\rho_n}\sigma\frac{\partial T}{\partial\sigma}+\left[\frac{\rho}{\rho_n}\frac{\partial T}{\partial\sigma}+\frac{\sigma\rho}{\rho_n}\frac{\partial^2 T}{\partial\sigma^2}+\sigma\frac{\partial T}{\partial\sigma}\frac{\partial}{\partial\sigma}\frac{\rho}{\rho_n}\right]\sigma'+\sigma\rho'\frac{\partial T}{\partial\sigma}\frac{\partial}{\partial\rho}\frac{\rho}{\rho_n}.$$

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The notion used here is conventional notation, and corresponds to the notation used in Ref. 5. We neglect the terms of the order of $\beta_T = -(\partial\rho/\partial T)/\rho$.[1]

Let us, following Ref. 11, multiply (1) by the left row eigenvector l of the A_{ij} matrix, i.e., by the quantities l_j defined by the relation $\Sigma_j l_j A_{ij} = \xi l_i$. For the various eigenvectors $l^{(\mu)}$ we have

$$\sum_j l_j^{(\mu)}(\varphi)\left(\frac{\partial\varphi_j}{\partial t}+\xi^{(\mu)}\frac{\partial\varphi_j}{\partial x}\right)=0, \quad j,\mu=1,2,3,4. \tag{3}$$

In order for the equations of motion to contain terms of order-in smallness—not higher than the second, the elements of the row vectors $l^{(\mu)}(\varphi)$ should contain the quantities φ_k in powers not higher than the first. The row vectors $l^{(\mu)}(\varphi)$ will be written out in their explicit term below.

This formulation is different in that all the variables φ_j in each of the four equations of the system (3) are differentiated in one and the same characteristic direction $\partial/\partial t + \xi\partial/\partial x$ in the (x,t) plane.

The characteristics $\xi^{(\mu)}$ are given by the following expressions:

$$\xi^{(1,2)}=v\pm\left(c_1+\frac{1}{2c_1}\frac{\partial^2 p}{\partial\rho^2}\rho'\right), \tag{4}$$

$$\xi^{(3,4)}=v+w\left(\frac{2\rho_s}{\rho}+\frac{\sigma}{\rho_n}\frac{\partial\rho_s}{\partial\sigma}\right)\pm\left\{c_2+\frac{\rho'}{2c_2}\frac{\partial T}{\partial\sigma}\frac{\partial}{\partial\rho}\frac{\rho_s}{\rho_n}\right.$$
$$\left.+\frac{\sigma'}{2c_2}\left[\frac{2\rho_s}{\rho_n}\sigma\frac{\partial T}{\partial\sigma}+\frac{\rho_s}{\rho_n}\sigma^2\frac{\partial^2 T}{\partial\sigma^2}+\sigma^2\frac{\partial T}{\partial\sigma}\frac{\partial}{\partial\sigma}\frac{\rho_s}{\rho_n}\frac{\rho^2}{\rho_n}\right]\right\}. \tag{5}$$

The formula (4) gives the local first-sound velocity; the formula (5), the local second-sound velocity. If we set $v = 0$, $\rho' = 0$, then the relation (5) goes over into the corresponding formula, obtained earlier by Khalatnikov,[6] for the local velocity of the wave profile.

If the Pfaffian form $\Sigma_j l_j^{(\mu)}(\varphi)d\varphi_j$ is integrable, i.e., if it is a total differential of the quantity $I_\mu(\varphi)$, then we can simplify the equations of the system (3) further:

$$\frac{\partial I_\mu}{\partial t}+\xi^{(\mu)}\frac{\partial I_\mu}{\partial x}=0, \quad \mu=1,2,3,4. \tag{6}$$

A remarkable property of the equations (6), which makes them extremely convenient for the investigation of specific problems, consists in the fact that each of them describes the conservation of the quantity I_μ along the characteristic direction $dx/dt = \xi^{(\mu)}(\varphi)$. The quantities I_μ are called Riemann invariants.

In the linear case the indicated scheme is easy to realize for He II. In this case the Pfaffian form is a sum of the differentials $d\varphi_i$ with constant coefficients. Such a form is, of course, integrable. The final result will be as follows:

$$\frac{\partial I_{1,2}^{(0)}}{\partial t}+c_1\frac{\partial I_{1,2}^{(0)}}{\partial x}=0, \quad \frac{\partial I_{3,4}^{(0)}}{\partial t}+c_2\frac{\partial I_{3,4}^{(0)}}{\partial x}=0. \tag{7}$$

The Riemann invariants are given in this case by the following expressions:

$$I_{1,2}^{(0)}=\rho'\pm\frac{\rho}{c_1}v, \quad I_{3,4}^{(0)}=\sigma'\pm\frac{\rho_s\sigma}{\rho c_2}w. \tag{8}$$

This is a classical result. The formulas (7) and (8) separate the wave modes (i.e., the first and second sounds) propagating to the right and left along the x axis (the upper sign corresponds to the wave propagating to the right).

In the nonlinear case, which we are interested in here, the Pfaffian form is, generally speaking, not integrable, and it is not possible to obtain Riemann invariants. Nevertheless, it turns out that we can introduce Riemann invariants for one important particular class of problems, namely, for waves running in one direction.

Let us recall what we mean by waves running in one direction (or, as they are also called, simple waves) in ordinary gas dynamics. The simple-waves tool developed by Riemann played a very important role in the solution of various gas-dynamical problems.[12] From the mathematical standpoint, simple waves are that particular case of the solution to the Euler equations in which the density and the velocity are connected by some functional relation: $v = v(\rho')$. To derive the evolutional equations for such waves, we use the following method. We substitute the function $v(\rho')$ into the Euler equations, writing the derivatives of the type $\partial v/\partial t$ as $(\partial v/\partial\rho')(\partial\rho'/\partial t)$, etc. As a result, we obtain a system of algebraic equations for the quantities $\partial\rho'/\partial t$ and $\partial\rho'/\partial x$. The consistency condition for this system allows us to determine the function $v(\rho')$.

But we cannot use a similar method in our case. Indeed, a natural generalization to the case of superfluid helium is the assumption that the variables are connected by relations of the form[2]: $v = v(\rho',\sigma')$, $w = w(\rho',\sigma')$. If further we seek the functions $w(\rho',\sigma')$ and $v(\rho',\sigma')$ from the consistency condition for the algebraic equations for the quantities $\partial\rho'/\partial t$, $\partial\rho'/\partial x$, $\partial\sigma'/\partial t$, and $\partial\sigma'/\partial x$, we have one condition for two functions, i.e., the problem is indeterminate.

Let us examine the nature of simple waves from a somewhat more profound standpoint. In the case of ordinary gas dynamics the above-described scheme for obtaining the Riemann invariants can be realized for isentropic flows. Indeed, in this case we have only two variables: (ρ' and v), and the Pfaffian form will have the following structure:

$$a(\rho', v)d\rho'+b(\rho', v)dv=\Pi. \tag{9}$$

It is clear that the quantities $I(\rho',v) = \text{const}$, which are the solutions to the differential equations $\Pi = 0$, will be the Riemann invariants. Thus, in gas dynamics we have two Riemann invariants, $I_1(\rho',v)$ and $I_2(\rho',v)$, which correspond to two different characteristics. If one of them is identically equal to a constant (zero for the sound cases), then the remaining invariant describes a simple wave. Thus, waves running in the same direction (simple waves), each of which is trivially derivable from the other in ordinary gas dynamics, have the following two basic properties: first, the existence of a functional relation between the variables and, second the identical vanishing of the wave propagating in the other direction. These two properties will be used to generalize the simple-waves method to the case of superfluid helium.

The computations below are organized as follows. We shall first of all demonstrate the integrability of the Pfaffian form $\Sigma_j l_j(\varphi)d\varphi_j$ for the first-sound mode, i.e., find the Riemann invariants, I_1 and I_2, that are conserved along the characteristics $\xi^{(1)}$ and $\xi^{(2)}$. By requiring that the invariant, I_2, that describes the wave running to the left vanish, we obtain the dependence of the quantity v on the density and entropy perturbations, ρ' and σ' respectively, i.e., we find the functional dependence $v(\rho',\sigma')$ in the wave running to the right. Further, assuming that the function $w(\rho',\sigma')$ exists, we, as in gas dynamics, substitute this function, as well as the function $v(\rho',\sigma')$ (already known) into the basic equations of motion (1). In this case, as in gas dynamics, we arrive at a system of four algebraic equations for the quantities $\partial\rho'/\partial t$, $\partial\rho'/\partial x$, $\partial\sigma'/\partial t$, and $\partial\sigma'/\partial x$. The solvability condition for this system will allow us to obtain the sought dependence $w = w(\rho',\sigma')$. This function will be used for the integration of the Pfaffian form corresponding to the eigenvalue $\xi^{(3)}$, which will allow us to determine the invariant I_3.

Let us explicitly write out in the Pfaffian forms corresponding to the first and second equations of the system (3):

$$\begin{aligned}\Pi^{1,2}=&\pm\left[\frac{c_1}{\rho}+\rho'\left(\frac{1}{2c_1\rho}\frac{\partial^2 p}{\partial\rho^2}-\frac{c_1}{\rho^2}\right)\right]d\rho'+dv\\ &\pm\frac{c_1}{c_1^2-c_2^2}\left[\frac{2\rho_s\rho_n}{\rho^2}-\rho\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}\right]w\,dw\\ &+\frac{1}{c_1^2-c_2^2}\left[\frac{2\rho_s\rho_n}{\rho^2}-\rho\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}\right]\frac{\rho c_2^2}{\rho_s\sigma}w\,d\sigma'.\end{aligned} \tag{10}$$

Let us at this stage take the basic step of perturbation theory, i.e., let us substitute the lower-order iterations into the higher-order ones. In the linear case the requirement that the wave running to the left vanish imposes the following relationships on the variables ρ', v, σ', and w [see (8)]:

$$v=(c_1/\rho)\rho', \quad w=(c_2\rho/\sigma\rho_s)\sigma'. \tag{11}$$

Let us substitute the second of the relations (11) into the nonlinear terms of the expression (10). In doing this we make an error only in the next, third, order in smallness. As is easy to see, the then resulting Pfaffian form is integrable. The Riemann invariants, I_1 and I_2, that then result have the following form:

$$\begin{aligned}I_{1,2}=&\pm\left[\frac{c_1}{\rho}\rho'+\left(\frac{1}{2c_1\rho}\frac{\partial^2 p}{\partial\rho^2}-\frac{c_1}{\rho^2}\right)\frac{\rho'^2}{2}\right]+v\\ &+\left[\frac{2\rho_s\rho_n}{\rho^2}-\rho\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}\right]\frac{\rho c_2}{\rho_s\sigma}\frac{c_2\pm c_1}{c_1^2-c_2^2}\frac{\sigma'^2}{2}.\end{aligned} \tag{12}$$

Let us, consistently continuing the proposed scheme, require that the wave running to the left vanish, i.e., let us set $I_2(\rho',v,\sigma')=0$. This leads to the following relation connecting the quantities v, ρ', and σ':

$$v=(c_1/\rho)\rho'+\gamma_1\rho'^2+\gamma_2\sigma'^2. \tag{13}$$

Thus, the condition for the absence of a wave propagating to the left establishes a strict relationship between the quantities ρ', σ', and v. In the linear case there is no functional relationship between the velocity v and the entropy perturbation σ', which indicates the independence of the wave modes. Using the relation (13), we can express the invariant I_1 in terms of the quantities ρ' and σ':

$$I_1=\rho'+\alpha_1\rho'^2+\alpha_2\sigma'^2. \tag{14}$$

In the formulas (12) and (13) we have, for brevity, introduced the following notation:

$$\begin{aligned}\alpha_1&=\frac{1}{2}\left(\frac{1}{2c_1^2}\frac{\partial^2 p}{\partial\rho^2}-\frac{1}{\rho}\right),\\ \alpha_2&=\frac{1}{2}\frac{\rho}{c_1}\left(\frac{\rho c_2}{\rho_s\sigma}\right)^2\frac{2c_1}{c_1^2-c_2^2}\left[\frac{2\rho_s\rho_n}{\rho^2}-\rho\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}\right],\\ \gamma_1&=\frac{1}{2}\left(\frac{1}{2c_1\rho}\frac{\partial^2 p}{\partial\rho^2}-\frac{c_1}{\rho}\right),\\ \gamma_2&=-\frac{1}{2}\left(\frac{\rho c_2}{\rho_s\sigma}\right)^2\frac{c_2-c_1}{c_1^2-c_2^2}\left[\frac{2\rho_s\rho_n}{\rho^2}-\rho\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}\right].\end{aligned} \tag{15}$$

Thus, we have realized the first part of the proposed scheme. We have found the quantity $I_1(\rho',\sigma')$ characterizing the wave propagating to the right along the x axis in accordance with Eq. (6) for the case $\mu = 1$. Here also there occur velocity (v) perturbations, but they are not arbitrary, but depend on ρ' and σ'.

Let us proceed to consider the evolution of the second-sound waves, i.e., the waves propagating along the characteristic $\xi^{(3)}$. In this case the corresponding Pfaffian form is not integrable, and it is not possible to obtain the Riemann invariant directly.

Let us now use the other property of the running waves, namely, the existence of a functional relationship between the variables. Let us, as suggested above, choose $v = v(\rho'\sigma')$ and $w = w(\rho',\sigma')$, using for the function $v(\rho',\sigma')$ the already found dependence (13). Let us substitute the quantities $v(\rho',\sigma')$ and $w(\rho',\sigma')$ into the basic equations (1), writing the derivatives of the type $\partial w/\partial t$ in the form $(\partial w/\partial\rho')(\partial\rho'/\partial t)+(\partial w/\partial\sigma')(\partial\sigma'/\partial t)$, etc. As a result we obtain the following equation:

$$\begin{aligned}&\frac{\partial\rho'}{\partial t}+(\rho+\rho')\left\{\frac{c_1}{\rho}\frac{\partial\rho'}{\partial x}+2\gamma_1\rho\frac{\partial\rho'}{\partial x}+2\gamma_2\sigma'\frac{\partial\sigma'}{\partial x}\right\}+\frac{c_1}{\rho}\rho'\frac{\partial\rho'}{\partial x}=0,\\ &\frac{c_1}{\rho}\frac{\partial\rho'}{\partial t}+2\gamma_1\rho'\frac{\partial\rho'}{\partial t}+2\gamma_2\sigma'\frac{\partial\sigma'}{\partial t}+A_{21}\frac{\partial\rho'}{\partial x}+\frac{c_1^2}{\rho}\frac{\partial\rho'}{\partial x}+A_{24}w_\sigma\frac{\partial\sigma'}{\partial x}=0,\\ &\frac{\partial\sigma'}{\partial t}+A_{31}\frac{\partial\rho'}{\partial x}+A_{33}\frac{\partial\sigma'}{\partial x}+A_{34}w_\rho\frac{\partial\rho'}{\partial x}+A_{34}w_\sigma\frac{\partial\sigma'}{\partial x}=0,\\ &w_\rho\frac{\partial\rho'}{\partial t}+w_\sigma\frac{\partial\sigma'}{\partial t}+\frac{c_1}{\rho}A_{42}\frac{\partial\rho'}{\partial x}+A_{43}\frac{\partial\sigma'}{\partial x}+A_{44}w_\sigma\frac{\partial\sigma'}{\partial x}=0.\end{aligned} \tag{16}$$

The equations (16) constitute a homogeneous system of algebraic equations for the quantities $\partial\rho'/\partial t$, $\partial\rho'/\partial x$, $\partial\sigma'/\partial t$, and $\partial\sigma'/\partial x$. The condition for the existence of nontrivial solutions, namely, the equality to zero of the determinant, allows us to determine the function $w(\rho',\sigma')$. But this condition is an extremely complicated partial differential equation. Here we can proceed in the following manner. Since we wish to retain in the final equations the terms of order—in smallness—not higher than the second, let us require that the dependence of the function $w(\rho',\sigma')$ on its arguments be not more complicated than the quadratic dependence. The most general form of such a dependence is the following:

$$w=\mu\sigma'+X\sigma'^2+Y\sigma'\rho'+Z\rho'^2. \tag{17}$$

There is no term linear in ρ' in this formula, since in the limiting case of small amplitudes the relative velocity does not depend on the density perturbations: the acoustic modes have been completely uncoupled. Let us substitute the relation (17) into the determinant of the system of equations (16). From the requirement that it be equal to zero we, can assuming that ρ' and σ' are linearly independent of each other, determine the coefficients μ, X, Y, and Z. There, however, arises here a difficulty of the following kind: In the zeroth- and first-order approximations the determinant identically vanishes for any function $w(\rho',\sigma')$. The reason for this is that the first two equations of the system (16) cease to be linearly independent the moment we use for the function $v(\rho',\sigma')$ the dependence (13), which is always fulfilled in the case of first sound. This can be verified directly by multiplying the first equation by $-c_1/\rho$ and adding the resulting equation to the second equation. (In doing this we should set in the linear terms $\sigma'_t=-c_2\sigma'_x$ and $\rho'_t=-c_1\rho'_x$: this gives rise to an error only in the next, third, order.) From the mathematical standpoint we are dealing with a situation in which it is necessary for the consistency of the system of equations in question that the rank of the matrix be equal not to three, but to two.[3] Practically, we should proceed here in this way. We should eliminate any of the first two equations of the system (16) by expressing, for example, $\partial\rho'/\partial t$ in terms of the remaining variables and then requiring that the remaining three equations also have, a null determinant. Equating the zeroth- and first-order terms in the determinant, we obtain expressions for μ, X, Y, and Z. Let us, omitting the simple, but tedious calculations, write out the final answer:

$$w=\frac{c_2\rho}{\sigma\rho_s}\sigma'+\frac{\rho}{4\sigma\rho_s\mu}\left[A_{43}{}^{\sigma}+(A_{44}{}^{w}+A_{33}{}^{w}+A_{34}{}^{\sigma})\mu^2\right]\sigma'^2$$
$$+\frac{\rho}{2\sigma\rho_s\mu}\left[A_{43}{}^{\rho}+(A_{44}{}^{v}-A_{33}{}^{v})\frac{c_1}{\rho}\mu-A_{34}{}^{\rho}\mu^2\right]\rho'\sigma'. \tag{18}$$

Here we have, for brevity, used the following notation: the $A_{ij}^{\varphi_k}$ denote the coefficients standing in front of the φ_k variables in the terms of the A_{ij}-matrix elements. The quantity $\mu=c_2\rho/\sigma\rho_s$, which corresponds to a transition to the linear case [see (8)].

Thus, we have obtained from the quantity w the functional ρ' and σ' dependence that is fulfilled in a wave running to the right. In order to obtain the evolutional equation for this wave, i.e., to determine the invariant I_3, we shall use the found dependence $w(\rho',\sigma')$, (18), as well as the earlier-found dependence $v(\rho',\sigma')$, (13), for the integration of the Pfaffian form $\Sigma_j l_j^{(3)}(\varphi)d\varphi_j$.

The explicit expression for this form will be the following:

$$\frac{w}{c_2^2-c_1^2}\left[\frac{c_2^2}{\rho_s}\frac{\partial\rho_s}{\partial\rho}+\frac{c_1^2}{2\rho}\left(1-\frac{\rho^2}{\rho_n}\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}\right)\right]d\rho$$
$$+\frac{wc_1}{c_2^2-c_1^2}\left[1-\frac{\rho^2}{\rho_n}\frac{\partial}{\partial\rho}\frac{\rho_n}{\rho}+\frac{\rho}{\rho_s}\frac{\partial\rho_s}{\partial\rho}\right]dv$$
$$+\left\{\frac{c_2\rho}{\rho_s}-w\left[\left(\frac{\sigma}{\rho_s}-\frac{\sigma}{\rho_n}\right)\frac{\partial\rho_s}{\partial\sigma}-\frac{\rho_s}{\rho}\right]+\left[\frac{A_{43}{}^{\sigma}}{c_2}-\xi_{\sigma}^{(3)}\frac{\rho}{\rho_s}\right]\sigma'\right.$$
$$\left.+\left[\frac{A_{43}{}^{\rho}}{c_2}-\xi_{\rho}^{(3)}\right]\rho'\right\}d\sigma'+dw. \tag{19}$$

Here, as before, the $A_{ij}^{\varphi_k}$ denote the coefficients standing in front of the quantities φ_k in the A_{ij} matrix and, similarly the $\xi_{\varphi_k}{}^{(3)}$ denote the coefficients in the expression (5) for the characteristic $\xi^{(3)}$.

If now in the expression (19) we replace v and w by the above-found functional relations $v(\rho',\sigma')$ and $w(\rho',\sigma')$, (13) and (18), we obtain in expression of the following form:

$$M(\rho',\sigma')\,d\rho'+N(\rho',\sigma')\,d\sigma'=\Pi_3. \tag{20}$$

Here M and N are certain expressions whose dependence on their arguments is not more complicated than the linear dependence.

Thus, we have been able to reduce the number of variables in the Pfaffian form from four to two. As has already been noted above, in the case of two variables the Riemann invariant is the solution to the differential equation $\Pi_3=0$. We omit the trivial computations connected with the solution of this equation, and write out the final answer:

$$I_3=\sigma'+\beta_1\sigma'^2+\beta_2\sigma'\rho'. \tag{21}$$

Here

$$\beta_1=\frac{1}{4}\left[\frac{\partial^2T}{\partial\sigma^2}\left(\frac{\partial T}{\partial\sigma}\right)^{-1}+\frac{\partial}{\partial\sigma}\left(\frac{\rho_n}{\rho}\right)\frac{2\rho_n}{\rho_s}\right],$$
$$\beta_2=\frac{1}{4}\left\{\frac{1}{\rho}\frac{c_1+c_2}{c_1-c_2}-\frac{c_1^2c_2}{c_1-c_2}\left(\frac{\rho}{\rho_s}-\frac{\rho}{\rho_n}\right)\frac{\partial}{\partial p}\frac{\rho_s}{\rho}\right\}. \tag{22}$$

Let us summarize the results obtained in this section. The evolution of waves propagating in one direction (simple waves) is, up to terms of second order in the deviations from equilibrium, governed by the following system of equations:

$$\frac{\partial I_1}{\partial t}+\xi^{(1)}(I_1,I_3)\frac{\partial I_1}{\partial x}=0,\quad \frac{\partial I_3}{\partial t}+\xi^{(3)}(I_1,I_3)\frac{\partial I_3}{\partial x}=0. \tag{23}$$

The quantities I_1 and I_3 are Riemann invariants, connected with the entropy and density perturbations, σ' and ρ' respectively, by the relations (14) and (21). The characteristics $\xi^{(1)}$ and $\xi^{(2)}$ can be expressed in terms of the quantities I_1 and I_3 with the aid of the formulas (13), (14), (21), and (22).

Let us draw attention to the nonsymmetric dependence of the invariants I_1 and I_3 on the perturbations ρ' and σ' (see (14) and (21)). This asymmetry is a consequence of the fact that the thermodynamic variables depend on the relative velocity $\mathbf{w}=\mathbf{v}_n-\mathbf{v}_s$ characterizing the second sound, but of course do not depend on the mean-mass velocity $\mathbf{v}=\mathbf{j}/\rho$.

2. PROPAGATION OF THE NONLINEAR PERTURBATIONS; DECAY OF AN ENTROPY PULSE

With the aid of the equations (23) we can investigate the problem of wave propagation from a wall, at which we prescribe entropy and density perturbations. We can also solve the initial-value problem, but in this case, as in ordinary gas dynamics, we require that the initial conditions be not arbitary, but connected by the relations (13) and (18).

As examples, let us consider the evolution of waves in the following two cases: a) we produce at the wall ($x = 0$) a density perturbation (we move a piston); b) we produce at the wall ($x = 0$) an entropy perturbation (we carry out pulsed heating).

Before proceeding to solve the formulated problem, it is useful to explicitly express the dependence of the perturbations ρ' and σ' on the invariants I_1 and I_3. Up to terms quadratic in the quantities I_1 and I_3 the sought dependences will have the following form:

$$\rho' = I_1 - \alpha_1 I_1^2 - \alpha_2 I_3^2, \tag{24}$$

$$\sigma' = I_3 - \beta_1 I_3^2 - \beta_2 I_1 I_3. \tag{25}$$

Here α_1, α_2, β_1, and β_2 are the same quantities figuring in the formulas (14) and (21).

In the case a) we have at the boundary the relations $\rho'(0,t) = \rho_0(t)$ and $\sigma' = 0$. In this case we have, in accordance with the expressions (14) and (21) for the invariants, only one invariant I_1, with $I_3 = 0$. This invariant, which is conserved on the characteristic $\xi^{(1)}$, describes a wave traveling with velocity close to c_1. As can be seen from the inversion formulas (24) and (25), only the density perturbation ρ' occurs in this wave, the entropy perturbation σ' being equal to zero. Thus, in the case when density perturbations ρ' are produced at the liquid boundary there propagate in the interior of the helium only density (and, of course, pressure, as well as mean-mass velocity) waves.[4]

In the case b) the situation is somewhat more interesting. If we produce at the wall entropy perturbations with amplitude of the order of σ_0' (here $\rho' = 0$), then, as can easily be seen from the formulas (13) and (19), both of the invariants I_1 and I_3 are nonzero. Since the waves connected with the invariants I_1 and I_3 propagate with different velocities $\xi^{(1)}$ and $\xi^{(3)}$, they separate in the (x,t) plane, as shown in Fig. 1. The wave in which only the invariant I_1 is nonzero moves ahead with velocity $\xi^{(1)}$ close to c_1. Behind it travels the wave in which only the invariant I_3 is nonzero. Related to the invariant I_1 is the density perturbation ρ', which is, in order of magnitude, equal to $\rho' = I_1 \approx \alpha_2 \sigma_0'^2$ (see the inversion formulas (24) and (25). On the other hand, the quantity I_3 is connected with both the density perturbation $\rho' = -\alpha_2 \sigma_0'^2$ and the entropy perturbation $\sigma' = \sigma_0' + \beta_1 \sigma_0'^2$.

Thus, the following picture should be observed in the

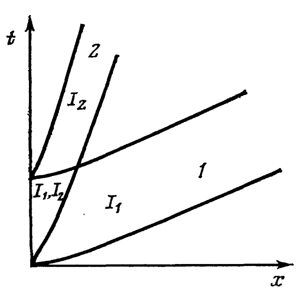

FIG. 1. Diagrammatic representation of the evolution of an entropy pulse in the (x,t) plane. The strip: 1) the region of propagation of the invariant I_1 (its slope $dx/dt \approx \xi^{(1)}$); 2) the region of propagation of the invariant I_3 (its slope $dx/dt \approx \xi^{(3)}$).

case of nonstationary heating of the wall. A precursor—a density wave—travels from the wall with the velocity of first sound. Behind it a "mixture" of density and entropy waves travels with velocity equal to the velocity of second sound.

Let us estimate the magnitude of the above-described effect. It can be seen from the expression (14) for the invariant I_1 and the inversion formula (24) that the extent of the conversion of the second sound into the precursor is determined by the coefficient α_2. If we characterize the entropy pumping by the quantity $w_0 = (c_2 \rho / \sigma \rho_s) \sigma_0'$, then the pressure perturbation in the precursor will be given by the following expression:

$$\delta p_{\rm pr} \approx \frac{1}{2} \left[\frac{2\rho_s \rho_n}{\rho} - \rho \frac{\partial}{\partial \rho} \frac{\rho_n}{\rho} \right] w_0^2. \tag{26}$$

Numerical estimates show that the second term in the square brackets is always greater than the first (cf. Ref. 10). For a heat pulse with amplitude of the order of 10 W/cm^2 and a temperature $T \approx 2$ K the pressure perturbation in the precursor $\delta p_{\rm pr} \approx 10^4$ g/cm·sec^2.

Notice that the pressure in the rear pulse differs from $\delta p_{\rm pr}$. Indeed, there exists in the rear sound-mixture pulse, in contrast to the precursor, a relative velocity, as result of which there will be a $(\partial p / \partial w^2)_{\rho,\sigma}$-related addition in the pressure. Furthermore, we must take the following circumstance into consideration. Since the second-sound velocity c_2 is significantly lower than the first-sound velocity, the nonlinear corrections to the velocity (see the expressions (4) and (5) for the characteristics $\xi^{(\mu)}$ change c_2 more drastically, i.e., $\Delta c_2 / c_2 \gg \Delta c_1 / c_1$. As a result, the rear pulse changes its shape more rapidly, forms a shock front and, as a consequence, attenuates rapidly. It is to be expected, therefore, that the pressure in the rear pulse will be significantly lower than the pressure in the precursor.

[1]As has already been noted, allowance for the thermal expansion also gives rise to interaction between the pressure and entropy perturbations. But for high amplitudes, specifically, for pulses of power $W \gtrsim 1$ W/cm^2, the interaction due to the nonlinear effects will predominate.

[2]In Ref. 6 it is suggested, by analogy with ordinary gas dynamics, that all the sought variables depend on one parameter, i.e., that ρ', v, σ', and w are functions of some parameter $l(x,t)$. This hypothesis does not fit the transition to the linear case; for then the quantities ρ' and v depend on the argument $x - c_1 t$, while the variables σ' and w should depend on $x - c_2 t$, and, consequently, all of them cannot be functions of the same parameter.

[3]In point of fact a similar situation obtains in the linear approximation. The matrix formed by the coefficients in this case has the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where A_1 and A_2 are 2×2 square matrices. The linear relations of the type (8) are also found from the requirements that the rank of the grand matrix be equal to two.

[4]To avoid any misunderstanding, let us make the following point. The nonlinear conversion of first sound into second sound is found in Refs. 1 and 2. There is, however, no contradiction here with the result obtained by us. The point is that, in the cited papers, by conversion is meant the instability of an initial high-power first-sound wave against small second-sound-mode perturbations. In the present paper we do not touch upon the question of stability of the solutions obtained.

[1]V. L. Pokrovskiĭ and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **71**, 1974 (1976) [Sov. Phys. JETP **44**, 1036 (1976)].
[2]N. I. Pushkina and R. V. Khokhlov, Pis'ma Zh. Eksp. Teor. Fiz. **19**, 672 (1976) [JETP Lett. **19**, 348 (1976)].

[3]S. K. Nemirovskiĭ, Fiz. Nizk. Temp. **4**, No. 1 (1978) [Sov. J. Low Temp. Phys. **4**, No. 1 (1978)].
[4]S. K. Nemirovskiĭ, Fiz. Nizk. Temp. **4**, No. 10 (1978) [Sov. J. Low Temp. Phys. **4**, No. 10 (1978)].
[5]I. M. Khalatnikov, Teoriya sverkhtekuchesti (Theory of Superfluidity), Nauka, Moscow, 1971.
[6]I. M, Khalatnikov, Dokl. Akad. Nauk SSSR **79**, 237 (1951).
[7]D. V. Osborne, Proc. Phys. Soc. **64**, 114 (1951).
[8]S. K. Nemirovskiĭ, Preprint No. 08-76, Inst. Teor. Fiz., Novosibirsk, 1976.
[9]S. Kitabatake and S. Yasuji, J. Phys. Soc. Jpn. Lett. **45**, 345 (1978).
[10]S. Putterman and S. Garret, J. Low Temp. Phys. **27**, 543 (1977).
[11]B. L. Rozhdestvenskiĭ and N. N. Yanenko, Sistemy kvazilineĭnykh uravneniĭ (Systems of Quasilinear Equations), Nauka, Moscow, 1968.
[12]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnoĭ sredy (Fluid Mechanics), Gostekhizdat, Moscow, 1954 (Eng. Transl., Pergamon, Oxford, 1959).

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