On the theory of vortices in a plasma

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We study the problem of the existence, in a plasma in a magnetic field, of nonlinear solitary waves similar to the solitary Rossby wave vortices in a rotating fluid. We formulate a prescription for a theoretical search for plasma vortices and exhibit a broad class of such vortices. We classify them as one- and two-potential vortices. The one-potential vortices which are similar to hydrodynamic vortices are described by a single scalar function which is the potential of the transverse electric field. The two-potential vortices are described by two scalar functions. They are connected with non-electrostatic (electromagnetic) branches of plasma oscillations. We discuss the problem of the spatial structure of the one-potential vortices and the features of the two-dimensional differential equation describing such vortices. We show that in the case of dipole vortices this equation reduces to a one-dimensional one for the radial part of the potential.

1. INTRODUCTION

Some time ago Larichev and Reznik[1] considering the problem of Rossby waves in a rotating fluid studied the nonlinear equation

$$\hat{D}_0\Delta_\perp\chi=\Lambda\partial\chi/\partial\eta. \tag{1.1}$$

Here χ is a function characterizing waves in the two-dimensional space $(x,\eta=y-ut)$ and is usually called the flux function, u is the phase velocity of the wave along y, $\Lambda(u)$ is a constant, $\Delta_\perp\equiv\partial^2/\partial x^2+\partial^2/\partial\eta^2$, and the operator $\hat{D}_0$ is given by the relation

$$\hat{D}_0=\partial/\partial\eta+[\nabla\chi,\ \nabla]_z. \tag{1.2}$$

The index z indicates the z-component of the vector product. The authors of Ref. 1 showed that Eq. (1.1) has a solitary wave type solution called by them a two-dimensional solitary Rossby wave. Because of the vortex nature of the motion of the fluid in such a wave the solution of Ref. 1 can be called a solitary Rossby wave vortex.

Hasegawa and Mima[2] studying the so-called oblique $(\partial/\partial z\neq 0)$ drift waves (or simply—drift waves) in a nonuniform plasma in a magnetic field found that such waves are also described by an equation such as (1.1). Hasegawa *et al.*[3] turned attention to the analogous problem of nonlinear drift and Rossby waves. Using these observations in Ref. 3 and the approach of Ref. 1 Meiss and Horton[4] showed the possibility of the existence of solitary oblique drift wave vortices analogous to the solitary Rossby wave vortices of Ref. 1. The authors of Ref. 4 assumed in that case that the field of the wave depended on the variables x, $\eta=y-ut+\alpha z$, where α is a constant so that in their case $\Lambda=\Lambda(u,\alpha)$. Pavlenko and Petviashvili[5] using an approach close to the one of Ref. 1 studied nonlinear channelled $(\alpha=0)$ waves in a nonuniform plasma in crossed magnetic and gravitational fields (normally a gravitational field is introduced in the problem of waves in a plasma to model the effects of magnetic field curvature[6]). The authors of Ref. 5 showed that such waves can also have the form of solitary vortices of the Rossby wave vortices kind.

A characteristic property of an equation such as (1.1) is that through integration it can be reduced to a linear equation with an integration constant (for details see section 3). In this connection we shall call the vortices described by equations such as (1.1) quasilinear (QL) vortices. The necessity to introduce some special term for the kind of vortices which we discuss is connected with the fact that such vortices do not exhaust all possible kinds of solitary waves with vortex characteristics of the particle motion. Examples of solitary vortices described by nonlinear equations which are in an essential way different from (1.1) are, in particular, found in Refs. 7 and 8.

In the present paper we want to draw attention to the fact that the class of plasma perturbations leading to the QL vortex problem is a very broad one and that in that sense the question of QL vortices is a fundamental one for the nonlinear theory of both nonuniform and uniform plasmas in a magnetic field. Moreover, we show that QL vortices may be met with in problems described not only by equations such as (1.1) but also by nonlinear equations (or sets of nonlinear equations) of a more complicated structure.

Characteristic for Eq. (1.1) (and also for all equations of this nature studied by us) is the presence of a term of the type $\hat{D}_0\Delta_\perp\chi$. This fact is one of the main pointers for developing actual plasma problems which reduce to the QL vortex problem. It is therefore useful to trace the "pedigree" of this term. To do this we turn to the equation for the two-dimensional motion (in the x,y-plane) of one of the plasma components (electrons or ions) assuming this component to be "cold" (with zero temperature) and collisionless:

$$Md\mathbf{V}/dt=e(\mathbf{E}+[\mathbf{V}\times\mathbf{B}_0]c/). \tag{1.3}$$

Here $d/dt=\partial/\partial t+\mathbf{V}\cdot\nabla$, $\mathbf{V}$ is the velocity, $\mathbf{E}$ the electric field of the wave, $\mathbf{B}_0\|\mathbf{z}$ the static magnetic field; e,M the charge and mass of the particles, c the velocity of light. Assuming that the motion is a low-frequency and long-wavelength one, $d/dt\ll eB_0/Mc=\omega_B$ (ω_B is the cyclotron frequency) we get from (1.3) an approximate equation for the velocity:

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$$\mathbf{V}=\mathbf{V}_E+\mathbf{V}_I. \tag{1.4}$$

Here $\mathbf{V}_E = c\mathbf{E}\times\mathbf{e}_z/B_0$ is the drift velocity of the particles in crossed electric and magnetic fields, $\mathbf{e}_z$ is the unit vector in the z-direction, $\mathbf{V}_I$ is the inertial particle velocity across the magnetic field given by the relation

$$\mathbf{V}_I=\frac{1}{\omega_B}\left[\mathbf{e}_z,\left(\frac{\partial}{\partial t}+\mathbf{V}_E\nabla\right)\mathbf{V}_E\right]. \tag{1.5}$$

Assuming the electric field to be a potential (electrostatic) one, $\mathbf{E} = -\nabla\varphi$ and substituting (1.4), (1.5) into the continuity equation for the corresponding plasma component,

$$dn/dt+n\operatorname{div}\mathbf{V}=0; \tag{1.6}$$

(n is the plasma density) we note that the quantity div$\mathbf{V}$ occurring in that equation is equal to

$$\operatorname{div}\mathbf{V}=-\frac{Mc^2}{eB_0^2}\left(\frac{\partial}{\partial t}+\frac{c}{B_0}[\nabla\varphi,\nabla]_z\right)\Delta_\perp\varphi. \tag{1.7}$$

It is clear that in the case of waves depending on x and $\eta = y - ut$ the right-hand side of (1.7) apart from a change in notation is nothing but the quantity $\hat{D}_0\Delta_\perp\chi$ in Eq. (1.1). Indeed, the kind of plasma waves which are of interest to us must at least have the following properties:

1) they must be low-frequency and long-wavelength waves at least as far as one of the plasma components (ions or electrons) are concerned;

2) in general, for such waves the transverse inertia of the corresponding plasma component must be important;

3) the electric field of the waves at right angles to the equilibrium magnetic field must be, at least approximately, potential.

The totality of these properties also is the above-noted pointer.

Yet another important pointer for a theoretical search for vortices is the fact that in the external region of the vortex $\Delta_\perp\chi = \varkappa^2\chi$ where $\varkappa$ is a real constant.[1] For this region of the vortex it thus follows from an equation such as (1.1) that

$$\Lambda(u,\alpha)=\varkappa^2. \tag{1.8}$$

This kind of relation is formally the same as the dispersion relation of the linear approximation for waves with frequency ω and wave vector k part from the substitution

$$\varkappa^2\to-k_\perp^2,\quad u\to\omega/k,\quad \alpha\to k_z/k_y. \tag{1.9}$$

In this connection one may call a relation such as (1.8) a modified dispersion equation (MDE). It is clear from (1.9) that only such waves in the plasma are of interest for our problem for which

$$\Lambda(u,\alpha)>0. \tag{1.10}$$

This inequality is the second of the above-mentioned pointers.

We illustrate the use of these two pointers by the example of purely electron waves. It is well known[9] that in a uniform plasma with a sufficiently high density ($\omega_{pe}^2 > \omega_{Be}^2$) in a magnetic field there is a branch of electrostatic low-frequency waves propagating almost at right angles to the magnetic field and being described by the dispersion equation $\omega^2 = \omega_{Be}^2 k_z^2/k_\perp^2$, where $\omega_{Be} = -eB_0/mc$ is the electron cyclotron frequency, $\omega_{pe}^2 = 4\pi e^2 n_0/m$ the square of the electron plasma frequency, n_0 the equilibrium plasma density; $-e,m$ the electron charge and mass. Such waves have the above-mentioned properties 1) to 3), but for them $\Lambda = -\alpha^2\omega_{Be}^2/u^2 < 0$. QL vortices are therefore not realized on this branch. When there is a gradient present of the equilibrium plasma density, $\partial n_0/\partial x \neq 0$, instead of the above-mentioned dispersion equation we have the following one:[6]

$$\omega^2+\omega k_y\varkappa_n\omega_{Be}/k_\perp^2=\omega_{Be}^2k_z^2/k_\perp^2, \tag{1.11}$$

where $\varkappa_n = \partial \ln n_0/\partial x$. In that case

$$\Lambda=(\alpha\omega_{Be}/u)^2(u\varkappa_n/\alpha^2\omega_{Be}-1). \tag{1.12}$$

It is clear that when α is sufficiently small (or $\varkappa_n$ sufficiently large) and u has the appropriate sign the condition $\Lambda > 0$ can be realized which is necessary for the existence of a QL vortex. Using (1.6) and the fact that when the above mentioned condition $\omega_{pe}^2 > \omega_{Be}^2$ holds the waves are quasineutral, $\tilde{n} = 0$ ($\tilde{n}$ is the perturbation of the electron density) we find that such a vortex is described by an equation of the form

$$\varkappa_n V_x+\operatorname{div}\mathbf{V}=0. \tag{1.13}$$

Substituting here $V_x = V_{Ex}$ (see (1.4)) and div$\mathbf{V}$ of the form (1.7) with the change $M\to m$, $e\to -e$ and assuming the wave to depend on x and $\eta = y - ut + \alpha z$ we bring (1.13) to the form

$$\hat{D}\Delta_\perp\varphi=\Lambda\partial\varphi/\partial\eta, \tag{1.14}$$

where Λ is given by Eq. (1.12) while the operator $\hat{D}$ is given by the relation

$$\hat{D}=\frac{\partial}{\partial\eta}-\frac{c}{uB_0}[\nabla\varphi\times\nabla]_z. \tag{1.15}$$

The formal identity of Eqs. (1.14), (1.15) with Eqs. (1.1), (1.2) indicates the constructive nature of the above indicated pointers.

In the light of what has been said the situation about the channelled vortex considered in Ref. 5 looks as follows. Channelled perturbations possess the properties 1) to 3). The dispersion equation of such perturbations has the form[6]

$$\omega^2=-\Gamma^2k_y^2/k_\perp^2, \tag{1.16}$$

where $\Gamma^2 = -g\varkappa_n$ is the square of the growth rate of the channelled instability, and $\mathbf{g}\|\mathbf{x}$ is the gravity force. From (1.16) and the correspondence rule (1.9) it follows that

$$\Lambda=\Gamma^2/u^2, \tag{1.17}$$

so that for $\Gamma^2 > 0$ condition (1.10) is satisfied.

The examples considered also indicate the important role of the nonuniformity of the plasma in the QL vortex problem. This role consists in that non-uniformity modifies "nonvortex" types of waves (in the context discussed here) into "vortex" waves or it leads to new types of waves which have "vortex" properties. The physical cause for this are the so-called gradient or drift effects—the same which lead to gradient or drift instabilities.[6]

It does, however, not follow from this that QL vortices are possible only in a nonuniform plasma. We consider, for instance, ion-sound waves with finite $k_\perp\rho_i$ (ρ_i is the ion Larmor radius). Qualitatively the dispersion equation of such waves can be written in the form

$$\omega^2=k_z^2c_s^2(1-k_\perp^2\rho_i^2), \tag{1.18}$$

where $c_s^2 = T_e/M$ is the square of the ion sound velocity, T_e the electron temperature, M the ion mass, $(k_\perp \rho_i)^2 \ll 1$. In that case

$$\Lambda=(u^2/\alpha^2c_s^2-1)/\rho_i^2, \tag{1.19}$$

so that $\Lambda > 0$ when $u > \alpha c_s$. From this it is clear that there is the possibility of ion sound QL vortices with finite $k_\perp \rho_i$. The analysis by Meiss and Horton[4] also confirms this.

The general considerations given here reveal, in our opinion, a wide field of activity for a theoretical analysis of QL vortices in nonuniform and uniform plasmas and they must stimulate the corresponding experimental studies.

The first thing which is necessary for such an analysis is a summary of information about the basic types of QL vortices. Such a summary of information with an exposition of the derivation of the equations which describe the corresponding types of vortices is given in section 2. Vortices described by the equations of section 2 can be split into two classes: one- and two-potential ones. One-potential vortices (which can be either electrostatic or electromagnetic) like the Rossby wave vortices[1] are characterized by a single scalar function which satisfies an equation like (1.14) with some expression for Λ. The general properties of the vortices described by these equations are discussed in section 3. The two-potential vortices are electromagnetic in an essential way. They are characterized by two scalar functions (two potentials) describing the electromagnetic field of the vortex. Some examples of two-potential vortices are discussed in section 4.

The results of this paper are discussed in section 5.

2. STARTING EQUATIONS FOR THE BASIC TYPES OF QUASI-LINEAR VORTICES

2.1 Low-frequency long-wavelenth vortices in a plasma with $T_i = 0$

We assume that the equilibrium state of the plasma is characterized by a density n_0, which is nonuniform along x ($\nabla n_0 \| \mathbf{x}$), a uniform electron temperature ($\nabla T_e = 0$), and a vanishing ion temperature ($T_i = 0$). We assume the equilibrium magnetic field $\mathbf{B}_0$ to be uniform and directed along $z(\mathbf{B}_0\|\mathbf{z})$. Moreover, we assume that there gravity acts on the plasma, $\mathbf{g}\|\mathbf{x}$.

Assuming the plasma pressure to be small compared to the pressure of the equilibrium magnetic field, $8\pi n_0 T_e/B_0^2 \ll 1$, we characterize the electric field of the perturbations by the quantities φ and ψ given by the relations

$$\mathbf{E}_\perp=-\nabla_\perp\varphi, \quad E_z=-\partial\psi/\partial z, \tag{2.1}$$

where the index $\perp$ indicates the vector components at right angles to the field $\mathbf{B}_0$. We assume the perturbations to depend on x and $\eta = y - ut + \alpha z$ (cf. section 1). We then find from the Maxwell equation $\partial \mathbf{B}_\perp/\partial t = -c\,\mathrm{curl}_\perp \mathbf{E}$ ($\mathbf{B}_\perp$ is the magnetic field of the perturbations)

$$\mathbf{B}_\perp=\alpha c[\nabla(\varphi-\psi)\times\mathbf{e}_z]/u. \tag{2.2}$$

Assuming the perturbations to be quasineutral we use the equation for the closing of the current $\mathbf{j}$, $\mathrm{div}\,\mathbf{j} = 0$, which we write in the form

$$\mathrm{div}\,\mathbf{j}_\perp+\partial j_z/\partial z=0. \tag{2.3}$$

Using the Maxwell equation $\mathrm{curl}_z \mathbf{B}_\perp = 4\pi j_z/c$ and Eq. (2.2) we find

$$j_z=-\frac{\alpha c^2}{4\pi u}\Delta_\perp(\varphi-\psi). \tag{2.4}$$

The equations given here are supplemented by the equations of two-fluid magnetohydrodynamics, namely the equations of motion for the ions and electrons (cf. (1.3)):

$$M d\mathbf{V}_i/dt=e(\mathbf{E}+[\mathbf{V}_i\times\mathbf{B}]/c)+M\mathbf{g}, \tag{2.5}$$

$$0=-T_e\nabla n-e(\mathbf{E}+[\mathbf{V}_e\times\mathbf{B}]/c) \tag{2.6}$$

and the electron equation of continuity (cf. (1.6))

$$\partial n/\partial t+\mathrm{div}\, n\mathbf{V}_e=0. \tag{2.7}$$

Here $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_\perp$ is the total magnetic field, $n = n_0 + \tilde{n}$ the total density, and $\tilde{n}$ the density perturbation; $\mathbf{V}_i$ and $\mathbf{V}_e$ the electron and ion velocities, $d/dt = \partial/\partial t + \mathbf{V}_i\nabla$; e and M the ion charge and mass. Using (2.5), (2.6) we evaluate $\mathbf{j}_\perp$. Substituting the result into (2.3) and using (2.4) we get

$$\hat{D}\Delta_\perp\varphi=\frac{\alpha^2c_A^2}{u^2}\hat{D}_1\Delta_\perp(\varphi-\psi)+\frac{gB_0}{uc}\frac{\partial}{\partial\eta}\frac{\tilde{n}}{n_0}. \tag{2.8}$$

Here $c_A^2 = B_0^2/4\pi n_0 M$ is the square of the Alfvén velocity, the operator $\hat{D}$ is given by Eq. (1.15), and

$$\hat{D}_1\equiv\frac{\partial}{\partial\eta}-\frac{c}{uB_0}[\nabla(\varphi-\psi),\nabla]_z. \tag{2.9}$$

Similarly, it follows from the equation for the longitudinal motion of the electrons, the electron equation of continuity, and the equation for the longitudinal motion of the ions, respectively, that

$$\hat{D}_1\left(\psi-\frac{T_e}{e}\frac{\tilde{n}}{n_0}\right)=\frac{V_{*e}}{u}\frac{\partial}{\partial\eta}(\psi-\varphi), \tag{2.10}$$

$$\hat{D}\frac{\tilde{n}}{n_0}=\frac{e}{T_e}\frac{V_{*e}}{u}\frac{\partial\varphi}{\partial\eta}+\frac{\alpha^2c^2}{4\pi e n_0 u^2}\hat{D}_1\Delta_\perp(\varphi-\psi)+\frac{\alpha}{u}\hat{D}_1V_{zi}, \tag{2.11}$$

$$\hat{D}\left(V_{zi}-\frac{e\alpha}{Mu}\psi\right)=0. \tag{2.12}$$

Here $V_{*e} = -cT_e\varkappa_n/eB_0$ is the electron drift velocity.

We note a few particular cases of the set of Eqs. (2.8), (2.10) to (2.12), corresponding to QL vortices.

Ion-sound and oblique drift wave vortices for $g = 0$. When $g = 0$, $\alpha \neq 0$, $(c, c_A) \to \infty$ it follows from the set of Eqs. (2.8), (2.10) to (2.12) that

$$\psi=\varphi,\quad \tilde{n}/n_0=e\varphi/T_e,\quad V_{zi}=e\alpha\varphi/Mu, \tag{2.13}$$

so that this set of equations reduces to a single equation for φ. This equation has the form (1.14) with Λ equal to

$$\Lambda=\mu/\rho_0^2, \tag{2.14}$$

$$\mu=1-V_{*e}/u-\alpha^2c_s^2/u^2,\quad \rho_0^2=T_e/M\omega_{Bi}^2. \tag{2.15}$$

Vortices with Λ of the form (2.14) were studied in Ref. 4.

Channelled vortices. When $\alpha = 0$ the set of Eqs. (2.8), (2.10) to (2.12) describes vortices with

$$\psi=V_{zi}=0,\quad \tilde{n}/n_0=-c\varkappa_n\varphi/uB_0. \tag{2.16}$$

In this case we are led to an equation for φ of the form (1.14) where now Λ is given by Eq. (1.17). Vortices with Λ of the form (1.17) were studied in Ref. 5.

Balloon vortices. As $\rho_0^2 \to 0$ and for arbitrary α the expressions for ψ, V_{zi}, and $\tilde{n}/n_0$ are given by Eqs. (2.16), while the equation for φ has the form (1.14) with Λ equal to

$$\Lambda = \Gamma^2/(u^2 - \alpha^2 c_A^2). \qquad (2.17)$$

The corresponding vortices are called balloon vortices.[10] As $\alpha \to 0$ they go over into channelled vortices; see above.

Oblique non-electrostatic vortices for $g = 0$ (coupled Alfvén and drift vortices). In contrast to the ion-sound and oblique drift wave vortices we assume $(\alpha c_A/u)^2$ to be finite. As c_s^2/c_A^2 is small, we can then assume the ratio $\alpha^2 c_s^2/u^2$ to be negligibly small. In the case $g = 0$ it then follows from Eqs. (2.8), (2.10) to (2.12) that

$$V_{zi} = 0, \quad \frac{\tilde{n}}{n_0} = \frac{e}{T_e}\left(\mu_0 \psi + \frac{V_{*e}}{u}\varphi\right), \qquad (2.18)$$

while the functions φ and ψ satisfy the following set of interconnected equations:

$$\hat{D}\Delta_\perp \varphi = \frac{\alpha^2 c_A^2}{u^2}\hat{D}_1 \Delta_\perp(\varphi - \psi), \qquad (2.19)$$

$$\hat{D}(\mu_0 \psi - \rho_0^2 \Delta_\perp \varphi) = 0, \qquad (2.20)$$

where $\mu_0 = 1 - V_{*e}/u$ (cf. (2.15)). The analysis of Eqs. (2.19), (2.20) is given in subsection 4.1

2.2. Low-frequency, long-wavelength vortices with $T_i \neq 0$

We consider how in the approach of subsection 2.1 ion-drift effects which are caused by the finite ion temperature $T_i \neq 0$ can be taken into account. We restrict ourselves to the case $\alpha c_s/u \ll 1$, i.e., we assume that $\mathbf{V}_{zi} = 0$.

When ion-drift effects are taken into account we must instead of (2.5) use the ion equation of motion in the form

$$Mn d\mathbf{V}/dt + \nabla \boldsymbol{\pi} = -\nabla p_i + en(\mathbf{E} + [\mathbf{V}_i \times \mathbf{B}]/c) + Mn\mathbf{g}, \qquad (2.21)$$

where p_i is the ion pressure, $\boldsymbol{\pi}$ is the tensor of the skew ion viscosity. Using the explicit form[6] of $\boldsymbol{\pi}$ we get

$$\operatorname{div} \mathbf{j}_\perp = -\frac{Mc}{B_0}\operatorname{rot}_z \left\{ n\left(\frac{\partial}{\partial t} + \mathbf{V}_E \nabla\right)(\mathbf{V}_E + \mathbf{V}_L) \right\} - g\frac{Mc}{B_0}\frac{\partial \tilde{n}}{\partial \eta} + \frac{\mathbf{B}_\perp}{B_0}\nabla j_z. \qquad (2.22)$$

Here $\mathbf{V}_L = \mathbf{e}_z \times \nabla p_i / Mn\omega_{Bi}$ is the average velocity of the ion Larmor current. We note that one can also obtain (2.22) by using Eqs. (7.44) from Ref. 6.

When we use the approximation $\rho_0^2 \to 0$, when $\psi = 0$, and $\tilde{n}$ is given by Eq. (2.16) it follows from (2.22) that

$$\operatorname{div} \mathbf{j}_\perp = -u\frac{Mc^2 n_0}{B_0^2}\left[\left(1 - \frac{V_{*i}}{u}\right)\hat{D}\Delta_\perp \varphi + \frac{\Gamma^2}{u^2}\frac{\partial \varphi}{\partial \eta}\right] + \frac{\alpha^2 c^3}{4\pi u^2 B_0^2}[\nabla \varphi \times \nabla]_z \Delta_\perp \varphi, \qquad (2.23)$$

where $V_{*i} = cT_i \varkappa_n / eB_0$ is the ion drift velocity (velocity of the equilibrium ion Larmor current). From (2.3), (2.4), and (2.23) there follows a dispersion equation of the form (1.14) with Λ equal to [cf. (1.17), (2.17)]

$$\Lambda = \Gamma^2/(u^2 - uV_{*i} - \alpha^2 c_A^2). \qquad (2.24)$$

The situation considered corresponds to drift-balloon vortices. In the case $\alpha = 0$ such vortices change to drift-channelled ones. The latter were first studied in Ref. 5. In that case the incorrect expression $[\hat{D} - (V_{*i}/u)\partial/\partial\eta]\Delta_\perp \varphi$ was used in Ref. 5 instead of the expression $(1 - V_{*i}/u)\hat{D}\Delta_\perp \varphi$ (see (2.23)).

2.3. Electron vortices

In section 1 we obtained Eq. (1.14) which describes electrostatic electron vortices. We now generalize (1.14) to the case when electromagnetic effects are important.

As in the derivation of Eq. (1.14) we assume that $\omega_{pe}^2 \gg \omega_{Be}^2$. Then $\tilde{n} = 0$ so that we supplement the term with that part of the Lorentz force which is caused by the wave magnetic field $\tilde{\mathbf{B}}$ (cf. (2.5)). Moreover, in contrast to subsection 2.1, we take into account not only $\tilde{\mathbf{B}}_\perp$ but also $\tilde{B}_z$, i.e., the component of the wave magnetic field along the direction of the equilibrium magnetic field $\mathbf{B}_0$. Taking $\tilde{B}_z$ into account is necessary when evaluating the expression $\operatorname{curl}_z \mathbf{E}_\perp$ as the contribution from the potential part of $\mathbf{E}_\perp$ (see (2.1)) vanishes in that expression. In contrast to subsection 2.1 we then obtain

$$\operatorname{div} \mathbf{V}_E = \frac{c}{B_0}\operatorname{rot}_z \mathbf{E}_\perp = -\frac{1}{B_0}\frac{\partial \tilde{B}_z}{\partial t}. \qquad (2.25)$$

We find the field $\tilde{B}_z$ by using the Maxwell equation $\operatorname{curl}_x \tilde{\mathbf{B}} = 4\pi \mathbf{j}_x / c$ which in the case considered means

$$\partial \tilde{B}_z / \partial y = -4\pi e n_0 V_{Ex}/c. \qquad (2.26)$$

It follows from (2.26) that

$$\tilde{B}_z = 4\pi e n_0 \varphi / B_0. \qquad (2.27)$$

From here on we proceed as in subsection 2.1. We then get the set of equations for φ and ψ (cf. (2.19), (2.20)):

$$\hat{D}\Delta_\perp \varphi = \left(\frac{\omega_{pe}^2}{c^2} + \frac{\varkappa_n \omega_{Be}}{u}\right)\frac{\partial \varphi}{\partial \eta} + \frac{\alpha^2 c_{Ae}^2}{u^2}\hat{D}_1 \Delta_\perp(\varphi - \psi), \qquad (2.28)$$

$$\hat{D}\left[\Delta_\perp(\varphi - \psi) + \frac{\omega_{pe}^2}{c^2}\psi\right] = 0. \qquad (2.29)$$

Here $c_{Ae}^2 \equiv B_0^2/4\pi mn$ is the square of the Alfvén velocity, evaluated using the electron mass.

We consider the limit as we go to the electrostatic case in (2.28) and (2.29). In that case (2.29) is replaced by the equation

$$\Delta_\perp(\varphi - \psi) = -(\omega_{pe}/c)^2 \varphi. \qquad (2.30)$$

We substitute (2.30) into (2.28), use the fact that $c_{Ae}^2 \omega_{pe}^2 / c^2 = \omega_{Be}^2$ and after that take $c^2 \to \infty$. We then are led to (1.14) with Λ of the form (1.12).

The set of Eqs. (2.28) and (2.29) reduces to (1.14) also when $\alpha = 0$. In that case

$$\Lambda = \omega_{pe}^2/c^2 + \varkappa_n \omega_{Be}/u. \qquad (2.31)$$

This case corresponds to transverse electromagnetic electron vortices. As $\omega_{pe}/c \to 0$ Eq. (2.31) goes over into Eq. (1.12) with $\alpha = 0$.

2.4. Short-wavelength drift vortices

It was shown in Refs. 11, 12 that there exist short-wavelength drift oscillations. We now consider the problem of QL vortices of such waves.

We write the electric field in the form (2.1). We neglect the longitudinal magnetic field $\tilde{B}_z = 0$ which is valid under the condition stipulated in subsection 2.1 that the plasma density be small compared to the magnetic field pressure. As before we assume the perturbations to be quasineutral and denote the density perturbation of each kind of particles (electrons and ions) by $\tilde{n}$. Assuming the perturbations as far as the ions are concerned to have a short wavelength we take $\tilde{n}$ to be of the Boltzmann form:

$$\tilde{n}=-en_0\varphi/T_i. \tag{2.32}$$

Substituting this $\tilde{n}$ into the electron equation of continuity we get (cf. (1.13))

$$-\frac{eB_0u}{cT_i}\left(1-\frac{V_{*i}}{u}\right)V_{Ex}+\operatorname{div}\mathbf{V}=0. \tag{2.33}$$

The remaining starting equations are the same as those used in the theory of electron vortices (see subsection 2.3). As a result we are led to a set of two equations for φ and ψ, one of which is Eq. (2.29) and the second one of which has the form (cf. (2.28))

$$\tilde{D}\Delta_\perp\varphi=\frac{1}{\rho_{ei}^2}\left(1-\frac{V_{*i}}{u}\right)\frac{\partial\varphi}{\partial\eta}+\frac{\alpha^2c_A^2}{u^2}\tilde{D}_1\Delta_\perp(\varphi-\psi). \tag{2.34}$$

Here $\rho_{ei}^2 = T_i/m\omega_{Be}^2$ is the square of the Larmor radius for the electrons with the ion temperature. Equations (2.29), (2.34) describe short-wavelength drift vortices. It is clear that the transition from the equations for electron vortices to the equations for short-wavelength drift vortices is accomplished through the substitution

$$\frac{\omega_{pe}^2}{c^2}+\frac{\varkappa_n\omega_{Be}}{u}\to\frac{1}{\rho_{ei}^2}\left(1-\frac{V_{*i}}{u}\right). \tag{2.35}$$

When $\alpha = 0$ the set (2.29), (2.34) reduces to a single equation for φ. This equation has the form (1.14) with

$$\Lambda=(1-V_{*i}/u)/\rho_{ei}^2. \tag{2.36}$$

When $\alpha\neq 0$ we get in the electrostatic approximation $(\Delta_\perp \gg (\omega_{pe}/c)^2)$ and using (2.30) from (2.34) again an equation for φ of the form (1.14), but now (cf. (2.14))

$$\Lambda=\frac{1}{\rho_{ei}^2}\left(1-\frac{V_{*i}}{u}-\frac{\alpha^2c_{se}^2}{u^2}\right), \tag{2.37}$$

where $c_{se}^2 = T_i/m$ is the square of the electron sound speed. Such kind of vortices are called electrostatic vortices of oblique drift oscillations.

3. ONE-POTENTIAL VORTICES

3.1. Integrability of Eq. (1.14)

Introducing the notation

$$F\equiv\Delta_\perp\varphi-\Lambda\varphi, \tag{3.1}$$

we note that Eq. (1.14) can be written in the form[1]

$$[\nabla F,\ \nabla(\varphi-uB_0x/c)]_z=0. \tag{3.2}$$

Hence

$$F=C(\varphi-uB_0x/c), \tag{3.3}$$

where C is a constant. It is thus clear that the nonlinear Eq. (1.14) can through integration be reduced to a linear one. We shall call this property of Eq. (1.14) integrability. This justifies the term "quasilinear vortex" introduced in section 1.

3.2 Exterior and interior regions of the vortex

We use for the further analysis apart from the Cartesian coordinates x, η polar coordinates $r = (x^2+\eta^2)^{1/2}$, $\theta = \arctan(\eta/x)$. We introduce the concept of the exterior and interior regions of the vortex, assuming that these regions are separated from one another by some closed curve $r = r(\theta)$. The prescription to find the function $r(\theta)$ will be given below. We assume that the constant in Eq. (3.3) has different values C^e, C^i in the exterior and interior regions and that $C^e = 0$. Equation (3.3) then means

$$\Delta_\perp\varphi=\begin{cases}\Lambda\varphi, & r>r(\theta)\\ \Lambda\varphi+C(\varphi-uB_0x/c), & r<r(\theta)\end{cases} \tag{3.4}$$

where $C\equiv C^i$. We assume that for $r = r(\theta)$ the potential φ, its derivative with respect to the normal $\mathbf{n}$ to the boundary between the two regions, $\mathbf{n}\nabla\varphi$, and $\Delta_\perp\varphi$ are continuous, i.e., that

$$\{\varphi,\ \mathbf{n}\nabla\varphi,\ \Delta_\perp\varphi\}|_i^e=0. \tag{3.5}$$

From the condition that φ and $\Delta_\perp\varphi$ are continuous and Eq. (3.4) it follows that the function $r(\theta)$ must satisfy the equation

$$\varphi[r(\theta),\ \theta]=(uB_0/c)r(\theta)\cos\theta. \tag{3.6}$$

3.3 Dipole vortex

The dipole vortex, i.e., a vortex such that

$$\varphi(r,\ \theta)=\Phi\cos\theta, \tag{3.7}$$

where $\Phi = \Phi(r)$ is a function of the radius r, is an important case of vortex. In that case condition (3.6) is satisfied for $r(\theta) = a$, where a is a constant, called the vortex radius. According to (3.6) this constant is determined by the equation

$$\Phi(a)=uB_0a/c. \tag{3.8}$$

One can treat Eq. (3.8) also as the condition on the potential (current function) of the vortex at the boundary between the exterior and interior regions $r = a$, where a is a free parameter. Such a treatment is the traditional one in the theory of dipole vortices.[4,5]

For the case of a dipole vortex Eq. (3.4) implies

$$\Delta_\perp\varphi=\begin{cases}(\beta/a)^2\varphi, & r>a\\ -(\gamma/a)^2\varphi+\dfrac{\gamma^2+\beta^2}{a^2}\dfrac{uB_0}{c}x, & r<a\end{cases}, \tag{3.9}$$

where β and γ are connected with Λ and C through the relations

$$\Lambda=\beta^2/a^2,\quad C=-(\beta^2+\gamma^2)/a^2. \tag{3.10}$$

It follows from (3.9) that

$$\Phi(r)=\Phi(a)\begin{cases}K_1(r\beta/a)/K_1(\beta), & r>a\\ \left(1+\dfrac{\beta^2}{\gamma^2}\right)\dfrac{r}{a}-\dfrac{\beta^2}{\gamma^2}\dfrac{J_1(r\gamma/a)}{J_1(\gamma)}, & r<a\end{cases} \tag{3.11}$$

Here J_1 and K_1 are a Bessel and a modified Bessel function of the second kind (Macdonald function). When $\Phi(r)$ is of the form (3.11) the first and third conditions (3.5) are satisfied for any γ (i.e., any C). To satisfy the second condition (3.5) it is necessary that the following relation holds between γ and β:[1]

$$K_2(\beta)/\beta K_1(\beta) = -J_2(\gamma)/\gamma J_1(\gamma). \tag{3.12}$$

This relation is sometimes called the dispersion equation of the dipole vortex.

Equation (3.12) has an infinite set of roots $\gamma_n = \gamma_n(\beta)$, $n = 1, 2, \ldots$. From the point of view of the theory of vortex stability the most important one is the one with the smallest value γ_n, $n = 1$.[4] A figure giving the function $\gamma_1 = \gamma_1(\beta)$ is given in Ref. 4.

We note also that the constant $\varkappa$ introduced in section 1 is connected with β and a through the relation $\varkappa^2 = \beta^2 a^2$.

3.4. Structure of the equation for the radial part of the dipole vortex potential

A notable property of the dipole vortex is that in that case Eq. (1.14) reduces to a one-dimensional one. We verify this by substituting (3.7) into (1.14). In terms of

$$X \equiv c\Phi/uB_0 r, \quad \xi \equiv r\Lambda^{1/2} \tag{3.13}$$

the one-dimensional equation obtained can be written in the form

$$(1-X)X''' + X'X'' - X' + \frac{3}{\xi}\left[X'^2 + (1-X)\left(X'' - \frac{X'}{\xi}\right)\right] = 0, \tag{3.14}$$

where the prime indicates derivation with respect to ξ. It is clear that the point $\xi = \xi_0$ where $X(\xi_0) = 1$ is a singular point of Eq. (3.14). Turning to (3.13) we note that the singular point is nothing but the boundary between the exterior and interior regions of the vortex, i.e., the point a.

We can also write Eq. (3.14) in the form

$$(1-X)^2\left(\frac{X'' - X + 3X'/\xi}{1-X}\right)' = 0. \tag{3.15}$$

When $\xi \neq \xi_0$ one can integrate Eq. (3.15) trivially which is in accordance with the integrability property of the two-dimensional equation discussed in subsection 3.1. It is clear from (3.15) that the mathematical reason for the discontinuous nature of the solution (3.11) is the presence of the above-mentioned singular point. It is also clear that to construct a smooth (analytic) solution it is necessary to supplement Eq. (3.15) with terms with higher derivatives of X with coefficients which do not vanish at $\xi = \xi_0$.

4. TWO-POTENTIAL QUASI-LINEAR VORTICES

4.1. Coupled Alfvén and electron drift vortices

We consider some of the consequences of Eqs. (2.19), (2.20). Similarly to (3.1) we introduce the notation

$$F_1 \equiv \Delta_\perp \varphi - \mu_0 \psi/\rho_0^2, \tag{4.1}$$

$$F_2 \equiv \mu_0 \psi - (\alpha c_A/u)^2 \rho_0^2 \Delta_\perp(\varphi - \psi). \tag{4.2}$$

We can then write Eqs. (2.19), (2.20) in the form

$$[\nabla F_1 \times \nabla(\varphi - uB_0 x/c)]_z = 0, \tag{4.3}$$

$$[\nabla F_2 \times \nabla[(\varphi - \psi) - uB_0 x/c]]_z = 0. \tag{4.4}$$

It is clear that Eqs. (4.3), (4.4) are integrable in the sense indicated in subsection 3.1. Similarly to (3.3) we get from them the following set of equations:

$$\Delta_\perp \varphi - \mu_0 \psi/\rho_0^2 = C_1(\varphi - uB_0 x/c), \tag{4.5}$$

$$\mu_0 \psi - (\alpha c_A/u)^2 \rho_0^2 \Delta_\perp(\varphi - \psi) = C_2(\varphi - \psi - uB_0 x/c), \tag{4.6}$$

where C_1 and C_2 are constants.

Spatially localized solutions of Eqs. (4.5), (4.6) are possible both in the case $C_1, C_2 \neq 0$, and in the cases when only one of the constants is nonvanishing, i.e., $C_1 \neq 0$, $C_2 = 0$ or $C_1 = 0$, $C_2 \neq 0$. When C_1, $C_2 \neq 0$ we are dealing with two singularities: one of them is characterized by Eq. (3.6) and the other one by an equation of the form

$$\varphi[r(\theta), \theta] - \psi[r(\theta), \theta] = uB_0 r(\theta)\cos\theta/c. \tag{4.7}$$

In the other two cases there is just one singularity characterized by Eq. (3.6) or (4.7).

As in the case of one-potential vortices we can introduce the concept of a two-potential dipole vortex. In the case of such a vortex the function φ is given by Eq. (3.7) while ψ has the form

$$\psi = \Psi \cos\theta, \tag{4.8}$$

where $\Psi = \Psi(r)$ is a function of the radius.

One can reduce Eqs. (4.5), (4.6) to a single fourth order equation for φ but we do not write down the result as it is obvious.

4.2 Electromagnetic electron vortices

We now turn to Eqs (2.28), (2.29) for electromagnetic electron vortices. We write

$$F \equiv \Delta_\perp(\varphi - \psi) + (\omega_{pe}/c)^2 \psi. \tag{4.9}$$

We can then write (2.29) in the form (3.2) and find that F satisfies an equation of the form of (3.3), i.e.,

$$\Delta_\perp(\varphi - \psi) + \frac{\omega_{pe}^2}{c^2}\psi = C\left(\varphi - \frac{uB_0}{c}x\right). \tag{4.10}$$

Substituting $\Delta_\perp(\varphi - \psi)$ from (4.10) into (2.28) we get

$$\hat{D}\Delta_\perp\varphi = \left(\frac{\omega_{pe}^2}{c^2} + \frac{\varkappa_n \omega_{Be}}{u}\right)\frac{\partial\varphi}{\partial\eta} - \frac{\alpha^2 c_{Ae}^2}{u^2}\left(\frac{\omega_{pe}^2}{c^2} - C\right)\hat{D}\psi. \tag{4.11}$$

Using the fact that $\partial\varphi/\partial\eta \equiv \hat{D}\varphi$ we write (4.11) in the form (3.2) with a different value of F. By analogy with (4.10) we then find

$$\Delta_\perp\varphi = \left(\frac{\omega_{pe}^2}{c^2} + \frac{\varkappa_n \omega_{Be}}{u}\right)\varphi - \frac{\alpha^2 c_{Ae}^2}{u^2}\left(\frac{\omega_{pe}^2}{c^2} - C\right)\psi + C_1\left(\varphi - \frac{uB_0}{c}x\right), \tag{4.12}$$

where C_1 is a constant similar to C.

Thus, similarly to subsection 4.1 instead of two nonlinear third order differential equations we have obtained two linear second order equations with some integration constants C and C_1. As in the case of Eqs. (4.5), (4.6), the set (4.11), (4.12) can be reduced to a single fourth order equation for φ. It is also clear that one can by using (4.11), (4.12) construct two one-dimensional equations for a dipole vortex (cf. subsections 3.3, 3.4).

We also note that in contrast to the situation for Alfvén vortices considered in subsection 4.1 when solutions with two singularities are possible, in the case of electromagnetic electron vortices there is only one singular point. Such a point may be double in the sense that two nonvanishing integration constants, C_1, $C \neq 0$ may correspond to it.

4.3 Electromagnetic short-wavelength drift vortices

According to subsection 2.4 the equations describing electron vortices basically remain valid also in the case of electromagnetic short-wavelength drift vortices except that one must in them perform the substitution (2.35). We can thus conclude without any additional analysis that the starting equations (Eqs. (2.29), (2.34)) of the theory of short-wavelength drift vortices are integrable in the sense indicated in subsection 3.1 and reduce to a set of two linear equations, one of which has the form (4.10) while the second one implies

$$\Delta_\perp \varphi = \frac{1}{\rho_{ei}^2}\left(1-\frac{V_{*i}}{u}\right)\varphi - \frac{\alpha^2 c_{Ae}^2}{u^2}\left(\frac{\omega_{pe}^2}{c^2}-C\right)\psi + C_1\left(\varphi-\frac{uB_0}{c}x\right). \quad (4.13)$$

The general properties of short-wavelength drift vortices are similar to the general properties of the electron vortices noted at the end of subsection 4.2.

5. DISCUSSION OF THE RESULTS

The low-frequency long-wavelength perturbations considered by us in subsections 2.1, 2.2, and 4.1 are the same as those animatedly discussed at the beginning of the sixties as the origin of anomalous plasma losses across a confining magnetic field (see, e.g., Refs. 14, 13). The initial non-linear theory of these waves was based upon ideas from turbulence theory.[14] Later in the framework of concepts of non-linear waves described by Korteweg–de Vries type equations the existence was proved of solitary drift waves in a plasma with $\nabla T_e \neq 0$.[8] Moreover, it was shown that there exist Alfvén solitons[7] with a spatial structure similar to that of Langmuir solitons. The anlaysis given in the present paper indicates the existence in a plasma confined by a magnetic field of a broad class of QL vortex type solitary waves and thereby leads to the idea of a more important role for solitary waves in the dynamics of such a plasma.

Short-wavelength drift oscillations considered in subsection 2.4 (see also subsection 4.3) had earlier been discussed in connection with the problem of the confinement of a plama by hot ions in adiabatic traps[11] and also in connection with the problem of high-frequency plasma heating,[15] the study of the dynamics of the plasma in the Earth's magnetopause region,[16] and in a number of other problems. Recently the possibility of the existence of solitons of such oscillations was proved which are analogous to the above noted low-frequency long-wavelength drift solitons.[17] Our analysis (see subsections 2.4 and 4.3) broadens the existing idea about possible kinds of short-wavelength drift solitons and thereby establishes prerequisites for the development of a more complete theory of non-linear processes in which short-wavelength drift oscillations participate.

Electron vortices (see section 1 and subsections 2.3, 4.2, and also Ref. 18) are, in particular, of interest because they, as compared to the electron-ion ones, can be studied experimentally in simpler laboratory set-ups than the electron-ion ones. Such studies are important for a proof of the reality of vortices in a plasma.

An appreciable number of the kinds of vortices considered by us in section 2 belong to the number of one-potential ones (see section 3). The electric potential of such vortices is apart from a factor the same as the current function characterizing Rossby wave vortices in a rotating fluid.[1] We paid considerable attention to one-potential dipole vortices. Such vortices can be considered to be the combination of a pair of vortices rotating in opposite directions. It follows from (3.8) that the characteristic magnitude of the speed of such a rotation is of the order of the vortex speed: $V \approx u$. This corresponds to the case of strong nonlinearity. The same estimate is valid also for all other species of vortices discussed by us. It is therefore clear that the term "quasilinear vortex" used by us must not be understood as weakly nonlinear (we explained the meaning of this term in sections 1,3).

Together with electrostatic vortices we considered also electromagnetic ones, i.e., vortices in which the perturbation of the magnetic field is important. Some of the electromagnetic vortices are described by the equations of the one-potential approximation (among them we have the balloon vortex of subsection 2.1 and also the electromagnetic transverse vortex of subsection 2.3). Of principal interest are also the two-potential vortices. Such vortices may be connected either with low-frequency long-wavelength plasma oscillation branches (subsection 4.1) or with short-wavelength drift (subsection 4.3) and electron (subsection 4.2) branches. Such kinds of vortices have apparently no analog in a normal liquid.

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