Higher orders of perturbation theory in classical mechanics

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It is well known that the perturbation theory series in classical mechanics are asymptotic series, i.e., the nth coefficient of perturbation theory increases rapidly as $n\to\infty$. For a number of problems, information about the behavior of this coefficient at large n is of interest. The present paper investigates a general method for calculating the asymptotic behavior of the higher orders of perturbation theory in classical mechanics. As an example, detailed calculations are made of the higher perturbation orders for the supplementary integral in the Hénon-Heiles model, which is a Hamiltonian with two degrees of freedom and describes the motion of galaxies in a magnetic field.

I. INTRODUCTION

The problems of classical mechanics can be divided naturally into two groups—integrable and nonintegrable, depending on whether or not the classical equations admit general analytic solution. It is well known[1] that an autonomous Hamiltonian system with n degrees of freedom is integrable if there exist $n-1$ single-valued functions $I_k(p, x)$ of the coördinates and momenta, called supplementary integrals, which together with the Hamiltonian $H(p, x)$ form a family of n independent functions whose Poisson brackets with one another vanish.

A simple example of integrable models is provided by the system with the Hamiltonian[2]

$$H(p,x)=\frac{1}{2}(p_1^2+p_2^2+\omega_1^2x_1^2+\omega_2^2x_2^2+(x_1^2+x_2^2)^2) \qquad (1)$$

with arbitrary frequencies ω_1 and ω_2, which has the supplementary integral

$$I(p,x)=(p_1x_2-p_2x_1)^2+\frac{\omega_2^2-\omega_1^2}{2}(p_1^2+\omega_1^2x_1^2+x_1^4-p_2^2-\omega_2^2x_2^2-x_2^4). \qquad (2)$$

One of the best known examples of nonintegrable systems is the problem of three gravitating bodies.[3]

The theory of integrable systems has by now been deeply studied and the inverse scattering method[4] makes it possible to construct numerous examples of such systems. Much less is known about nonintegrable systems.

In the investigation of nonintegrable problems, much importance attaches to the various approximate methods, among which perturbation theory is one of the most reliable and popular. For this method to be applicable, the Hamiltonian of the problem must decompose into two parts,

$$H(p,\ x;\ \lambda)=H_0(p,\ x)+\lambda H_1(p,\ x;\ \lambda), \qquad (3)$$

where $H_0(p, x)$ is the unperturbed Hamiltonian, for which the solutions of the equations of motion are assumed known, $\lambda H_1(p, x; \lambda)$ is the Hamiltonian of the perturbation, and λ is the expansion parameter (coupling constant). If λ is small, then any (or almost any) quantity I of interest can be represented in the form of a perturbation series in λ:

$$I=\sum_n I_n\lambda^n \qquad (4)$$

and in each particular case one can give an algorithm for the successive calculation of the coefficients I_n.

The calculation of the finite orders of perturbation theory is almost identical for integrable and nonintegrable problems. However, it is well known[3,5–8] that for nonintegrable problems the series obtaind by such a perturbation theory are only asymptotic series, i.e., the coefficients I_n increase with increasing n faster than a^n with any fixed a.

For some problems, it is of interest to know the asymptotic behavior of these coefficients as $n\to\infty$. The most natural of these problems is the problem of extending the region of applicability of perturbation theory. This problem is intimately related to the problem of summing perturbation series and in the simplest variant can be formulated as follows: Suppose the first few terms of the perturbation series for some quantity are known; using information about the behavior of the higher orders of perturbation theory, the problem is to construct a function that approximates the required quantity for values of the coupling constant as large as possible.

In the analogous (simpler) problems of quantum mechanics and quantum field theory (see Refs. 9 and literature quoted there), the use of asymptotic expressions for I_n at large n made it possible to increase by one or two orders of magnitude the range of coupling constants in which perturbation theory gives good results. For problems of classical mechanics, in which the coupling constant is usually fixed, extension of the region of applicability of perturbation theory means that the time during which the motion can be described by perturbation theory can be significantly increased.

In Ref. 10, a method was proposed for calculating the higher orders of perturbation theory for the simplest nontrivial case of nonintegrable models of classical mechanics—two-dimensional area-preserving mappings.[1)] As an illustration of the method, asymptotic estimates were obtained in Ref. 10 for the perturbation theory coefficients for a mapping of the form

$$x'=-y,\qquad y'=x+y^3. \qquad (5)$$

As usual in problems of classical mechanics, the results of the calculations depend strongly on the arithmetic nature of the frequencies of the unperturbed motion. To study this dependence, the proposed method was used in Ref. 12 to

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determine the higher orders of perturbation theory for a formal integral of a more general two-dimensional area-preserving mapping:

$$x'=\cos\alpha(x+y^3)-y\sin\alpha, \qquad y'=\sin\alpha(x+y^3)+y\cos\alpha \tag{6}$$

for different α.

The aim of the present paper is to generalize the proposed method to the more interesting case of Hamiltonian systems with several degrees of freedom.

The method to be discussed below applies in principle to all nonintegrable problems of classical mechanics with a small parameter in which a large number of terms of perturbation theory is important. In the first place, these include various problems in celestial mechanics, the physics of colliding-beam accelerators, and plasma physics, for which it is necessary to take into account many coefficients of the perturbation series in order to achieve the required accuracy over a long time interval (for a discussion of analytic programs for calculating the coefficients of the perturbation series in these and other theories, see the review of Ref. 13).

To avoid additional technical difficulties, we shall consider the model Hénon-Heiles system[14] as an example in this paper. This model was originally proposed to describe the motion of galaxies in a magnetic field and has been the subject of numerous numerical and analytic investigations (see Refs. 5, 14, and 15–19 and the literature in them).

The Hénon-Heiles model is a Hamiltonian system with two degrees of freedom and Hamiltonian

$$H(p,x)=\frac{1}{2}\left(p_1^2+p_2^2+x_1^2+x_2^2+2x_1^2x_2-\frac{2}{3}x_2^3\right), \tag{7}$$

its equations of motion having the form

$$d^2x_1/dt^2+x_1=-2x_1x_2, \qquad d^2x_2/dt^2+x_2=x_2^2-x_1^2. \tag{8}$$

The unperturbed part of the Hénon-Heiles Hamiltonian describes two noninteracting oscillators with equal frequencies, and cubic powers of the coordinates give the perturbation. If a coupling constant is introduced in front of these terms, it can be eliminated from the equations of motion by a simple scale transformation of the coordinates and the momenta, so that the role of a coupling constant in this model is played by powers of the coordinates and momenta, or

$$r^2={}^1/_2(p_1^2+p_2^2+x_1^2+x_2^2), \tag{9}$$

as must be for perturbation theory around a point of equilibrium. Note also that the region of perturbation theory for the model (7) corresponds to the region $E\to 0$, where E is the energy of the system.

As in Refs. 10 and 12, we restrict the discussion to the perturbation theory for the supplementary integral. It is known[5,15–17,20] that for any Hamiltonian system of the type (7) with arbitrary relationship between the frequencies of the unperturbed Hamiltonian the supplementary integral can be represented in the form of a series with respect to homogeneous polynomials:

$$I(p,x)=\sum_n I_n(p,x), \tag{10}$$

where

$$I_n(p,x)=\sum_{k+l+r+t=n} b_{klrt}x_1^k x_2^l p_1^r p_2^t.$$

The results of a numerical calculation of the first five terms of this expansion for the Hénon-Heiles model (with $n=4, 5, 6, 7, 8$) are given in Ref. 16. For integrable systems, an expansion of the type is convergent or frequently even finite, as in (2). For nonintegrable problems, the coefficients I_n increase rapidly as $n\to\infty$. It is precisely the behavior of I_n at large n that will interest us in what follows.

By definition, nonintegrable models do not admit single-valued supplementary integrals. It was shown in Ref. 10 that, nevertheless, for nonintegrable systems one can formally construct a function which does not change under the influence of the equations of motion and whose perturbation theory expansion is identical to the ordinary expansion obtained from recursion relations, as in Refs. 15–17 and 20. However, the function constructed in this way will have a singularity of the type of the square root of some quadratic form near each periodic trajectory of the considered problem. To determine the nature of this singularity (i.e., the quadratic form in the radicand), it is necessary to linearize the equations of motion near the given periodic trajectory and find the monodromy matrix of the linearized equations. To the $r\to 0$ region of perturbation theory there correspond trajectories with period $T\to\infty$. Since the set of such trajectories is dense, the supplementary integral has singularities on a dense set, which leads to divergence of the series (10).

The most complicated problem when this scheme is applied to real problems is the finding of an expression for the monodromy matrix for long periodic trajectories. As is shown in Ref. 10, in the limit $T\to\infty$ the quantities in which we are interested decrease faster than any fixed power of the coupling constant, and it is a difficult problem to find them analytically.

The paper is arranged as follows.

In Sec. 2, we briefly discuss the general properties of the Hénon-Heiles model and the construction of the perturbation theory for it.

Section 3 is devoted to the application of perturbation theory to the construction of long periodic solutions.

In Sec. 4, we determine the supplementary integral outside the framework of perturbation theory, and in Sec. 5 we find the explicit form of its singularity near a given periodic trajectory.

In Sec. 6, we construct a function that has given singularities near all periodic trajectories of the considered problem and, expanding it in a perturbation theory series, find the asymptotic expressions that we seek for the higher orders of the supplementary integral.

In Appendix A, we recall the definition of the monodromy matrix for the solutions of linearized equations and give without proof an expression that relates the trace of the monodromy matrix to the exact Fourier components of the considered periodic trajectory. Unfortunately, periodic solutions of Hamiltonian systems of general form have been little studied, and we limit ourselves to rough estimates for the higher Fourier components, on the basis of which we obtain an approximate expression for the trace of the monodromy matrix of the long periodic trajectories.

In Appendix B, we give explicit expressions for the first three coefficients in the expansion of the solution of the Hénon-Heiles equations (8) in the perturbation theory series.

2. GENERAL PROPERTIES OF THE HENON-HEILES MODEL

In discussing the Hénon-Heiles model, it is convenient to introduce the complex coordinate $z = x_2 - ix_1$, and then the equations of motion (8) become

$$d^2z/dt^2+z=\bar{z}^2; \tag{11}$$

here and below, $\bar{z}$ denotes the complex conjugate. The well-known C_3 symmetry of the model (see, for example, Ref. 19 and the references in it) is obvious from (11). Namely, if $z(t)$ is a solution of (11), then

$$z'(t)=e^{i\omega}z(t), \tag{12}$$

where $\omega = 2\pi/3$, is also a solution of these equations with the same energy. We note one further simple symmetry. If $z(t)$ is a solution of Eqs. (11), then

$$z'(t)=\overline{z(-t)} \tag{13}$$

satisfies the same equations.

The Hénon-Heiles Hamiltonian describes finite motion for $0<E\leqslant 1/6$. To describe the nature of the trajectories in this model (as in any model with two degrees of freedom), it is convenient to introduce the standard plot of the Poincaré intersection[3,11,5] of the phase space with the plane $x_1 = 0$. If we eliminate p_1 by using the equation $H(p, x) = E$, where H is given by Eq. (7), and set $x_1 = 0$, then as coordinates on the resulting two-dimensional surface we can use x_2 and $p_2 = \dot{x}_2$. This means that any trajectory will be determined by its point of intersection with the chosen plane. Figure 1 shows the Poincaré plot in the Hénon-Heiles model for $E = 1/11$. The boundary of the admissible region, which for small E is an oval determined by the condition $E = (\dot{x}_2^2 + x_2^2 - 2x_2^3/3)/2$ is the projection of the simple periodic solution

$$x_1=0, \quad d^2x_2/dt^2+x_2=x_2^2. \tag{14}$$

We shall denote this periodic trajectory by the letter π_1. The letters π_2 and π_3 in Fig. 1 are the images of this trajectory under the transformation (12). For small E, all these three trajectories are stable; π_4, π_5, and π_6 are the hyperbolic partners corresponding to these elliptic trajectories. All these six trajectories arise (as we shall see below) as a result of break up of the resonance torus with $N_1 = N_2 = 1$. It follows from Birkhoff's theorem[20] that when the resonance torus breaks up elliptic and hyperbolic trajectories alternate, and the existence of the supplementary symmetry (12) has the consequence that from the one resonance torus six trajectories are formed. It is also easy to prove the existence of the periodic trajectories π_7 and π_8, which are stable at small E.

The trajectories π_1–π_8 are the simplest periodic trajectories in the Hénon-Heiles model. There exist more complicated trajectories. We shall be interested in periodic trajectories surrounding the points π_7 and π_8. Figure 1 shows some of these trajectories. These trajectories are characterized by two integers: the number of intersections with the Poincaré plane and the number of revolutions around the point π_7 (or π_8). Below, we shall relate these numbers to the numbers N_1 and N_2 that characterize the resonance torus in perturbation theory.

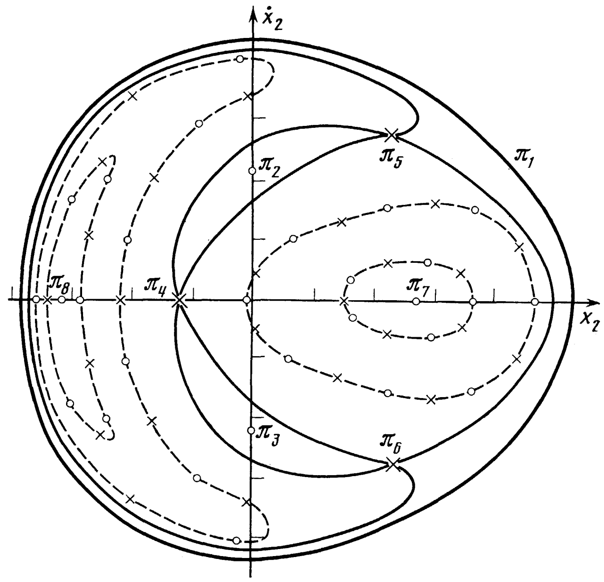

FIG. 1. The Poincaré plot for the Hénon-Heiles model for $E = 1/11$. Shown are the positions of the simplest periodic trajectories π_1–π_8 and the two trajectories with $(N_1, N_2) = (5, 4)$ and $(8, 7)$. The continuous curves show schematically the separatrices of the trajectories π_4–π_6. The true picture of the separatrices, including their splittings, is complicated and we do not give it. The broken curves show schematically the section of the (nonexist) resonance tori in perturbation theory: the crosses represent hyperbolic trajectories, and the open circles elliptic trajectories.

The mapping induced on the Poincaré plane by the trajectories of the Hamiltonian system are very similar to the explicit two-dimensional area-preserving mappings of the type (6), and the method used in Refs. 10 and 12 to calculate a formal integral for these mappings will be used to calculate the higher orders of perturbation theory for the supplementary integral of Hamiltonian systems with two degrees of freedom.

There exist different but related variants of perturbation theory in Hamiltonian systems.[3,1,5,20] The most popular perturbation theory, which can be used for any ratio of the frequencies of the unperturbed Hamiltonian,[1,5,8] consists of using a sequence of canonical transformations

$$x_i=x_i(\eta, \xi), \quad p_i=p_i(\eta, \xi) \tag{15}$$

with $x_i dp_i + \xi_i d\eta_i = d\Phi(p, \eta)$ to reduce the considered Hamiltonain $H(p, x)$ to the simplest possible form, the so-called normal form $H(\eta, \xi)$.

For an irrational ratio of the frequencies,[20] the normal form is a function of $(\eta_1^2 + \xi_1^2)$ and $(\eta_2^2 + \xi_2^2)$, which is equivalent to expanding the solutions of the equations of motion in a series of almost periodic functions:

$$x_i(t)=\sum_{m_1,m_2} x^{(i)}_{m_1,m_2} e^{i(m_1\Omega_1+m_2\Omega_2)t}. \tag{16}$$

For a rational ratio of the frequencies, the normal form is more complicated, but for $n = 2$ is always integrable. We emphasize that the transformation (15), which reduces the

Hamiltonian to normal form, is itself specified in the form of formal asymptotic series, and it therefore does not follow from integrability of the Hamiltonian in normal form that the original Hamiltonian is integrable. The general theory of the reduction of a Hamiltonian to normal form is discussed in detail in Ref. 16. We give the result only for the Hénon-Heiles model (for which the unperturbed frequencies are equal). By means of polynomial canonical transformations of the type (15), the Hénon-Heiles Hamiltonian (7) can be reduced to a normal form that is a function of the three variables I_1, I_2, I_3 (and not four: $\eta_1, \eta_2, \xi_1, \xi_2$):

$$H(\eta,\xi)=H(I_1,I_2,I_3)=\sum_n H_n(\eta,\xi) \tag{17}$$

and

$$H_n(\eta,\xi)=\sum_{k+l+m=n} C_{klm} I_1^k I_2^l I_3^m,$$

where

$$I_1=\eta_1^2+\xi_1^2,\quad I_2=\eta_2^2+\xi_2^2,\quad I_3=\eta_1\xi_2-\eta_2\xi_1. \tag{18}$$

To simplify the expressions, it is also convenient to introduce two further variables:

$$I_4=\eta_1\eta_2+\xi_1\xi_2 \text{ and } h^2={}^1/_2(\eta_1^2+\xi_1^2+\eta_2^2+\xi_2^2), \tag{19}$$

which are related to I_1, I_2, I_3 by

$$^1/_4(I_1-I_2)^2+I_3^2+I_4^2={}^1/_4(I_1+I_2)^2=h^4. \tag{20}$$

It is readily verified that the Poisson brackets

$$\{I_i, h^2\}=0 \tag{21}$$

for $i = 1, 2, 3, 4$. Therefore, any function $H(I_1, I_2, I_3)$ commutes with h^2, and, therefore, h^2 is an explicit supplementary integral for the Hamiltonian in the normal form (17). In other words, the Hamiltonian in the normal form is integrable.

There exists[16] a simple algorithm for successive determination of the coefficients of the canonical transformations (15) and all terms of the normal form. Some of the first terms of this expansion, obtained from the results of Ref. 16, are given in Appendix B [Eqs. (B1) and (B2)]. In Ref. 16, the following definition is chosen for the supplementary integral:

$$I(p,x)=E-h^2. \tag{22}$$

We shall also follow this definition, although sometimes other definitions are more convenient (see Refs. 15 and 17). If we express h^2 in terms of the old variables, then for it it is sufficient to substitute ξ_i and η_i in the form of the expansions (B1) in $h^2 = (\eta_1^2 + \xi_1^2 + \eta_2^2 + \xi_2^2)/2$, and we obtain an expansion of the supplementary integral in a perturbation series. The first three terms of this expansion are given in (B3).

Since the Hamiltonian in the normal form (17) is integrable, it is in principle not difficult to find explicit solutions of the equations of motion and, in particular, expressions for action-angle variables. The corresponding expressions for the Hénon-Heiles model are given in (B6)–(B8).

3. PERTURBATION THEORY FOR PERIODIC TRAJECTORIES

The expressions of perturbation theory are formally valid for any ratio of the exact frequencies Ω_1 and Ω_2 [cf. (B8)]. According to these expressions, the trajectories of the considered problem always lie on two-dimensional tori determined by the conditions $H = \text{const}$ and $I = \text{const}$. The question of whether such a perturbation theory is convergent (and meaningful) is more complicated. It is known[6–8,3] that for the majority of Hamiltonians the series of this perturbation theory are only asymptotic series. In accordance with the well-known Kolmogorov-Arnol'd-Moser theorem,[7,8,5] it is possible to construct a convergent variant of perturbation theory by ensuring, through the choice of the initial conditions, that the ratio Ω_1/Ω_2 of the frequencies is a "good" irrational number for which

$$\left|\frac{\Omega_1}{\Omega_2}-\frac{m}{n}\right|>\frac{c}{n^\gamma} \tag{23}$$

for all m and n. Here, c and $\gamma > 2$ are certain constants. This means that the irrational tori [tori on which $\Omega_1/\Omega_2 =$ irrational number satisfying (23)] are only slightly deformed by a small perturbation, and the trajectories on them are described as before by almost periodic functions of the type (16). The resonance tori, i.e., tori on which

$$\Omega_1=N_1\Omega_0,\quad \Omega_2=N_2\Omega_0, \tag{24}$$

where N_1 and N_2 are mutually prime integers, are in the general case broken up, in accordance with Birkhoff's theorem,[20] leaving only a finite number of periodic trajectories. The fate of the remaining tori for which Ω_1/Ω_2 is an irrational number not satisfying (23) has been studied very little.

Thus, for a rational ratio of the frequencies the expressions of perturbation theory describe a nonexist object—the resonance torus. Therefore, for periodic trajectories perturbation theory must be used with care.

On the basis of the existence of the symmetry (12) of the Hénon-Heiles system, one can show that for sufficiently large N_1 and N_2 this system has periodic trajectories satisfying the initial conditions

$$x_1(0)=0,\quad \dot{x}_2(0)=0. \tag{25}$$

On the Poincaré section we have chosen (see Fig. 1) these trajectories are represented by points passing through the abscissa. Using this property, we arrive at the following perturbation theory scheme for periodic trajectories. We fix the value of the energy. We require that the initial values always satisfy the conditions (25). We construct the perturbation theory series by the manner described above. By the choice of $x_2(0)$ we attempt to satisfy the resonance condition $\Omega_1/\Omega_2 = N_1/N_2$. Thus, we describe an actually existing periodic trajectory (and not a complete resonance torus); it is natural to assume that the perturbation theory (or some modification of it such as Newton's method[7,8]) applies and makes it possible to obtain convergent expressions for such a trajectory. Unfortunately, the author does not know of rigorous results on convergent approximations for periodic trajectories (in this connection, see Ref. 18). At the least, the method of successive canonical transformations used in the proof of the Kolmogorov-Arnol'd-Moser theorem[7,8,5] cannot be assumed to hold for the description of periodic trajectories.

The question of periodic trajectories is of interest in its own right and requires further investigations.

We use this scheme (for want of a better) to describe the periodic trajectories satisfying (25). For fixed energy we must, to satisfy the resonance condition (24), choose in a definite manner the value of the supplementary integral or the associated quantity $R(p, x)$ [see (B9)]. Using (B8) and (B7), we find that for this R must be equal to R_p, where

$$R_p = \frac{6}{7}\frac{N_1-N_2}{N_1+N_2}\left(1-\frac{3}{4}E\right). \tag{26}$$

The conditions $R = R_p$, $E = \text{const}$ determine a two-dimensional resonance torus. As we said above, we shall seek only a periodic trajectory satisfying the condition (25). Using the explicit expressions for R [see (B7)] and E, we can readily find its approximate position.

We denote by $\dot{x}_{1p}$ and x_{2p} the coordinates of the initial point of this trajectory (we recall that the other two coordinates vanish). Then in the lowest order, we obtain from (B6) and (B7)

$$\dot{x}_{1p}=\rho_1+\rho_2, \quad x_{2p}=\rho_1-\rho_2, \tag{27}$$

where

$$\rho_1=[(E+R_p)/2]^{1/2}, \quad \rho_2=[(E-R_p)/2]^{1/2}.$$

If necessary, it is not difficult to write down the following terms in the expansion of these quantities in a perturbation theory series using (44) and (45) for $\varphi_1 = \varphi_2 = 0$.

There exist different types of periodic trajectory. They differ from one another in their position on the Poincaré plane. Some trajectories surround the elliptic (at low energies) trajectories π_7 and π_8 (see Fig. 1). Others surround π_1, π_2, and π_3. In the following sections, we shall see that the most important for our purposes are the trajectories of the first type surrounding π_7 and π_8. One can show that the difference $N_1 - N_2$ determines the number of revolutions around the point π_7 (and π_8). The main contribution to the asymptotic estimates of the perturbation theory coefficients will be given, as for area-preserving mappings of the type (6) (see Refs. 10 and 12), by the trajectories that go round these points once. Such trajectories correspond to

$$N_1-N_2=\pm 1. \tag{28}$$

The number of intersections of such a periodic trajectory with the Poincaré plane is equal to the larger of the numbers N_1 and N_2.

Knowing one periodic trajectory, we can in general obtain two other trajectories by means of the transformation (12). Among the periodic trajectories are some for which the transformation (12) reduces to a shift in time:

$$e^{i\omega}z(t)=z(t+\delta). \tag{29}$$

In the following sections we shall see that precisely these trajectories are of interest to us. It is easy to show that if the property (29) is to hold it is necessary that

$$N_1+N_2=3p, \tag{30}$$

where p is an integer. Combining these conditions with the condition (28), we find that for the trajectories in which we are interested the number p must be odd. As we shall see below, the remaining trajectories will make an exponentially small contribution to the higher orders of perturbation theory, and we shall not discuss them.

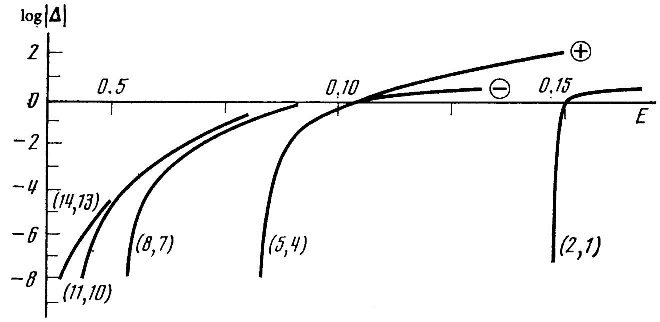

FIG. 2. Dependence of $|\Delta|$ on the energy for various periodic trajectories in the Hénon-Heiles model. The numbers give the values of (N_1, N_2) for the corresponding periodic trajectories. For $(N_1, N_2) = (5, 4)$ we have given the value of $|\Delta|$ for the elliptic ($-$) and hyperbolic ($+$) trajectories.

In accordance with Birkhoff's theorem,[20] the break up of a resonance torus with fixed N_1 and N_2 gives rise to two types of trajectory—elliptic and hyperbolic. The arguments given in Appendix A and the results of numerical calculation for the Hénon-Heiles system (see Fig. 2) and for the area-preserving mappings (5), (6) (Refs. 10 and 12) show that for small Δ we have $|\Delta_{(+)} + \Delta_{(-)}| \lesssim \Delta^2$, where $\Delta_{(-)}$ and $\Delta_{(+)}$ are the values of Δ for the elliptic and hyperbolic trajectories, respectively. Therefore, for small Δ we can assume that the equation $\Delta_{(+)}(E) = -\Delta_{(-)}(E)$ holds.

We calculated Δ numerically for the periodic trajectories satisfying (28) and (30) with $(N_1, N_2) = (2, 1)$, $(5, 4)$, $(8, 7)$, $(11, 10)$, $(14, 13)$. The results of the calculations are given in Fig. 2, in which we have plotted the dependence of $\log|\Delta|$ on the energy for the Hénon-Heiles system. Some comments on the figure are appropriate. If at a certain energy there exists a periodic trajectory with fixed N_1 and N_2, then with decreasing energy this trajectory will approach closer and closer to the trajectories π_7 and π_8 (see Fig. 1). At the energy determined by the condition $P_2 = 0$, where P_2 is the action type variable defined in (B7), this trajectory disappears, and Δ tends to zero, as predicted by Eq. (A14). This explains the logarithmic poles in Fig. 2. In accordance with what we have said above, we have given only the absolute magnitude of Δ. When Δ is small, the difference between $|\Delta_{(+)}|$ and $|\Delta_{(-)}|$ is very small. In Fig. 2, we have given as an illustration $|\Delta_{(+)}|$ and $|\Delta_{(-)}|$ for the trajectory (5, 4) for fairly high energies. To determine the sign of Δ, we calculate the monodromy matrix at a point on the abscissa (see Fig. 1) lying between the points π_4 and π_7. Then it is found numerically that the signs of Δ alternate, as is shown in Fig. 1 for two neighboring trajectories, and for the trajectory (5, 4) at low energies $\Delta > 0$.

For the numerical calculation of Δ we used the following procedure. Fixing the energy, we sought an initial point satisfying the conditions (25) whose trajectory intersects the Poincaré plane N_1 times ($N_1 > N_2$) and passes once around the point π_7.

Knowing this periodic trajectory, we solved numerically the linearized equations (A2) with different initial conditions, and then determined the monodromy matrix in accordance with (A4). Since the quantities in which we are interested are small, the calculations must be made with high accuracy, which makes the calculations at large N_1 and N_2 difficult.

4. DETERMINATION OF THE SUPPLEMENTARY INTEGRAL OUTSIDE THE FRAMEWORK OF PERTURBATION THEORY

To be able to speak meaningfully of the singularities of the supplementary integral, it is first necessary to determine it outside the framework of perturbation theory. There are various ways in which this can be done. We shall basically follow the arguments of Refs. 10 and 12. We note first that there is an obvious lack of uniqueness in a calculation of the supplementary integral by perturbation theory. Indeed, if $I(p, x)$ is some supplementary integral determined in the form of the series (10), then $F(I, E)$, where E is the energy and $F(u, v)$ is an arbitrary function of two variables that can be represented in the form of a double series in u and v, will also be a supplementry integral. One of the convenient methods of eliminating this nonuniqueness is to fix the value of the integral on the plane $x_1 = 0, p_2 = 0$, i.e., the function

$$I(p_1, 0, x_2, 0) = I_0(p_1, x_2) \tag{31}$$

is assumed known.

It is readily seen that, specifying the function $I_0(p_1, x_2)$, we thereby completely eliminate the above nonuniqueness. This means that any method of calculating $I(p, x)$ perturbatively leads to a quite definite function $I_0(p_1, x_2)$ and, conversely, choosing a function $I_0(p_1, x_2)$ compatible with perturbation theory, we can, in principle, uniquely determine all the remaining coefficients of the series (10).

Now suppose we know the function $I_0(p_1, x_2)$, i.e., we know the value of the supplementary integral on the plane $x_1 = 0, p_2 = 0$. We consider the set of trajectories of the given system that begin on this plane:

$$\begin{gathered} x_i = x_i(t), \quad p_i = p_i(t), \\ x_1(0) = 0, \quad x_2(0) = x_2^{(0)}, \quad p_1(0) = p_1^{(0)}, \quad p_2(0) = 0, \end{gathered} \tag{32}$$

and we set

$$I(p, x) = I_0(p_1^{(0)}, x_2^{(0)}). \tag{33}$$

The function $I(p, x)$ constructed in this way is defined at the points of the phase space that are the images of the plane we have chosen, and it is obviously an integral of the considered problem. Of course, there exist regions of the phase space that are inaccessible from the plane $x_1 = 0, p_2 = 0$. Among them are, for example, trivial regions near π_2 and π_3 in Fig. 1, these being associated with the existence of the C_3 symmetry (12) of the Hénon-Heiles system. To determine the integral in these regions, we shall consider not only the trajectories (32) that occur in the definition (33) but also their partners obtained from (32) by the transformation (12). In addition, near some hyperbolic trajectories there exist regions that are not reached by the trajectories (32) for real $x_2^{(0)}$ and $p_1^{(0)}$. To determine the integral in these regions it is necessary to assume that $p_1^{(0)}$ and $x_2^{(0)}$ in (32) are complex numbers. Since we consider polynomial equations of motion, this complexification does not give rise to difficulties.

By its construction, the function $I(p, x)$ is, on the one hand, an integral of the problem and, on the other, its expansion in the series (10) is identical to the expansion obtained from the ordinary recursion relations [because the fixing of the function $I_0(p_1, x_2)$ uniquely determines all the coefficients of the perturbation series]. However, nonintegrable problems do not admit supplementary integrals with "reasonable" analytic properties, and we shall see below that the integral defined in (33) has singularities of a known form near every periodic trajectory of the Hamiltonian system. Since the set of such trajectories in the region of applicability $r \to 0$ (or $E \to 0$) of perturbation theory is dense, the integral has singularities on a dense set, which leads to divergence of series of the type (10).

The reason for these singularities is the intersection of the images of different points of the plane $x_1 = 0$, $p_2 = 0$ under the influence of the equations of motion of the considered problem. One might think that the existence of such singularities is due to our definition (32), (33). However, because an analytic supplementary integral does not exist in nonintegrable problems, all redefinitions of it outside the framework of perturbation theory must lead to a complicated structure of singularities. A minimal requirement on all redefinitions is that the expansion of the integral in the perturbation series should be identical to the ordinary expansion obtained from the recursion relations. It seems that our choice (32), (33) is one of the simplest and most natural.

5. THE SINGULARITY OF THE SUPPLEMENTARY INTEGRAL NEAR A PERIODIC TRAJECTORY

We consider a small neighborhood of a periodic trajectory to which there corresponds in perturbation theory the resonance torus (24) with certain N_1 and N_2. If $I(p, x)$ is an integral of the considered system of equations, it must, of course, remain an integral near this periodic trajectory as well. This means that it must also be an integral for solutions of the linearized equations of motion (A2). Knowing the monodromy matrix (A4), we can readily write down the general form of the integral of the linearized system.

Indeed, let us consider in more detail the monodromy matrix for trajectories satisfying the initial conditions (32). The general form of the monodromy matrix determined at the point satisfying (25) is given in (B10). To investigate this matrix, it is convenient to go over to the variables

$$u_1 = \eta_1 + \gamma\xi_2, \quad u_2 = \mu\xi_1 + \nu\eta_2, \quad u_3 = \nu\eta_1 + \mu\xi_2, \quad u_4 = \eta_2 + \gamma\xi_1, \tag{34}$$

where $\mu = b\gamma + q$, $\nu = c\gamma - b$, and the remaining quantities are determined in (B10). In the lowest order, we find from (B11) that

$$\mu = \frac{7}{6} T(x_{2p}^2 - \dot{x}_{1p}^2), \quad \nu = \frac{7}{6} T \frac{x_{2p}}{\dot{x}_{1p}} (x_{2p}^2 - \dot{x}_{1p}^2). \tag{35}$$

In the variables u_i, the monodromy matrix (B10) takes the form

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b^2 + cq & 1 & 0 & 0 \\ 0 & 0 & 1 + {}^1/_2\Delta & \Delta(1 + \Delta/4) \\ 0 & 0 & 1 & 1 + {}^1/_2\Delta \end{pmatrix}, \tag{36}$$

where $\Delta = \mathrm{Sp}\, M - 4$. The monodromy matrix of the periodic trajectories of any Hamiltonian system with two degrees of freedom can be reduced to a similar form. It can be seen from (36) that there exist two simple invariants of the matrix M:

the trivial

$$h_0(\xi,\eta)=u_1=\eta_1+\gamma\xi_2 \tag{37}$$

and the nontrivial

$$h(\xi,\eta)=u_3^2-\Delta(1+\Delta/4)u_4^2. \tag{38}$$

For the two-dimensional area-preserving mappings of the type (5), (6) considered in Refs. 10 and 12, the first integral did not exist but only the second.

The first integral has a very simple meaning. Since the energy for autonomous Hamiltonian systems is, of course, an integral of the motion,

$$\delta E=\frac{\partial E}{\partial p_i}\eta_i+\frac{\partial E}{\partial x_i}\xi_i \tag{39}$$

is an integral for the linearized system. For the Hénon-Heiles model, E is determined in (7), and for trajectories satisfying the conditions (25) this expression is equivalent to (37).

The supplementary integral depends formally on four variables. However, because of the energy conservation it will be a function of E and three variables. Below, we shall consider it on an isoenergy surface, i.e., we shall admit only those variations for which δE in (39) is equal to zero.

Like perturbation theory for Hamiltonian systems near a position of equilibrium, it is possible to construct a perturbation theory near a periodic trajectory. The nature of the series of this perturbation theory will depend on the eigenvalues of the monodromy matrix, but in all cases when we are on an isoenergy surface the first term in the expansion of any formal integral in a series in the deviations from the initial point of the chosen periodic trajectory will be a function of the quadratic form $h(\xi\eta)$ defined in (38):

$$\left(I(p_i^{(0)}+\eta_i,x_i^{(0)}+\xi_i)-I(p_i^{(0)},x_i^{(0)})\right)\Big|_{\substack{x_1^{(0)}=0,\ p_2^{(0)}=0\\ E=\text{const}}}=f(h(\xi,\eta)). \tag{40}$$

If we take $I(p,x)$ in this equation to be the supplementary integral defined in (32), (33), we can readily find the form of the function $f(h)$. Indeed, on our chosen plane $I(p,x)$ is the fixed function $I_0(p_1,x_2)$ [see (33)]. Therefore, near any point on this plane we must have

$$\left(I(p_i^{(0)}+\eta_i,x_i^{(0)}+\xi_i)\Big|_{\substack{x_1^{(0)}=0,\ \xi_1=0\\ p_2^{(0)}=0,\ \eta_2=0}}-I_p^{(0)}\right)\Big|_{E=\text{const}}=\left[\left(\frac{\partial I^{(0)}}{\partial p_1}\right)\eta_1+\left(\frac{\partial I^{(0)}}{\partial x_2}\right)\xi_2\right]_{E=\text{const}}=ku_3, \tag{41}$$

where

$$k=\frac{1}{\gamma\nu-\mu}\left(\gamma\frac{\partial I^{(0)}}{\partial p_1}-\frac{\partial I^{(0)}}{\partial x_2}\right),$$

in which $I_p^{(0)}=I_0(\dot x_{1p},x_{2p})$ is the value of the integral on the periodic trajectory. If we assume that $I(p,x)$ is defined in (22), then from (35) we find the value of k in the lowest order:

$$k=x_{2p}\dot x_{1p}/(x_{2p}^2-\dot x_{1p}^2)T. \tag{42}$$

Comparing (41) with (38) and (40), we find that

$$\left(I(p_i^{(0)}+\eta_i,x_i^{(0)}+\xi_i)-I_p^{(0)}\right)\Big|_{\substack{x_1^{(0)}=0,\ p_2^{(0)}=0\\ E=\text{const}}}=k[u_3^2-\Delta(1+\Delta/4)u_4^2]^{1/2}. \tag{43}$$

Using the explicit form of the monodromy matrix around other points of the given periodic trajectory or simply using the invariance of $I(p,x)$ under translation along the trajectory, we can find the form of the singularity of the supplementary integral near an arbitrary point of the periodic trajectory on the isoenergy surface.

To do this, we note that in perturbation theory each periodic trajectory lies on the two-dimensional torus determined by the equations $E=\text{const}$ and $I(p,x)=I_p=\text{const}$. Let $(p_i^{(0)},x_i^{(0)})$ be some point of a periodic trajectory. On the three-dimensional equal-energy surface $E=\text{const}$, we introduce a triplet of mutually perpendicular vectors. We take the first parallel to the tangent to the periodic trajectory at the point, the second parallel to the $I(p,x)$ gradient vector calculated by perturbation theory, and the third vector to lie on the torus and be perpendicular to the other two. Then from (37) we can find that the singular part of the supplementary integral has the form

$$(I(p_i^{(0)}+\eta_i,x_i^{(0)}+\xi_i)-I_p^{(0)})_{E=\text{const}}=[(\delta_E I)^2-\Delta(1+\Delta/4)(\delta S)^2]^{1/2}, \tag{44}$$

$$\delta_E I=(I-I_p)_{E=\text{const}}=\left(\frac{\partial I}{\partial p_i}\eta_i+\frac{\partial I}{\partial x_i}\xi_i\right)_{E=\text{const}},$$

where $\delta_E I$ is proportional to the distance to the periodic trajectory calculated along the second vector, and δS is the distance along the third vector normalized in such a way that for points satisfying the condition (25) we have $\delta S=ku_4$, where k is given by (41). It is easy to find the first terms in the perturbation expansion of δS. If for the periodic trajectory N_1 and N_2 are large, then in the lowest order

$$(\delta S)^2=\frac{(\rho_1-\rho_2)^2}{2T^2N_1N_2}(l_p-\cos(N_2\Phi_1-N_1\Phi_2)), \tag{45}$$

where ρ_1 and ρ_2 are determined in (27), and Φ_1 and Φ_2 are angle type variables obtained by solving the equations of motion by perturbation theory and they are normalized such that $\Phi_1=\Phi_2=0$ for points satisfying (25). The lowest orders of perturbation theory for Φ_i are given in (B6); $l_p=l_p(E)$ is a constant, specific to each periodic trajectory. For the correct description of the singularity, l_p must be equal to 1. We introduced this additional constant to describe the arbitrariness in the determination of the supplementary integral (see below). If we expand Φ_1 and Φ_2 near the periodic trajectory determined by the condition $\cos(N_2\Phi_1-N_1\Phi_2)=1$, we readily see that we obtain a correctly normalized $(\delta S)^2$.

Note that $\cos(N_2\Phi_1-N_1\Phi_2)$ in (45) is a so-called resonance integral. It is a single-valued function of the coordinates and momenta and does not change under the influence of the equations of motion (since $\dot\Phi_1=\Omega_1$, $\dot\Phi_2=\Omega_2$, and $\Omega_1/\Omega_2=N_1/N_2$). As a result, $(\delta S)^2$ remains invariant under translation along the resonance torus. Thus, near every periodic trajectory of the Hamiltonian system for which $\Delta \neq 0$ the supplementary integral determined outside the framework of perturbation theory has a singularity of the type of the square root of a quadratic form that depends on the deviations from the given periodic trajectory. Knowing Δ, we can explicitly find the coefficients of this quadratic form using perturbation theory.

6. CONSTRUCTION OF ASYMPTOTIC EXPRESSIONS FOR THE SUPPLEMENTARY INTEGRAL

To obtain asymptotic estimates of the perturbation theory coefficients for the supplementary integral, it is necessary to construct a function having the above singularities near all periodic trajectories of the considered problem. Because Δ is small, we can simplify this problem. We expand (44) in powers of Δ and retain only the first nontrivial term:

$$I_{\rm sing} = -\frac{\Delta}{2}\frac{(\delta S)^2}{(I-I_p)_E}. \tag{46}$$

This expression does not mean that near the periodic trajectory the integral has a pole. The correct singularity is indicated in (44). However, if we expand the expression (44) in a series in the coupling constant, the higher orders of this expansion will be identical to the higher orders of the expansion (46). For our purposes, this is sufficient. Since the function (46) has a pole type singularity near the periodic trajectory, the function

$$I^{(as)}(p,x) = -\sum_p \frac{\Delta_p}{2}\frac{(\delta S)^2}{(I-I_p)_E}, \tag{47}$$

where the summation is extended to all periodic trajectories, will have such singularities near each periodic trajectory (Δ_p in this expression is the value of Δ for the given periodic trajectory).

In principle, the expression (47) solves the posed problem. It has the correct singularities (in this connection, see the remark above) near each periodic trajectory and its expansion with respect to the coupling constant gives us the asymptotic estimates in which we are interested. Before we turn to the explicit expressions, let us consider the normalization of the integral. As we mentioned in Sec. 4, the integral is determined up to an arbitrary function of two variables. We recover the integral from knowledge of the singularities with respect to the coupling constant. In such an approach, the arbitrariness in $I(p, x)$ is manifested as the possibility of adding to the function (47) some polynomial (or an entire function) in the coupling constant. Since this additional function does not have singularities at finite values of the coupling constant, the coefficients of its expansion in a peturbation theory series decrease with increasing order of the perturbation theory. Therefore, in this case nothing need be added to the function (47). However, since the function $I_0(p_1, x_2)$ in (32), (33) is to a large degree arbitrary, it could be that the suplementary integral will have "redundant" singularities due to the singularities of the function $I_0(p_1, x_2)$, i.e., the perturbation series for $I_0(p_1, x_2)$ will be only an asymptotic series. An example of such a case is given in Ref. 12. It was to take into account this possibility that we introduced the additional constant $l_p(E)$ in the definition of $(\delta S)^2$ in (45). We choose a constant l_p, since the residue at the pole for $I = I_p$ must be the same along an entire periodic trajectory.

To obtain asymptotic expressions for the perturbation theory coefficients, it is necessary to expand (47) in a series with respect to the coupling constant (in a series in homogeneous polynomials). Only $I(p, x)$ in the denominators in (47) depends on the coupling constant. In what follows, it is convenient to go over from the variables (p_1, p_2, x_1, x_2) to the variables $(r, \theta, \varphi_1, \varphi_2)$ in accordance with

$$x_1=\rho_1\sin\varphi_1+\rho_2\sin\varphi_2;\quad p_1=\rho_1\cos\varphi_1+\rho_2\cos\varphi_2, \tag{48}$$
$$x_2=\rho_1\cos\varphi_1-\rho_2\cos\varphi_2,\quad p_2=-\rho_1\sin\varphi_1+\rho_2\sin\varphi_2,$$

where $\rho_1 = r\cos\theta$, $\rho_2 = r\sin\theta$ [cf. (9)]. It is obvious that r plays the part of a coupling constant.

We calculate first the difference $(I-I_p)E$:

$$(I-I_p)_E=(\partial I/\partial r)_p(r-r_p(\theta,\varphi)), \tag{49}$$

where $r_p(\theta, \varphi)$ is the value r at which $(I-I_p)_E = 0$ (or $R = R_p$). Using the expressions (26) and (B7), we obtain

$$r_p(\theta,\varphi)=a_p(\theta)+r_1(\theta,\varphi)a_p^2(\theta)+r_2(\theta,\varphi)a_p^3(\theta), \tag{50}$$

where

$$a_p^2(\theta)=\frac{6(N_1-N_2)}{7(N_1+N_2)\cos 2\theta},$$

$$r_1(\theta,\varphi)=-\frac{g_{11}\cos^2\theta-g_{21}\sin^2\theta}{\cos 2\theta},$$

$$r_2(\theta,\varphi)=-\frac{g_{12}\cos^2\theta-g_{22}\sin^2\theta}{\cos 2\theta}+\frac{5}{2}r_1^2(\theta,\varphi)-\frac{3}{8}.$$

The quantities $g_{ij}(\theta, \varphi)$ are determined in (B7). Note that there exist two values of r_p corresponding to the two different signs of a_p. Below, we shall label by the indices (+) and (−) the quantities corresponding to the two signs of a_p. For $(\partial I/\partial r)_p$, the lowest order of perturbation theory is sufficient: $(\partial I/\partial r)_p = 7a_p^3\cos^2 2\theta/3$. Fixing the values of the angles θ, φ_1, and φ_2, we simultaneously fix the value of the resonance energy E_p:

$$E_p=r_p^2+r_p^3E_3(\theta,\varphi), \tag{51}$$

where $E_3(\theta, \varphi)$ is given by (B5).

As is clear from the derivation of (43) and (44), the residue at the pole for $I = I_p$ must be calculated on the periodic trajectory nearest the given point (p, x). The conditions determining the periodic trajectory $E = \text{const}$, $I = I_p$ (or $R = R_p$) and $N_2\Phi_1 - N_1\Phi_2 = 2\pi k$, where k is an integer, fix ρ_1 and ρ_2 (or r_p and θ) in the lowest order and give relations between φ_1 and φ_2: $N_2\varphi_1 - N_1\varphi_2 = 2\pi k$. To satisfy the last relation, we must add to φ_1 and φ_2 certain terms of order $1/N \propto a_p^2$. Since a dependence on φ_j arises only in the higher terms in the perturbation theory expansion of h and E, the additional terms will have a higher order of smallness than the terms retained in (50) and (51), and they can be ignored in the obtaining of the principal term of the asymptotic expressions (see below). Therefore, by φ_i in (50) and (51) we can understand the given φ_1 and φ_2.

It follows from (50) that a_p is the real expansion parameter for the periodic trajectories. For $|a_p|$ to be small, two conditions must be satisfied:

1) For fixed value of the difference $N_1 - N_2$, the sum $N_1 + N_2$ must be large;

2) $\cos 2\theta$ must not be very small.

The second condition eliminates the region of phase space near the periodic trajectories π_1–π_6 (see Fig. 1), which is natural, since we have not taken into account the periodic trajectories surrounding the elliptic points π_1–π_3. In addition, we note that the third terms in the expansion of the action-angle variables (B6) and (B7) contain terms which increase on the approach to the trajectories π_7 and π_8. Therefore, the expression (52) given below for $I_n^{(as)}$ applies in the complete phase space except for small (for small a_p) regions near the periodic trajectories π_1–π_8. When obtaining an asymptotic expression for the supplementary integral in these regions, it is necessary to use other expansion parameters. Thus, to describe the regions around the trajectories π_1–π_6 (π_7–π_8) it is necessary to assume that $\cos 2\theta$ (respectively, $\sin 2\theta$) is small and instead of the series (10) to consider series with respect to these parameters.

Note that for the model (11), using the transformation $z(t) \to \overline{z(t)}$, we can always construct from one periodic trajectory another with interchanged N_1 and N_2. To these two trajectories there correspond real and imaginary values of a_p in (50).

Taking into account Eqs. (49), (50), and (45), we find

$$I_n^{(as)}(p, x) = \sum_n r^n I_n^{(as)}(\theta, \varphi), \qquad (52)$$

where

$$I_n^{(as)}(\theta, \varphi) = \sum_p g_p(\theta) \left[\frac{\Delta_p(E_p^{(+)})}{(r_p^{(+)})^n} W_p^{(+)}(\theta, \varphi) - \frac{\Delta_p(E_p^{(-)})}{(r_p^{(-)})^n} W_p^{(-)}(\theta, \varphi) \right],$$

$$W_p^{(\pm)} = l_p^{(\pm)} - \cos(N_2\Phi_1^{(\pm)} - N_1\Phi_2^{(\pm)}),$$

$$g_p(\theta) = \frac{3(1-\sin 2\theta)}{112\pi^2 N_1^2 N_2^2 \cos^2 2\theta a_p^2},$$

and $a_p(\theta)$ and $r_p^{(\pm)}(\theta, \varphi)$ were determined in (50). The two terms in the square brackets correspond to the two different signs of a_p in (50). The minus sign in front of the second term is due to the fact that in accordance with (A14), (A13), and (B7) the quantity Δ transforms under $a_p \to -a_p$ as follows:

$$\Delta(E_p^{(+)}) \to (-1)^{N_1+N_2} \Delta(E_p^{(-)}).$$

Since we consider periodic trajectories for which N_1 and N_2 satisfy the conditions (28) and (30), the sum $N_1 + N_2$ is an odd number, and $(-1)^{N_1+N_2} = -1$, i.e., the two different values of r_p in (50) in this case correspond to two different types of trajectory—elliptic and hyperbolic (see Fig. 1). The $\Phi_1^{(\pm)}$ and $\Phi_2^{(\pm)}$ in (52) are angle type variables determined by perturbation theory. The first terms of their series expansions are given in (B6). In the calculations, one must substitute in place of r in these expressions the quantities $r_p^{(\pm)}$ determined in (50). Equation (52) gives the explicit dependence of Δ on the energy. Namely, for the resonance values of the energy $E_p^{(\pm)}$ determined in (45) the traces of the monodromy matrix in (52) must be calculated. Knowing Δ for a given periodic trajectory as a function of the energy, calculated, for example, numerically as in Fig. 2, we can readily find directly from (52) the contribution of this trajectory to the higher perturbation orders.

In principle, Eq. (52) solves the problem we have posed of calculating the higher orders of perturbation theory for the supplementary integral. However, the sum in (52) is taken over all periodic trajectories, and we must therefore consider what trajectories make the main contribution at large N_1, N_2 (and small a_p).

For an estimate, we use the expression (A14) for Δ. Omitting all the correction factors, we find that the trajectory corresponding to N_1 and N_2 makes a contribution to $I_n^{(as)}$ proportional to

$$S_n = (dP_1)^{N_2}(dP_2)^{N_1} a_p^{-n}, \qquad (53)$$

where in the lowest order [see (B7)] $P_1 = a_p \cos\theta$, $P_2 = a_p \sin\theta$. Bearing in mind that for the trajectories in which we are interested $N_1 \approx N_2 \propto N$ and $a_p^2 = c/N$, where $c = |3(N_1 - N_2)/(7\cos 2\theta)|$, we find

$$S_n(N) = \exp(U(N)), \qquad (54)$$

where

$$U(N) = \left(\frac{n}{2} - N\right) \ln\left(\frac{N}{c}\right) + N \ln q \text{ and } q = \frac{d^2}{2} \tan 2\theta.$$

The function S_n attains a maximum at $N = N_m$ determined from the condition $dU/dN = 0$, from which we obtain a rough estimate for N_m and $S_n(N_m)$:

$$N_m \approx \frac{n}{2\ln(ne/2cq)}, \quad S_n(N_m) \approx \left(\frac{n}{2}\right)! \frac{1}{(c\ln(ne/2qc))^n}. \qquad (55)$$

It can be seen from these expressions that when q is of order unity the dependence of N_m and $S_n(N_m)$ on q is weak and the main contribution to the sum over the periodic trajectories in (52) is made by one trajectory, for which $N = N_m$.

It is clear from the expression for $S_n(N_m)$ that the main contribution for fixed sum $N_1 + N_2$ is made by the trajectories with the minimal difference $N_1 - N_2 = \pm 1$. As we discussed in Sec. 3, this condition means that the corresponding trajectory passes once around the point π_7 (or π_8). [Of course, the relation (30) must also be satisfied, since it is only in this case that the expression (A14) for Δ holds.]

It is clear from the same estimates how many perturbation theory coefficients must be retained to obtain the principal term of the asymptotic formula. Since $r_p \propto 1/N^{1/2}$, in Φ_1 and Φ_2 it is necessary to take into account the first three terms, and since $n \propto N_m$, three perturbation coefficients are also important in r_p. In all the remaining quantities, the lowest order of perturbation theory is sufficient.

The expression (52) is the main result of this paper. It relates the parameters of periodic trajectories to the higher perturbation theory coefficients for the supplementary integral. Some of these quantities, such as r_p, Φ_1, Φ_2, can be obtained by ordinary perturbation theory. Others, like Δ, must either be calculated numerically or estimated by means of expressions of the type (A14).

A rough estimate for I_n at large n can be obtained by combining (55) and (52):

$$I_n^{(as)}(\theta,\varphi)=\left(\frac{n}{2}\right)!\frac{g_p(\theta)}{(c\ln(ne/2qc))^n}$$

$$\times[W_p^{(+)}e^{-nr_1a_p}-(-1)^n W_p^{(-)}e^{nr_1a_p}], \qquad (56)$$

where we have taken into account only one periodic trajectory, for which $N_1+N_2\approx 2N_m$ [and, of course, N_1 and N_2 satisfy (28) and (30)]. It would be interesting to compare the asymptotic expression (52) with the results of numerical calculation. Unfortunately, the results of numerical calculations of the perturbation theory coefficients for the supplementary integral have been published only up to $n=8$.[16] It can be seen from (55) that the saddle point trajectory in this case will be the trajectory with $(N_1, N_2)=(2, 1)$, which according to the estimates lies outside the region of the perturbation theory (the expansion parameter for it is too large). The authors of Refs. 15 and 17 say that the method that they developed in these papers makes it possible to calculate the first 18 terms of the perturbation theory series, but results are not given.

7. CONCLUSIONS

In conclusion, we emphasize the following:

1) The method proposed for calculating the higher orders of perturbation theory can evidently be used in any problem of classical mechanics (at least, with two degrees of freedom). We have used it to calculate the higher orders of perturbation theory for a supplementary integral, but it can be modified to obtain asymptotic estimates for other quantities too, for example, to solve the equations of motion.

2) The success of the method depends to a large degree on the possibility of obtaining asymptotic estimates for the monodromy matrix or for the higher coefficients of the Fourier expansion of the periodic solutions. These questions have so far been little studied. Analytic and numerical study of the properties of the periodic solutions of Hamiltonian systems is undoubtedly of great interest. If we restrict ourselves to asymptotic estimates for not too high orders of perturbation theory (for example, $n\lesssim 20$), the main contribution will be made by only a few periodic trajectories, the parameters of which can be found numerically. This has all the more point for real Hamiltonian systems, for which the determination of very high orders of perturbation theory is a difficult and tedious task.

3) It would be interesting to study the higher orders of perturbation theory for real problems of celestial mechanics and (or) the physics of colliding-beam accelerators, for which a large number of perturbation theory coefficients is needed. For example, many coefficients of the perturbation theory for the motion of the Moon are known (see the review of Ref. 13 and the references in it). One would believe that, as in simpler problems,[9] allowance for the higher orders of perturbation theory will make it possible to extend greatly the region of applicability of perturbation theory in classical mechanics. For a fixed coupling constant, this means that the time during which the description of the motion by perturbation theory is admissible can be greatly increased.

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APPENDIX A: MONODROMY MATRIX OF PERIODIC SOLUTIONS OF A HAMILTONIAN SYSTEM

Let $x_i^{(0)}(t)$ and $p_i^{(0)}(t)$ be a periodic solution of the Hamiltonian equations of motion ($i=1, ..., n$)

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial x_i} \qquad (A1)$$

with period T: $x_i^{(0)}(t+T)=x_i^{(0)}(t)$, $p_i^{(0)}(t+T)=p_i^{(0)}(t)$. We set

$$x_i(t)=x_i^{(0)}(t)+\xi_i(t), \quad p_i(t)=p_i^{(0)}(t)+\eta_i(t)$$

and linearize Eqs. (A1) near the chosen periodic solution. Retaining the terms linear in ξ_i and η_i, we obtain a system of equations for the deviations from the periodic trajectory (equations in variations):

$$\frac{d\xi_i}{dt}=\left(\frac{\partial^2 H}{\partial p_i\,\partial x_j}\right)_0\xi_j+\left(\frac{\partial^2 H}{\partial p_i\,\partial p_j}\right)_0\eta_j,$$
$$\frac{d\eta_i}{dt}=-\left(\frac{\partial^2 H}{\partial x_i\,\partial x_j}\right)_0\xi_j-\left(\frac{\partial^2 H}{\partial x_i\,\partial p_j}\right)_0\eta_j. \qquad (A2)$$

All the derivatives in these expressions are calculated on the chosen periodic solution and are periodic functions with period T. We introduce the quantities $\Psi_a(t)$($a=1,..., 2n$) such that

$$\Psi_a(t)=\xi_i(t) \text{ for } a=2i-1, \quad \Psi_a(t)=\eta_i(t) \text{ for } a=2i. \qquad (A3)$$

If T is the period of the functions $x_i^{(0)}$ and $p_i^{(0)}$ and, therefore, of the coefficients in (A2), the monodromy matrix M_{ab}(a, $b=1,..., 2n$) of the linear equations (A2) is determined by

$$\Psi_a(t_0+T)=M_{ab}\Psi_b(t_0). \qquad (A4)$$

By virtue of the conservation of phase space, $\mathrm{Det}\,M=1$. The main quantity in which we shall be interested is the trace of the monodromy matrix or, more precisely,

$$\Delta=M_{aa}-2n. \qquad (A5)$$

One can show that when the period $T\to\infty$ the quantity Δ decreases faster than any fixed power of the coupling constant, and therefore it is difficult to obtain an asymptotic expression for Δ.

We expand the given periodic solution in a Fourier series:

$$x_j^{(0)}(t)=\sum_{m=-\infty}^{+\infty}x_m^{(j)}e^{im\Omega_0 t}, \qquad (A6)$$

where $\Omega_0=2\pi/T$. As was done in Ref. 12 for area-preserving mappings, it is possible to relate Δ to the Fourier components of the given periodic solution.

We give without derivation the leading term of the corresponding expression for $n=2$:

$$\Delta=\mathrm{Sp}\,M-4=\alpha_{(+)}B+\alpha_{(-)}C, \qquad (A7)$$

where

$$\alpha_{(\pm)}=4\pi N_1N_2\Omega_0\sum_{p=\pm1}p\sum_{\substack{m_1+m_2+m_3=N_2p\\ n_1+n_2+n_3=-N_1p}}(m_3N_1\pm n_3N_2)\,\Phi(m_i,n_i), \qquad (A8)$$

and $\Phi(m_i, n_i)$ is a polynomial in $x^{(i)}_m$ that is associated with the particular form of the potential energy of the considered problem. For the Hénon-Heiles model,

$$\Phi(m_i, n_i)=2x^{(1)}_{m_1,n_1}x^{(1)}_{m_2,n_2}x^{(1)}_{m_3,n_3}+x^{(2)}_{m_1,n_1}x^{(2)}_{m_2,n_2}x^{(2)}_{m_3,n_3}-x^{(1)}_{m_1,n_1}x^{(1)}_{m_2,n_2}x^{(2)}_{m_3,n_3}, \qquad (A9)$$

where $x^{(i)}_{m,n}\equiv x^{(i)}_{mN_1+nN_2}$, and B and C are certain quantities that can be calculated by ordinary perturbation theory. For the Hénon-Heiles model, the first terms in the expansion of these quantities are

$$B=0,\quad C=-{}^7/_6T; \qquad (A10)$$

N_1 and N_2 are the integers that characterize the resonance torus on which the given periodic trajectory lies in accordance with perturbation theory (see Sec. 3). Equation (A8) with $p=\pm1$ for the Hénon-Heiles model is valid under the condition

$$N_1+N_2\equiv 0\ (\mathrm{mod}\ 3). \qquad (A11)$$

Periodic trajectories for which N_1 and N_2 do not satisfy this condition have a Δ much less than (A7), and they are not important for our purposes. Equations (A7)–(A10) enable us, if we know the Fourier components of the periodic solution, to calculate the trace of the monodromy matrix. Since simple iterative methods can be used to determine the Fourier components of the periodic solutions,[18] this may be the most convenient method of numerical calculation of the trace of the monodromy matrix.

To obtain analytic expressions for Δ, it is necessary to know the high (for $N_1, N_2\to\infty$) Fourier components of the considered periodic solution. Unfortunately, little is known about the properties of periodic solutions of Hamiltonian systems, and we restrict ourselves here to rough estimates.

One can give arguments that suggest that for $x^{(i)}_{m,n}$ the following estimate is natural:

$$x^{(i)}_{m,n}\equiv x^{(i)}_{mN_1+nN_2}=\frac{1}{N_1N_2}a_1^{|m|}a_2^{|n|}g^{(i)}(N_1m,N_2n), \qquad (A12)$$

in which the form of the function $g^{(i)}(y_1, y_2)$ is fixed and the quantities a_i are proportional to the action type variables of the considered problem:

$$a_i=dP_i, \qquad (A13)$$

where d is a constant. For the Hénon-Heiles model, the first terms in the expansion of P_i are given by (B7).

Substituting (A12) and (A13) in (A7), we obtain

$$\Delta_{N_1,N_2}=DN_1N_2a_1^{N_2}a_2^{N_1}, \qquad (A14)$$

where D is a constant. This expression, like (A7), is valid when the condition (A11) is satisfied. For other N_1 and N_2, as in the similar cases for area-preserving mappings,[10,12] the value of Δ is much smaller and, roughly speaking, proportional to the square of (A14). Because of the roughness of our estimates, we cannot rule out that D in (A14) and d in (A13) are certain slowly varying functions of E and (or) N_1 and N_2.

For small N_1 and N_2, we can calculate numerically Δ and make a comparison with (A14). Figure 2 gives the results of calculation of Δ for a number of trajectories. For example, for $(N_1, N_2)=(2, 1)$ it is found numerically that in the range of energies for which $0\leqslant|\Delta_{2,1}|\leqslant1$ the dependence $\Delta(E)$ can be well approximated by the straight line

$$\Delta_{2,1}(E)=-392(E-0.1487), \qquad (A15)$$

which does not contradict (A14).

For large (N_1, N_2), reasonable agreement between the estimate (A14) and the results of numerical calculation can be achieved for $D\approx0.1$ and $d\approx4$.

APPENDIX B: PERTURBATION THEORY FOR THE HÉNON-HEILES MODEL

If p_i, x_i $(i=1, 2)$ satisfy the Hamilton equations with the Hamiltonian (7), then the quantities canonically conjugate to them:

$$\xi_1=x_1+\frac{4}{3}p_1p_2+\frac{2}{3}x_1x_2+\frac{17}{72}x_2p_1p_2-\frac{77}{144}x_1p_1^2-\frac{111}{144}x_1p_2^2+\frac{7}{144}(x_1x_2^2+x_1^3)+\dots,$$

$$\xi_2=x_2-\frac{2}{3}p_2^2+\frac{2}{3}p_1^2-\frac{1}{3}x_2^2+\frac{1}{3}x_1^2-\frac{111}{144}x_2p_1^2+\frac{17}{72}x_1p_1p_2+\frac{7}{144}(x_2^3+x_1^2x_2)+\dots, \qquad (B1)$$

$$\eta_1=p_1-\frac{2}{3}x_2p_1-\frac{2}{3}x_1p_2-\frac{17}{144}(p_1^3+p_1p_2^2)+\frac{57}{144}x_2^2p_1-\frac{7}{72}x_1x_2p_2+\frac{43}{144}x_1^2p_1+\dots,$$

$$\eta_2=p_2+\frac{2}{3}p_2x_2-\frac{2}{3}x_1p_1-\frac{17}{144}(p_2^3+p_1^2p_2)+\frac{43}{144}x_2^2p_2-\frac{7}{72}x_1x_2p_1+\frac{57}{144}x_1^2p_2+\dots$$

satisfy the Hamilton equations with the Hamiltonian

$$H(\eta,\xi)=h^2-\frac{5}{12}h^4+\frac{7}{12}I_3^2+\frac{101}{3456}I_1^3-\frac{235}{3456}I_2^3+\frac{773}{1152}I_2I_4^2+\frac{73}{1152}I_3^2I_2-\frac{907}{1152}I_1I_4^2-\frac{263}{1152}I_1I_3^2+\dots, \qquad (B2)$$

where I_i and h^2 are determined in (18)–(20).[16] (For our purposes, the terms of higher order in ξ, η, and H are not needed.)

The first terms in the expansion of the supplementary integral (22) in a perturbation series have the following form[2)]:

$$I(p,x)\equiv E-h^2=-\frac{5}{12}E^2+\frac{7}{12}L^2-\frac{49}{3456}I_1^3 -\frac{385}{3456}I_2^3\pm\frac{623}{1152}I_2I_4^2 +\frac{203}{1152}I_2I_3^2-\frac{1057}{1152}I_1I_4^2-\frac{133}{1152}I_1I_3^2+O(\lambda^7), \quad \text{(B3)}$$

where

$$L(p,x)=\eta_1\xi_2-\eta_2\xi_1=p_1x_2-p_2x_1+p_1x_1^2-p_1x_2^2 +\frac{2}{3}p_1^3-2p_2x_1x_2 -2p_2^2p_1+\frac{2}{3}(p_1x_2-p_2x_1)(x_1^2+x_2^2-p_1^2-p_2^2). \quad \text{(B4)}$$

The quantities I_i are determined in (18) and (19) with replacement of $(\eta,\ \xi)$ by $(p,\ x)$; E is the energy: $E=(p_1^2+p_2^2+x_1^2+x_2^2+2x_1^2x_2-2x_2^3/3)/2$; $O(\lambda^7)$ means that in (B3) the homogeneous polynomials to sixth degree inclusively are correct.

To solve the Hénon-Heiles equations, it is convenient to go over from the variables (p,x) to the variables $(r,\theta,\varphi_1,\varphi_2)$ in accordance with (48). We begin by rewriting the energy (7) in the new coordinates:

$$E=r^2+r^3E_3(\theta,\varphi), \quad \text{(B5)}$$

where

$$E_3(\theta,\varphi)=-{}^1/_3\cos^2\theta\cos 3\varphi_1+\cos^2\theta\sin\theta\cos(2\varphi_1-\varphi_2) -\cos\theta\sin^2\theta\cos(2\varphi_2-\varphi_1)+{}^1/_3\sin^3\theta\cos 3\varphi_2.$$

We give without derivation the first three perturbation theory terms for the action-angle variables. The angle type variables are

$$\Phi_1=\varphi_1+rf_{11}(\theta,\varphi)+r^2f_{12}(\theta,\varphi),$$
$$\Phi_2=\varphi_2+rf_{21}(\theta,\varphi)+r^2f_{22}(\theta,\varphi), \quad \text{(B6)}$$

where

$$f_{11}(\theta,\varphi)=\frac{1}{6}\cos\theta\sin 3\varphi_1-\sin\theta\sin(2\varphi_1-\varphi_2) +\frac{1}{2}\frac{\sin^2\theta}{\cos\theta}\sin(2\varphi_2-\varphi_1),$$

$$f_{21}(\theta,\varphi)=-\frac{1}{2}\frac{\cos^2\theta}{\sin\theta}\sin(2\varphi_1-\varphi_2)+\cos\theta\sin(2\varphi_2-\varphi_1) -\frac{1}{6}\sin\theta\sin 3\varphi_2,$$

$$f_{12}(\theta,\varphi)=-\frac{1}{72}\cos^2\theta\sin 6\varphi_1+\frac{1}{6}\sin 2\theta\cos 3\varphi_1\sin(2\varphi_1-\varphi_2) +\frac{1}{8}\frac{\sin^4\theta}{\cos^2\theta}\sin(4\varphi_2-2\varphi_1) +\frac{1}{2}\sin^2\theta\left(\frac{1}{3}\cos 3\varphi_1\sin(\varphi_1-2\varphi_2) -\sin(4\varphi_1-2\varphi_2)\right)-\frac{\sin^3\theta}{\cos\theta}\sin(2\varphi_1-\varphi_2)\cos(\varphi_1-2\varphi_2) +\frac{1}{18}\frac{\sin 2\theta}{\cos^2 2\theta}(1+2\cos^2 2\theta-3\cos 2\theta)\sin(3\varphi_1-3\varphi_2),$$

$$f_{22}(\theta,\varphi)=\frac{1}{2}\cos^2\theta\left(\frac{1}{3}\cos 3\varphi_2\sin(\varphi_2-2\varphi_1) -\sin(4\varphi_2-2\varphi_1)\right) +\frac{1}{6}\sin 2\theta\cos 3\varphi_2\sin(2\varphi_2-\varphi_1)-\frac{1}{72}\sin^2\theta\sin 6\varphi_2 +\frac{\cos^3\theta}{\sin\theta}\cos(2\varphi_1-\varphi_2)\sin(\varphi_1-2\varphi_2)+\frac{1}{8}\frac{\cos^4\theta}{\sin^2\theta}\sin(4\varphi_1-2\varphi_2) -\frac{1}{18}\frac{\sin 2\theta}{\cos^2 2\theta}(1+2\cos^2 2\theta+3\cos 2\theta)\sin(3\varphi_1-3\varphi_2).$$

The action variables (canonically conjugate to Φ_1 and Φ_2 are P_1^2 and $P_1^{(2)}$) are

$$P_1=r\cos\theta(1+rg_{11}(\theta,\varphi)+r^2(g_{12}(\theta,\varphi)-{}^1/_2g_{11}^2(\theta,\varphi))),$$
$$P_2=r\sin\theta(1+rg_{21}(\theta,\varphi)+r^2(g_{22}(\theta,\varphi)-{}^1/_2g_{21}^2(\theta,\varphi))), \quad \text{(B7)}$$

where

$$g_{11}(\theta,\varphi)=-\frac{1}{6}\cos\theta\cos 3\varphi_1+\sin\theta\cos(2\varphi_1-\varphi_2) +\frac{\sin^2\theta}{2\cos\theta}\cos(\varphi_1-2\varphi_2),$$

$$g_{21}(\theta,\varphi)=-\frac{\cos^2\theta}{2\sin\theta}\cos(\varphi_2-2\varphi_1) -\cos\theta\cos(\varphi_1-2\varphi_2)+\frac{1}{6}\sin\theta\cos 3\varphi_2,$$

$$g_{12}(\theta,\varphi)=-\frac{5}{24}\cos^2\theta+\frac{1}{2}\sin^2\theta-\frac{\sin\theta\cos 2\theta}{\cos\theta}\cos(\varphi_1+\varphi_2) +\frac{1}{8}\frac{\sin^4\theta}{\cos^2\theta}+\frac{\sin^3 2\theta\cos(3\varphi_1-3\varphi_2)}{12\cos 2\theta\cos^2\theta}$$

$$g_{22}(\theta,\varphi)=\frac{1}{8}\frac{\cos^4\theta}{\sin^2\theta}+\frac{\cos\theta\cos 2\theta}{\sin\theta}\cos(\varphi_1+\varphi_2) -\frac{5}{24}\sin^2\theta+\frac{1}{2}\cos^2\theta-\frac{\sin^3 2\theta\cos(3\varphi_1-3\varphi_2)}{12\cos 2\theta\sin^2\theta}.$$

The equations of motion in the action-angle variables are

$$d\Phi_i/dt=\Omega_i,\quad dP_i/dt=0, \quad \text{(B8)}$$

with the frequencies

$$\Omega_1=1-\frac{5}{6}h^2+\frac{7}{6}R-\frac{67}{144}h^4-\frac{7}{144}R^2-\frac{7}{72}Rh^2,$$
$$\Omega_2=1-\frac{5}{6}h^2-\frac{7}{6}R-\frac{67}{144}h^4-\frac{7}{144}R^2+\frac{7}{72}Rh^2$$

and $R=P_1^2-P_2^2$, $h^2=P_1^2+P_2^2$ [h^2 was determined in (B.3)]. Also helpful is the expression relating $I(p,x)$ in (B3) to E and R:

$$I=\frac{7}{12}R^2-\frac{5}{12}E^2-\frac{217}{432}E^3+\frac{7}{16}R^2E. \quad \text{(B9)}$$

We need the general form of the monodromy matrix of periodic trajectories satisfying the initial conditions (25). Using the symmetry (13), we can show that the monodromy matrix at this point must have the form

$$M=\begin{pmatrix} 1+\rho & c & b & \rho/\gamma \\ -\gamma^2\alpha & 1+\rho & -\gamma\delta & -\gamma\alpha \\ \gamma\alpha & -\rho/\gamma & 1+\delta & \alpha \\ \gamma\delta & -b & q & 1+\delta \end{pmatrix}, \tag{B10}$$

where

$$\gamma=\frac{x_{2p}-x_{2p}^2}{\dot{x}_{1p}},\quad \alpha=-\Delta\left(1+\frac{\Delta}{4}\right)Q,$$

$$\rho=\frac{1}{2}\Delta\gamma(c\gamma-b)Q,$$

$$\delta=-\frac{1}{2}\Delta(b\gamma+q)Q,\quad Q=\frac{1}{c\gamma^2-2b\gamma-q}$$

in which $\dot{x}_{1p}$ and x_{2p} are the coordinates of the chosen periodic trajectory. In the lowest order, they are given by Eqs. (27). The parameters b, c, q can be expanded in an ordinary perturbation theory series. The first terms of this expansion are

$$c=\frac{T}{6}(7x_{2p}^2-5\dot{x}_{1p}^2),\quad b=\frac{T}{3}x_{2p}\dot{x}_{1p},$$

$$q=-\frac{T}{6}(7\dot{x}_{1p}^2-5x_{2p}^2), \tag{B11}$$

where T is the period of the given trajectory.

The properties of $\Delta = \mathrm{Sp}\, M-4$ were discussed in Appendix A.

[1)]We shall not here discuss the connection between Hamiltonian systems and area-preserving mappings. For this, see, for example, Ref. 5. We merely mention that these theories have much in common but the dimension of the phase space at which the phenomenon of nonintegrability arises is at least four for autonomous Hamiltonian systems and only two for area-preserving mappings. This reduction in the dimension greatly simplifies the calculations and is one of the reasons why area-preserving mappings are considered as examples. In addition, study of mappings of the type (6) is also of independent interest, since such mappings arise in the study of the motion of charged particles in colliding-beam accelerators.[11]

[2)]The sign of term I (185) in Table IV of Ref. 16 must be changed.

[1]V. I. Arnol'd, Matematicheskie metody klassicheskoĭ mekhaniki; English translation: Mathematical Methods of Classical Mechanics (Graduate Texts in Mathematics, Vol. 60), Springer, New York (1978).
[2]R. Garnier, Rend. Circ. Mat. Palermo **43**, 155 (1919).
[3]H. Poincaré, Les Méthodes Nouvelles de la Mécanique Céleste, Paris, Vol. 1 (1892), Vol. 2 (1899) (Russian translations published by Nauka, Moscow (1971, 1972).
[4]V. E. Zakharov, S. V. Manakov, and L. P. Pitaevskiĭ, Teoriya solitonov: Method obratnoĭ zadachi (The Theory of Solitons: The Inverse Scattering Method), Nauka, Moscow (1980).
[5]J. K. Moser, "Lectures on Hamiltonian systems," Mem. Am. Math. Soc. **81**, 1 (1968) (Russian translation published on book by Mir, Moscow (1973)).
[6]C. L. Siegel, Ann. Math. **42**, 806 (1941).
[7]A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **98**, 527 (1954).
[8]V. I. Arnol'd, Dokl. Akad. Nauk SSSR, **137**, 225 (1961); Usp. Mat. Nauk **18**, 91 (1963).
[9]L. N. Lipatov, Zh. Eksp. Teor. Fiz. **72**, 411 (1977) [Sov. Phys. JETP **45**, 216 (1972)]; J. Zinn-Justin, Phys. Rep. **49**, 205 (1979); E. B. Bogomolny, V. A. Fateyev, and L. N. Lipatov, Sov. Sci. Rev. Phys. Rev. **2**, 247 (1980).
[10]E. B. Bogomol'nyĭ, Yad. Fiz. **37**, 444 (1983) [Sov. J. Nucl. Phys. **37**, 266 (1983)].
[11]Nonlinear Dynamics and the Beam-Beam Interaction (eds. M. Month and J. C. Herrera), Am. Inst. Phys. Conf. Proc. (1979), p. 57.
[12]E. B. Bogomolny, CERN Preprint TH-3490 (1982).
[13]V. P. Gerdt, O. N. Tarasov, and D. V. Shirkov, Usp. Fiz. Nauk **130**, 113 (1980) [Sov. Phys. Usp. **23**, 59 (1980)].
[14]M. Hénon and C. Heiles, Astron. J., **69**, 73 (1964).
[15]G. Z. Contopoulos, Astrophys. **49**, 273 (1960); Astron. J. **68**, 763 (1963); Astron. J. Suppl. **13**, 122 (1966); G. Contopoulos and M. Moutsoulas, Astron. J. **70**, 817 (1965).
[16]F. G. Gustavson, Astron. J. **71**, 670 (1966).
[17]A. Giorgilli and L. Galgani, Celest. Mech. **17**, 267 (1978).
[18]C. R. Eminhizer, R. H. G. Helleman, and E. W. Montroll, J. Math. Phys. **17**, 121 (1976); R. H. G. Helleman, Statistical Mechanics and Statistical Methods, New York (1977), p. 343; A. C. Bountis, D. Ph. Thesis, Univ. of Rochester, Preprint (1978).
[19]R. C. Churchill. G. Recelli, and D. L. Rod, Lect. Notes Phys. **93**, 76 (1979).
[20]G. D. Birkhoff, Dynamical Systems, New York (1927) (Russian translation published by OG1Z, Moscow (1941).

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