Size effects in disordered conductors

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The dependences of the quantum corrections to the conductivity and to the density of states on the sample size are investigated. It is shown that the known experiments on the voltage dependences of the resistance of a thin wire on its length and of tunnel resistance are in good quantitative agreement with the developed theory.

1. INTRODUCTION

Two length scales are connected with the quantum corrections to kinetic and thermodynamic effects in disordered conductors. The first, $L_\varphi = (D\tau_\varphi)^{1/2}$, is the length over which dephasing of the electron wave function results from inelastic processes or of spin-spin scattering from paramagnetic centers. The second, $L_{T,\varepsilon} = (D/T,\varepsilon)^{1/2}$, is the correlation function of the wave functions of electrons that differ in energy by ε or T (we assume $\hbar = 1$ and $k = 1$). Here D is the electron diffusion coefficient, τ_ϕ is the dephasing time, and T is the temperature. The first length is connected with weak-localization effects,[1,2] and the second with effects of interaction between electrons.[3,4] The existence of these macroscopic scales leads to size-effect phenomena both in localization effects and in interaction effects when the sample dimensions become comparable with these lengths.

The effective dimensionality $d_{\rm loc}$ of a sample relative to weak-localization effects is determined by the relation between its geometric dimensions and the length L_φ. The effective dimensionality $d_{\rm int}$ of the sample relative to interaction effects is determined by the relation between the sample dimensions and the length $L_{T,\varepsilon}$.

Given a sample of length L, width b, and thickness a, with $L < b < a$, the dimensionality at $L_\varphi < a$ is $d_{\rm loc} = 3$. With decreasing temperature, the effective dimensionality decreases as L_φ increases: $d_{\rm loc} = 2$ at $b < L_\varphi < a$ and $d_{\rm loc} = 1$ at $L > L_\varphi > b$. At $L_\varphi > L$ we finally have $d_{\rm loc} = 0$. The variation of $d_{\rm int}$ is similar. Since usually $L_\varphi > L_T$, the effective dimensionality of the sample relative to weak-localization effects is not higher than relative to interaction effects ($d_{\rm int} \geqslant d_{\rm loc}$).

We note that weak localization and electron-electron interaction influence differently the kinetic and thermodynamic properties. Thus, whereas the former is determined by both effects, anomalies in the latter are connected only with interaction effects.

We investigate in this paper size effects in the conductivity and in the density of states, inasmuch as experimental data are available for only these two quantities.[5–7] We shall not consider all the particular cases, but discuss only those that, from our point of view, are most instructive or have already been realized in experiment.[5–7]

2. BASIC RESULTS

Tunnel-resistance anomalies in a tunnel junction, which are connected with a minimum of the density of states on the Fermi surface,[3] were investigated in tunnel junctions in which one electrode was a thin film of In_2O_{3-x}.[5] Variation of the applied voltage V revealed a transition from $d_{\rm int} = 2$ at $(D/eV)^{1/2} > a$ to $d_{\rm int} = 3$ at $(D/eV)^{1/2} < a$. In the three-dimensional case, a square-root dependence of the tunnel-junction resistance on the applied voltage V was observed. This agrees with the theory of Ref. 3, but the effect was approximately twice as large. At low voltages and in very thin films, a logarithmic dependence of the tunnel-junction resistance on the voltage V was observed. This behavior does not agree with theory of Ref. 4, according to which the dynamic screened Coulomb interaction between electrons leads in the two-dimensional case to a doubly logarithmic energy dependence of the density of states, namely,

$$\delta\nu(\varepsilon) = -\frac{\nu_0}{4\pi\mu\tau}\ln\left(\frac{\varepsilon a^2}{D^3\varkappa_2^4}\right)\ln\left(\frac{\varepsilon a^2}{D}\right). \qquad (1)$$

Here ν_0 is the density of states on the Fermi level for particles with given spin, μ is the Fermi energy, $\varkappa_2$ is the reciprocal screening length in the two-dimensional case,

$$\varkappa_2 = \varkappa^2 a/2, \quad \varkappa^{-1} = (8\pi e^2\nu_0)^{-1/2}, \qquad (2)$$

$\varkappa^{-1}$ is the Debye screening length, and τ is the momentum-relaxation time. An attempt was made[5] to explain the experimental results with account taken of only the screened Coulomb interaction between the electrons. The factor preceding the logarithm, however, turned out to be approximately a decade less than observed in experiment.

We show in the present paper that the experimental results can be explained quantitatively by taking into account, first, the interaction between the electrons from both electrodes of the tunnel junction, and second, by recognizing that in electron tunneling an important role is played by states near the junction interface over distances much shorter than $L_V = (D/eV)^{1/2}$, i.e., that the anomaly in the tunnel resistance is connected with the local density of states of the interacting electrons near the sample surface.

For a two-dimensional sample, when $L_\varepsilon \gg a$, expression (1) is valid under the condition of a Coulomb interaction between the electrons at large distances. In a tunnel junction, however, just as in MIS structures,[8] the bare interaction is not Coulombic. Indeed, in a tunnel junction in which one of the electrodes is a thin film and the other is bulk metal, both the film and bulky-electrode electrons participate in the screening of the interaction potential. If it is assumed that the conductivity σ of the bulky electrode is much higher than that of the film, i.e., that the Maxwellian relaxation time in

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the bulky electrode $\tau_M = (4\pi\sigma)^{-1}$ is much less than in the thin film, such a situation can be described by introducing image forces. Owing to the presence of image forces the interaction between the electrons in the film is dipolar at distances much larger than the insulator thickness Δ and the film thickness a. As a result, the energy dependence of the density of states takes at $\varkappa_2^{-1} < \Delta < L_\varepsilon$ the form

$$\delta\nu(\varepsilon) = (4\pi^2 aD)^{-1} \ln(2\varkappa_2\Delta) \ln(\varepsilon a^2/D). \quad (3)$$

The first logarithm in (3) is a reflection of the long-range character of the potential at distances shorter than Δ. Using the data of Ref. 5, viz., $D \approx 10$ cm^2/sec, $R_\square = \rho/a = 500\ \Omega$ (ρ is the resistivity), and $\Delta = 30$ Å, we get $\ln(2\varkappa_2\Delta) \approx 6$–7, in good agreement with experiment.

In MIS structures, owing to the purely two-dimensional character of the electrons on the surface, as well as in a tunnel junction at $a \ll \Delta$, there is an energy region where the density of states has the doubly logarithmic form (1). For a tunnel junction of two two-dimensional dirty conductors, expression (1) is valid for the density of states.

In a semi-infinite sample the local density of states (i.e., the density of states averaged over a small volume with linear dimensions much larger than the mean free path l but much smaller than L_ε) is of the form

$$\delta\nu(\varepsilon) = \delta\nu^{(j=0)}(\varepsilon, z) + \delta\nu^{(j=1)}(\varepsilon, z), \quad (4)$$

$$\delta\nu^{(j=0)} = \frac{\varepsilon^{1/2}}{2^{1/2}\pi^2 D^{3/2}} + \frac{1}{8\pi^2 Dz}\left\{2\int_{x_\varepsilon}^{x_{1/\tau}} \frac{dx}{x} e^{-x}\cos x + \ln\varepsilon\tau\right\}, \quad (5)$$

$$\delta\nu^{(j=1)} = \frac{3(1-(1+F/2)^{1/2})}{2^{5/2}\pi^2}\frac{\varepsilon^{1/2}}{D^{3/2}} + \frac{3}{8\pi^2 Dz}\int_{x_\varepsilon}^{\tilde{x}_\varepsilon} \frac{dx}{x} e^{-x}\cos x, \quad (6)$$

where $F/2$ is the exact dimensionless amplitude of the interaction at zero frequency difference, z is the distance from the sample surface, $x_{1/\tau} = z(2/D\tau)^{1/2}$, $x_\varepsilon = (2\varepsilon/D)^{1/2}$, $\bar{x}_\varepsilon = x_\varepsilon(1+F/2)^{1/2}$. The term $\delta\nu^j$ is a correction to the density of states and is necessitated by interaction of a particle and a hole with combined spin j.[9]

It follows from (4)–(6) that

$$\delta\nu(\varepsilon, z=0) = 2\delta\nu(\varepsilon, z\to\infty) = [1 - {}^3/_2((1+F/2)^{1/2}-1)]\varepsilon^{1/2}/2^{1/2}\pi^2 D^{3/2}. \quad (7)$$

Thus, allowance for the interface leads to a doubling of the local density of states near it, compared with that in the bulk; this was apparently observed in Ref. 5.

The dependence of the density of states at the Fermi level ($\varepsilon = 0$) on the distance to the interface at $z \gg l$ is of the form

$$\delta\nu(0, z) = -\frac{1}{4\pi^2 Dz}\ln\left(\frac{2z\gamma\sqrt{3}}{l}\right) + \frac{3}{16\pi^2 Dz}\ln\left(1+\frac{F}{2}\right), \quad (8)$$

where $\ln\gamma \approx 0.577$ is the Euler constant and l is the mean free path.

We discuss now size effects in the conductivity. Masden and Giordano[6] investigated the dependence of the resistance of thin $Au_{40}Pd_{60}$ wires on their length. It was observed that at $T \gtrsim 1$ K the resistance became dependent on the length, thus indicating that L_T or L_φ becomes comparable with the sample length. The temperature dependence of the resistance of the longest wires was of the form $T^{-1/2}$, which seems to point to a substantial role of the interaction effect.

The total correction to the conductivity is a sum of two contributions: the correction $\delta\sigma_{\rm loc}$ due to weak localization effects,[2] and the correction $\delta\sigma_{\rm loc}$ due to interaction effects.[3,4]

In a wire of length L and cross-section area $S_\perp$ the localization correction is[1)]

$$\delta\sigma_{loc} = -\frac{e^2}{\pi S_\perp} L_\varphi\left(\operatorname{cth}\frac{L}{L_\varphi} - \frac{L_\varphi}{L}\right). \quad (9)$$

We emphasize that this expression does not agree, owing to a numerical coefficient, with the expression obtained in Ref. 10 at $L \ll L_\varphi$; this is due to the effect of the boundary conditions on the current contacts.

The expression for $\delta\sigma_{\rm int}$ at $L_T < L$ can be written in the form

$$\delta\sigma_{int} = -\frac{3e^2 L_T \zeta(3/2)}{(2\pi)^{3/2} S_\perp}\left[1 - \frac{3}{2}\frac{(1+F/2)^{1/2}-1}{(1+F/2)^{1/2}+1}\right] + \frac{e^2 L_T^2}{6S_\perp L}\left[5 - \frac{3}{2}\frac{F+4(1+F/2)^{1/2}-4}{F+4(1+F/2)^{1/2}+4}\right] - \frac{5\sqrt{2}}{\pi^{5/2}}\zeta\left(\frac{5}{2}\right)\frac{e^2 L_T^3}{S_\perp L^2}, \quad (10)$$

where $\zeta(x)$ is the Riemann zeta function. Expression (10) is valid accurate to terms exponentially small in L/L_T.

At $L < T_T$ we have

$$\delta\sigma_{int} = -\frac{e^2 L}{\pi S_\perp}\left\{0{,}36 - \frac{3}{2}\chi\left(\frac{1}{(1+F/2)^{1/2}}\right)\right\}, \quad (11)$$

where

$$\chi(x) = \frac{(x+1)^2}{6x}\ln\frac{1+x}{2x} - \frac{(x-1)^2}{6x}\ln\frac{|1-x|}{2x} - 2\int_0^\infty \frac{dt}{t}\left\{\frac{1}{t^2} - \frac{\operatorname{cth} t}{t}\right\}\left\{1 - \operatorname{sh}^2 t\left[\operatorname{sh}\frac{2xt}{x+1}\operatorname{sh}\frac{2t}{x+1}\right]^{-1} + \operatorname{sh}^2 t\left[\operatorname{sh}\frac{2xt}{x-1}\operatorname{sh}\frac{2t}{x-1}\right]^{-1}\right\}.$$

As $x \to 1$ we have $\chi(x) \approx (1-x)/3$.

Expressions (9)–(11) describe qualitatively the experimentally observed dependence of the resistance on the length of the wire.[6] Comparison of the theoretical expressions with experiment 6 does not permit separation of the localization and interaction effects. It would be useful here to investigate in thin wires the magnetoresistance due to suppression of the localization contribution at $H > H_{loc} = \sqrt{3\hbar c/eaL_\varphi}$ (Ref. 11), as well as to the suppression of the contribution of terms proportional to F in $\delta\sigma_{\rm int}$ at $\omega_s > T$, where ω_s is the Zeeman splitting of the electron states.

Masden and Giordano[7] investigated also the dependence of the resistance of $Au_{40}Pd_{60}$ and Pt films with dimensions $L \gg b$, the conductivity being measured along the short side b. At $T \approx 1.5$ K the additional resistance became dependent on b at $b \approx 1$–2 μm.

For this geometry, the localization contribution to the conductivity is

$$\delta\sigma_{loc} = -\frac{e^2}{\pi\hbar}\left\{\ln\frac{2L_\varphi}{l} + 2\sum_{n=1}^{\infty} K_0\left(\frac{2b}{L_\varphi}n\right) - \frac{\pi}{2}\frac{L_\varphi}{b}\right\}, \quad (12)$$

where $k_0(x)$ is a Macdonald function.

The contribution of interaction effects at $b < L_T$ is

$$\delta\sigma_{int} = -\frac{5e^2}{2\pi^2}\left[1 - 3\frac{2+F}{5F}\ln\left(1+\frac{F}{2}\right)\right]\ln\frac{b}{l}, \quad (13)$$

and at $b > L_T$

$$\delta\sigma_{int} = -\frac{5e^2}{(2\pi)^2}\ln\frac{\gamma}{2\pi T\tau}\left[1 - 3\frac{2+F}{5F}\ln\left(1+\frac{F}{2}\right)\right.$$
$$+\frac{e^2}{(2\pi)^{5/2}}\zeta\left(\frac{3}{2}\right)\frac{L_T}{b}(16-3\pi\tilde\beta) + \frac{e^2\pi}{36}\left(\frac{L_T}{b}\right)^2$$
$$\left.-\frac{5e^2\zeta(5/2)}{3(2\pi)^{3/2}}\left(\frac{L_T}{b}\right)^3 + 0\left(\frac{L_T^5}{b^5}\right)\right], \quad (14)$$

where

$$\tilde\beta = \frac{5}{2} - \frac{8}{\pi}\frac{(1+F/2)^{1/2}}{F^2}\left[4K\left(\left(\frac{F}{2+F}\right)^{1/2}\right)\right.$$
$$\left.-(F+4)E\left(\left(\frac{F}{2+F}\right)^{1/2}\right)\right]. \quad (15)$$

and $K(x)$ and $E(x)$ are complete elliptic integrals of the first and second kind, respectively.

It was stated in Ref. 7 that the conductivity has a logarithmic temperature dependence, and the factor preceding the logarithm decreases with decreasing film width. From (12) and (14) it can be seen that the theory does not predict this behavior. To draw final conclusions concerning the discrepancies between theory and experiment, however, experiments must be performed in a temperature range wider than the 1.5–10 K covered in Ref. 7.

The film thickness in the experiments is very frequently such that the situation is not purely two-dimensional. At the end of this section we present therefore for the conductivity expressions that describe the transition from the two- to the three-dimensional case.

The localization contribution to the conductivity is of the form[12]

$$\delta\sigma_{loc} = -\frac{e^2}{2\pi^2 a}\ln\left(\frac{L_\varphi}{l}\frac{\operatorname{sh}(a/l)}{\operatorname{sh}(a/L_\varphi)}\right). \quad (16)$$

The correction to the temperature dependence of the conductivity, necessitated by the interaction between the electrons, takes at $a > L_T$ the form

$$\delta\sigma_{int} = \frac{e^2}{2\pi^2 L_T}\cdot 0.915\left\{\left[\frac{2}{3} - \frac{8}{F}\left[(1+F/2)^{3/2} - 1 - 3F/4\right]\right]\right\}$$
$$+\frac{5e^2}{8\pi^2 a}\left[1 - \frac{3}{5}\frac{F+2}{F}\ln\left(1+\frac{F}{2}\right)\right]\ln T\tau. \quad (17)$$

At $a < L_T$, σ_{int} is given by Eq. (13) with $b \to a$.

3. DERIVATION OF BASIC RELATIONS

We show in this section how to obtain the main results described above. All the corrections to the conductivity and to the density of states are connected with the contribution of the diffusion modes: the diffusion and the cooperon.

In the coordinate representation the equation for the diffusion is

$$(-i\omega - D\nabla^2)D_\omega(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'). \quad (18)$$

In general form, the homogeneous boundary conditions for Eq. (18) are

$$D\frac{\partial D_\omega(\mathbf{r},\mathbf{r}')}{\partial r_n} - WD_\omega(\mathbf{r},\mathbf{r}') = 0. \quad (19)$$

The first term of (19) describes the particle flux to the sample surface, and the second describes the surface relaxation of the fluctuations at a rate W. In the general case the diffusion describes both fluctuations of the density of particles with a given energy and fluctuations of the spin density.[13] In the first case it determines the corrections $\delta\nu^{j=0}$ and $\delta\sigma_{int}^{j=0}$ to the local density of the states and to the conductivity, respectively. These corrections are connected with particle-hole interaction at a total spin $j = 0$. In the second case it describes the corrections $\delta\nu^{j=1}$ and $\delta\sigma_{int}^{j=1}$ connected with particle-hole interaction with total $j = 1$. In the general case these two types of fluctuation have unequal surface relaxation rates that can differ greatly. Thus, for example, deposition of paramagnetic or heavy atoms on the surface (as, e.g., in the experiments of Bergmann[14]) lead to an effective spin relaxation.

We shall consider, however, the simplest situation; 1) $w = 0$ on the free surface and the boundary condition is

$$\frac{\partial D_\omega(\mathbf{r},\mathbf{r}')}{\partial r_n} = 0; \quad (20)$$

2) on a boundary with the bulky contact we have

$$D_\omega(\mathbf{r}, \mathbf{r}') = 0. \quad (21)$$

The last condition means that in the bulky metal of the junction all the fluctuations are suppressed.

The equation for the cooperon $C(\mathbf{r}, \mathbf{r}')$ is

$$(-D\nabla^2 + 1/\tau_\varphi)C(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'). \quad (22)$$

The boundary conditions are similar to (19), but the surface relaxation rate is different. We shall again, however, confine ourselves to the following boundary conditions: on the boundary with the good contact

$$C(\mathbf{r}, \mathbf{r}') = 0 \quad (23)$$

and on the free boundary

$$\partial C(\mathbf{r}, \mathbf{r}')/\partial r_n = 0. \quad (24)$$

Generalization to the case of arbitrary boundary conditions is quite trivial, albeit unwieldy.

1. Density of states

The correction to the local density of states, necessitated by the dynamically screened Coulomb interaction, can be represented in the form[3]

$$\delta\nu^{j=0}(\varepsilon, \mathbf{r})$$
$$= -4\nu_0 \operatorname{Im}\int_{\varepsilon}^{\tau^{-1}}\frac{d\omega}{2\pi}\int d\mathbf{r}_1\, d\mathbf{r}_2\, D_\omega(\mathbf{r},\mathbf{r}_1)\, V_\omega^{j=0}(\mathbf{r}_1\mathbf{r}_2)\, D_\omega(\mathbf{r}_2\mathbf{r}). \quad (25)$$

Here $V_{\omega}^{j=0}(\mathbf{r},\mathbf{r}')$ is the potential of the dynamically screened Coulomb interaction. The equation for it in the coordinate representation is

$$V_{\omega}^{j=0}(\mathbf{r},\mathbf{r}')=V_0(\mathbf{r},\mathbf{r}')-\int d\mathbf{r}_1\,d\mathbf{r}_2\,V_0(\mathbf{r},\mathbf{r}_1)\Pi_{\omega}(\mathbf{r}_1\mathbf{r}_2)V_{\omega}^{j=0}(\mathbf{r}_2\mathbf{r}'), \tag{26}$$

where the polarization operator is

$$\Pi_{\omega}(\mathbf{r}\mathbf{r}')=-2\nu_0 D\nabla^2 D_{\omega}(\mathbf{r},\mathbf{r}')=2\nu_0[i\omega D_{\omega}(\mathbf{r},\mathbf{r}')+\delta(\mathbf{r}-\mathbf{r}')]. \tag{27}$$

In the case of a tunnel junction, the bare interaction $V_0(\mathbf{r},\mathbf{r}')$ includes both the direct Coulomb interaction and the image forces.

The correction to the local density of states, connected with the interaction of a particle and hole with $j=1$, is of the form

$$\delta\nu^{j=1}(\varepsilon,\mathbf{r})=6\nu_0\,\mathrm{Im}\int_{\varepsilon}^{\tau^{-1}}\frac{d\omega}{2\pi}\int d\mathbf{r}_1\,d\mathbf{r}_2\,D_{\omega}(\mathbf{r}\mathbf{r}_1)V_{\omega}^{j=1}(\mathbf{r}_1\mathbf{r}_2)D_{\omega}(\mathbf{r}_2\mathbf{r}), \tag{28}$$

where

$$V_{\omega}^{j=1}(\mathbf{r},\mathbf{r}')=\frac{F}{2+F}\delta(\mathbf{r}-\mathbf{r}')+\frac{F}{4+2F}\int d\mathbf{r}_1\,\Pi(\mathbf{r},\mathbf{r}_1)V_{\omega}^{j=1}(\mathbf{r}_1,\mathbf{r}'). \tag{29}$$

a) *Density of states in two-dimensional case.* We consider the density of states in a two-dimensional case at $j+0$. We transform to the Fourier representation in the coordinates in the film plane. At $L_{\omega}=(D/\omega)^{1/2}>a$ the solution of Eq. (18) with the boundary conditions (20) is

$$D_{\omega,q}(z,z')=1/a(Dq^2-i\omega) \tag{30}$$

and

$$\Pi_{\omega,q}(z,z')=2\nu_0[i\omega/a(Dq^2-i\omega)+\delta(z-z')], \tag{31}$$

where q is a two-dimensional wave vector in the film plane. We note that according to (30) $D_{\omega,q}(z,z')$ does not depend on the transverse coordinates z and z' $(-a/2\leqslant z,z'\leqslant a/2)$ and therefore to find $\delta\nu^{j=0}$, according to (25), we need know only

$$\langle\overline{V_{\omega,q}^{j=0}}\rangle=\int\frac{dz\,dz'}{a^2}V_{\omega,q}^{j=0}(z,z').$$

When the image forces are taken into account, the bare interaction takes the form

$$V_0(z,z')=\frac{2\pi e^2}{q}\left[\exp(-q|z-z'|)-\exp(-q|z+z'|)\frac{k\,\mathrm{cth}\,q\Delta-1}{k\,\mathrm{cth}\,q\Delta+1}\right];$$

here k is the dielectric constant of the insulating linear in the tunnel junction. In a two-dimensional case the significant values are $q\gg a^{-1}$ and at $\Delta\lesssim a$ we have

$$V_0(z,z')=2\pi e^2[z+z'-|z-z'|]. \tag{32}$$

Solving (26) with the bare potential (32) we obtain at $\varkappa_2\Delta>1$

$$V_{\omega,q}^{j=0}=\frac{1}{2\nu_0 a}\frac{-i\omega+Dq^2}{Dq^2-i\omega/2\varkappa_2\Delta}. \tag{33}$$

Using (30) and (33) we obtain after integrating with respect to q and the correction to the density of states in (3).

b) *Surface effect in the density of states.* We consider now the density of states for a semi-infinite sample. This situation arises in experiment when the film that makes up the tunnel junction is thick enough, so that $a>(D/eV)^{1/2}$. We investigate the dependence of the local density of states on the distance to the boundary.

In the three-dimensional case we can neglect in (26) the left-hand side compared with the bare interaction, and as a result $V_{\omega}^{j=0}$ ceases to depend on the bare interaction and is determined from the equation

$$\int d\mathbf{r}_1\Pi_{\omega}(\mathbf{r},\mathbf{r}_1)V_{\omega}^{j=0}(\mathbf{r}_1\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'). \tag{34}$$

To find the correction to the local density of state we must find the quantity

$$S^j(\mathbf{r},\mathbf{r}')=\int d\mathbf{r}_1\,d\mathbf{r}_2 D_{\omega}(\mathbf{r},\mathbf{r}_1)V_{\omega}^j(\mathbf{r}_1\mathbf{r}_2)D_{\omega}(\mathbf{r}_2\mathbf{r}'). \tag{35}$$

Expressions (25) and (28) contain the quantity $S_j(\mathbf{r},\mathbf{r}')$ with coinciding arguments. To find this quantity we must solve Eqs. (29) and (34). We consider for example $S^{j=0}$. We represent $V_{\omega}^j(\mathbf{r},\mathbf{r}')$ in the form

$$V_{\omega}^{j=0}(\mathbf{r},\mathbf{r}')=\frac{1}{2\nu_0}\delta(r-r')+Y(\mathbf{r},\mathbf{r}'). \tag{36}$$

Using (27) for $\Pi_{\omega}(\mathbf{r},\mathbf{r}')$ and (18) for $D_{\omega}(\mathbf{r},\mathbf{r}')$ we obtain for $Y(\mathbf{r},\mathbf{r}')$ the equation

$$D\nabla^2 Y(\mathbf{r},\mathbf{r}')=(i\omega/2\nu_0)\delta(\mathbf{r}-\mathbf{r}'). \tag{37}$$

Substituting (36) in (35) and expressing $D_{\omega}(\mathbf{r},\mathbf{r}')$ with the aid of (18) we get, after integrating by parts and taking (37) into account,

$$S^{j=0}(\mathbf{r},\mathbf{r}')=\frac{1}{2i\nu_0\omega}\left[D_{\omega}(\mathbf{r},\mathbf{r}')-\frac{2\nu_0}{i\omega}Y(\mathbf{r},\mathbf{r}')\right]=\frac{1}{2i\nu_0\omega}[D_{\omega}(\mathbf{r},\mathbf{r}')-D_{\omega=0}(\mathbf{r},\mathbf{r}')]. \tag{38}$$

We have similarly

$$S^{j=1}(\mathbf{r},\mathbf{r}')=-\frac{1}{2i\nu_0\omega}[D_{\omega}(\mathbf{r},\mathbf{r}')-D_{\omega(1+F/2)}(\mathbf{r},\mathbf{r}')]. \tag{39}$$

We note that (38) and (39) are valid under general boundary conditions of the form (19).

For a semi-infinite sample the solution of (18) with boundary condition (20) is

$$D_{\omega}(r,r')=\int\frac{d^2q}{(2\pi)^2}\exp[i\mathbf{q}(\rho-\rho')]\times\frac{e^{-\varphi z}\theta(z-z')\,\mathrm{ch}\,\varphi z'+e^{-\varphi z'}\theta(z'-z)\,\mathrm{ch}\,\varphi z}{D\varphi}, \tag{40}$$

where $\varphi^2=q^2-i\omega/D$, $\mathbf{r}=(\rho,z)$. Using expressions (38)–(40) we obtain for the local density of states expressions (5) and (6).

2. Localized corrections to the conductivity

Effects of weak localization lead to conductivity corrections in the form[2]

$$\delta\sigma_{loc}(\mathbf{r})=-(2De^2/\pi)C(\mathbf{r},\mathbf{r}), \tag{41}$$

where $C(\mathbf{r},\mathbf{r})$ is defined in (22). For a thin wire its solution with boundary conditions (23), (24) is

$$C(\mathbf{r},\mathbf{r}')=\frac{L_\varphi}{2DS}\left[\operatorname{ch}\left(\frac{L-|z-z'|}{L_\varphi}\right)-\operatorname{ch}\left(\frac{z+z'}{L_\varphi}\right)\right]\left[\operatorname{sh}\frac{L}{L_\varphi}\right]^{-1}, \tag{42}$$

where L is the length of the wire, z is the coordinate along the wire, and S is the cross section area.

For a narrow film with distance b between the contacts along the z axis we have

$$C(\mathbf{r},\mathbf{r}')=\int\frac{dq}{2\pi}\exp[iq(x-x')]\frac{L_q}{2D\operatorname{sh}(b/L_q)}\times\left[\operatorname{ch}\left(\frac{b-|z-z'|}{L_q}\right)-\operatorname{ch}\left(\frac{z+z'}{L_q}\right)\right], \tag{43}$$

where q is a wave vector parallel to the contacts and $L_q^{-2}=q^2+L_\omega^{-2}$.

For a film of thickness a.

$$C(\mathbf{r},\mathbf{r}')=\int\frac{d^2q}{(2\pi)^2}\exp[i\mathbf{q}(\boldsymbol{\rho}-\boldsymbol{\rho}')]\frac{L_q}{2D\operatorname{sh}(a/L_q)}\times\left[\operatorname{ch}\left(\frac{y+y'}{L_q}\right)+\operatorname{ch}\left(\frac{a-|y-y'|}{L_q}\right)\right], \tag{44}$$

where y is the coordinate normal to the film surface and q is a two-dimensional wave vector in the plane of the film.

It can be seen from (41) and (42)–(44) that the correction to the conductivity depends on the coordinates. However, since (41) is a small correction to the total conductivity, the observed quantity is a mean value over the volume

$$\delta\sigma_{loc}=\int\frac{dr}{V}\delta\sigma_{loc}(r). \tag{45}$$

Using (41)–(45) we can obtain all the expressions (9), (13), and (16) for the localization corrections to the conductivity.

3. Conductivity of interacting electrons

Allowance for the interaction between the electrons leads, in contrast to the localization corrections, to a nonlocal connection between the current and the electric field:

$$\delta j_i(\mathbf{r})=\int d\mathbf{r}'\,\delta\sigma_{int}^{ik}(\mathbf{r},\mathbf{r}')E_k(\mathbf{r}'). \tag{46}$$

The quantity $\delta\sigma_{int}^{ik}(\mathbf{r},\mathbf{r}')$ contains besides the local contribution proportional to $\delta(r-r')$ a term that attenuates exponentially as $\mathbf{r}-\mathbf{r}'\to\infty$ not over the mean free path l, but over the length L_T, and has a power-law falloff at $l\ll|\mathbf{r}-\mathbf{r}'|\ll L_T$. This correction falls off in this region like $|\mathbf{r}-\mathbf{r}'|^{-4}$ in the three-dimensional case, like $|\mathbf{r}-\mathbf{r}'|^{-2}$ in the two-dimensional case, and like $\ln(|\mathbf{r}-\mathbf{r}'|/L_T)$ in the one-dimensional case.

In a uniform electric field, when the corrections to the conductivity are small, we have

$$\delta\sigma_{int}^{ik}=\frac{1}{V}\int d\mathbf{r}\,d\mathbf{r}'\delta\sigma_{int}^{ik}(\mathbf{r},\mathbf{r}'). \tag{47}$$

Calculations similar to those in Ref. 3, but in the coordinate representation, yield

$$\delta\sigma_{int}^{ik}(\mathbf{r},\mathbf{r}')=-\frac{2\nu_0(De)^2}{\pi i}\int_{-\infty}^{\infty}d\omega\frac{d}{d\omega}\left(\frac{\omega}{\exp(\omega/T)-1}\right)\times\left[F_{ik}^{j=0}(\mathbf{r},\mathbf{r}')-\frac{3}{2}F_{ik}^{j=1}(\mathbf{r},\mathbf{r}')\right], \tag{48}$$

$$F_{ik}^{j}=\frac{\partial^2S^j}{\partial r_i\,\partial r_k'}D_\omega+S^j\frac{\partial^2D_\omega}{\partial r_i\,\partial r_k'}-\frac{\partial S^j}{\partial r_i}\frac{\partial D_\omega}{\partial r_k'}-\frac{\partial S^j}{\partial r_k'}\frac{\partial D_\omega}{\partial r_i}. \tag{49}$$

In all the cases considered by us the directions of the electric field and of the current coincide. Using the boundary conditions (20) and (21) we get

$$\delta\sigma_{int}=\frac{2(De)^2}{\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\frac{d}{d\omega}\left(\frac{\omega}{\exp(\omega/T)-1}\right)\frac{1}{V}\int d\mathbf{r}\,d\mathbf{r}'\times\frac{\partial D_\omega(z,z')}{\partial z}\left[\frac{\partial D_\omega(z,z')}{\partial z'}+2\frac{\partial D_{\omega=0}(z,z')}{\partial z'}-3\frac{\partial D_{\omega(1+F/2)}(z,z')}{\partial z'}\right]. \tag{50}$$

Expression (50) cannot be calculated in closed form, but the asymptotic values of $\delta\sigma_{\rm int}$ are given in all limiting cases by expressions (19), (11), (13) and (14), (17).

In conclusion, we examine the result of taking into account for the conductivity cooperon boundary conditions analogous to (19).

For a thin wire, the localization correction to the conductivity is of the form

$$\delta\sigma_{loc}=-\frac{e^2}{\pi S}L_\varphi\operatorname{cth}\frac{L}{L_\varphi}\left\{\left(1+\frac{\alpha}{L_\varphi}\operatorname{th}\frac{L}{2L_\varphi}\right)\times\left(1+\frac{\alpha}{L_\varphi}\operatorname{cth}\frac{L}{2L_\varphi}\right)\right\}^{-1}\left\{1+\left(\frac{\alpha}{L_\varphi}\right)^2+2\frac{\alpha}{L_\varphi}\operatorname{th}\frac{L}{L_\varphi}-\frac{L_\varphi^2-\alpha^2}{L_\varphi L}\operatorname{th}\frac{L}{L_\varphi}\right\}, \tag{51}$$

where $\alpha^{-1}=W_C/D$ and W_C is the rate of surface relaxation on the contacts.

In the limiting case $\alpha\to0$ we obtain from (51)

$$\delta\sigma_{loc}=-\frac{e^2}{\pi S}L_\varphi\left\{\operatorname{cth}\frac{L}{L_\varphi}+\frac{L_\varphi}{L}\right\}.$$

[1)]An expression for the localization correction to the conductivity for arbitrary conditions on the current contacts is given at the end of the article.

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