Nonlinear dynamics of wave packets in a dissipative medium

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The effects of linear and nonlinear dissipation (instability) on the evolution of envelope solitons of a quasimonochromatic wave are investigated. Equations for the amplitude of the soliton and the wave number of the high-frequency population are derived in which the spectral dependence of the instability growth rate and the interaction with the induced mean flow (low-frequency mode) are taken into account. The rate of amplitude attenuation and the variation of the wave number of gravitational waves at the surface of deep water during viscous dissipation that occurs in a nonlinear regime are calculated. It is shown that the scattering of the mean flow by irregularities in the bottom may have a considerable effect on the evolution of wave packets of sufficiently high amplitude. The necessity of taking into account the ion-sound attenuation that slows down the solitons is demonstrated for Langmuir solitons in a nonisothermal plasma.

1. INTRODUCTION

The necessity of investigating nonlinear quasimonochromatic wave packets (envelope solitons) is obvious for a broad class of physical problems.[1] Solitons in conservative systems are, as a rule, long-lived formations; they are stable to small perturbations and interact weakly with one another. Moreover, solitons in certain situations are the final result of the evolution of arbitrary initial perturbations (completely integrable systems, see Ref. 1). In this connection, it is of considerable interest to investigate the effect of small nonconservative corrections, dissipation and instability, on the soliton. The action of a weak nonconservatism changes slightly the shape of the envelope soliton, but leads to a slow evolution of its parameters: the amplitude and the wave number of the high-frequency population. The possibility of a change in the latter parameter over a narrow range of wave numbers under the action of linear dissipation of a certain special form has been noted for Langmuir waves[2] and wind waves on deep water.[3] In the present work we have investigated the change in the spectrum of the envelope solitons of a weakly nonlinear wave in a wide range under the action of linear and nonlinear dissipation (instability) with an arbitrary spectral dependence. The phenomenon considered here, of an adiabatic spectral shift of the solitons over a wide range of wave numbers, is very important, since it can lead to the transfer of energy over the spectrum and to an important restructuring of the nonlinear oscillations. This phenomenon, in particular, should play a significant role in strong wave turbulence in dissipative media with nondecaying spectrum, where, because of the modulation instability, the generation of solitons of the envelope of quasimonochromatic waves is possible.[4]

2. EVOLUTION OF SCHRÖDINGER SOLITONS

The self-modulation of a weakly nonlinear quasimonochromatic wave $\exp(-i\omega t + ikx)$ is described by a set of equations for the complex amplitude $a(x,t)$ and for the perturbation of the refractive index $n(x,t)$:

$$i\left(\frac{\partial a}{\partial t}+v\frac{\partial a}{\partial x}\right)+\varkappa\frac{\partial^2 a}{\partial x^2}+\alpha_0 a|a|^2=\beta n a+i\hat{D}_v(a)+i\hat{N}(a),$$
$$\frac{\partial^2 n}{\partial t^2}-c^2\frac{\partial^2 n}{\partial x^2}=\delta\frac{\partial^2}{\partial x^2}(|a|^2)+\hat{D}_s\left(\frac{\partial n}{\partial t}\right) \qquad (1)$$

(for the definition of $\varkappa$, α_0, β, δ and other parameters, see below). Here v is the group velocity of the high-frequency waves, c is the phase velocity of the low-frequency waves (the dispersion of which is neglected), $\hat{D}_v$ and $\hat{D}_s$ are linear operators that determine the dissipation and the instability, respectively, of the high-frequency and low-frequency waves, N is the operator of nonlinear dissipation which limits the growth of the high-frequency waves. In the conservative approximation ($\hat{D}_v = \hat{D}_s = \hat{N} = 0$) the evolution of the amplitude $a(x,t)$ is determined by two nonlinear mechanisms. The first is the local nonlinear correction to the frequency and is determined, in particular, by the generation of the higher harmonics and their reaction to the high-frequency wave. The other mechanism is the interaction with the low-frequency waves (or, in the more general case, with the mean flows) that arise under the action of the high-frequency wave packet. The system (1) characterizes the dynamics of the Langmuir waves in a nonisothermal plasma,[5] the interaction of the surface and internal waves in the ocean,[6] and so on. In the next section we shall consider the evolution of solitons for this system. Here we shall limit ourselves to the investigation of solitons within the framework of the single-wave approximation, when the amplitude obeys the nonlinear equation

$$i\left(\frac{\partial a}{\partial t}+v\frac{\partial a}{\partial x}\right)+\varkappa\frac{\partial^2 a}{\partial x^2}+\alpha a|a|^2=i\mu\hat{R}(a),$$
$$\mu\hat{R}(a)=\gamma_0 a-i\gamma_1\frac{\partial a}{\partial x}-\gamma_2\frac{\partial^2 a}{\partial x^2}-\left[\rho_0 a|a|^2-i\frac{\rho_1}{2}\left(|a|^2\frac{\partial a}{\partial x}-a^2\frac{\partial a^*}{\partial x}\right)+ia\hat{L}(|a|^2)\right], \qquad (2)$$

where μ is a small parameter determining the value of the nonconservatism. This equation can, in particular, be ob-

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tained from the system (1) ($\alpha = \alpha_0 + \beta\delta/c^2$) for slow motions with characteristic velocity $V \ll c$ in the quasistatic approximation, when the terms $\partial^2 n/\partial t^2$ in (1) can be neglected.

Equation (2) characterizes the self-action of a weakly nonlinear wave with a narrow spectrum in a medium with weak dissipation. The right side of this equation represents the first terms of the expansion in powers of the amplitude and wave number for the dissipation operator in the single-wave approximation. Terms in the operator $\hat{R}(a)$ that are linear in the amplitude are obtained in the expansion of the growth rate $\gamma(k)$ in the operator

$$\hat{D}_v(a) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \gamma(k) a_k e^{ikx}\, dk$$

[a_k is the Fourier transform of the complex amplitude $a(x)$] in powers of the wave number:

$$\gamma_0 = \gamma(k=0), \quad \gamma_1 = [d\gamma/dk]_{k=0}, \quad \gamma_2 = {}^1/_2 [d^2\gamma/dk^2]_{k=0}.$$

The first two nonlinear terms in $\hat{R}(a)$ represent the analogous expansion of the operator $\hat{N}(a)$ of the nonlinear dissipation of a high-frequency wave in the cubic approximation in the amplitude, while

$$\rho_0 = \rho(k=0), \quad \rho_1 = [d\rho/dk]_{k=0},$$

where $\rho(k)$ is the Landau coefficient, which determines the limitation of the intensity of the high-frequency oscillations in the spatially homogeneous regime. Actually, if we take the solution (2) in the form $a = A \exp(ikx)$ ($k = \text{const}$), then it is not difficult to establish the fact that the intensity $|A|^2$ should satisfy the equation

$$\frac{d}{dt}|A|^2 = 2(\gamma_0 + \gamma_1 k + \gamma_2 k^2)|A|^2 - 2(\rho_0 + \rho_1 k)|A|^4.$$

Finally, the last term in $\hat{R}(a)$ characterizes the nonlocal mechanism of nonlinearity, which is connected with the dissipation of the mean flow, which latter is induced by the wave packet. In particular, the linear operator $\hat{L}$ can be expressed, in the quasistatic approximation, in terms of the operator $\hat{D}_s$ of dissipation of the low-frequency wave in the system (1). Expanding $\hat{L}$ in a series in powers of the wave number, we restrict ourselves to the approximation: $\hat{L}(|a|^2) = s\partial |a|^2/\partial x$.

It should be noted that we take ir to account the dispersion of the nonlinear dissipation in Eq. (2), but the dispersion of the conservative nonlinearity is lacking, i.e., those terms are omitted that are determined by the small imaginary corrections to the coefficients ρ_1 and s. Account of these corrections can lead only to a change in the shape of the profile of the nonlinear waves, but have no effect on the evolution of its characteristic parameters.

In the conservative approximation, when $\hat{R} \equiv 0$ in Eq. (2), an arbitrary bounded perturbation decays into solitons—stationary waves of the form

$$a = A \exp[-{}^1/_2 i\alpha A^2 t - i\varkappa q^2 t + iq(x - Vt)]\ \mathrm{ch}^{-1}[\tilde{k}(x - Vt)] \quad (3)$$

(where $\tilde{k} = A(\alpha/2\varkappa)^{1/2}$, $V = v + 2\varkappa q$) and into nonsoliton oscillating wave trains, the amplitude of which decreases with passage of time.[7] Under the action of a weak nonconservatism, the amplitude of the soliton A and the correction to the wave number of the high-frequency population q changes slowly. The equations of the evolution of these parameters can be obtained with the help of the asymptotic method.[8] However, in first approximation in the parameter μ, these equations can be found from the laws of conservation of the number of quasiparticles.

$$N = \int_{-\infty}^{+\infty} |a|^2\, dx$$

and of the quasimomentum[1)] (Refs. 3 and 9)

$$P = \frac{1}{2i}\int_{-\infty}^{+\infty} (a^* \partial a/\partial x - a\, \partial a^*/\partial x)\, dx.$$

Namely,

$$\frac{dN}{dt} = 2\mu \operatorname{Re}\int_{-\infty}^{+\infty} a^* \hat{R}(a)\, dx, \qquad \frac{dP}{dt} = 2\mu \operatorname{Im}\int_{-\infty}^{+\infty} a^* \frac{\partial}{\partial x}\hat{R}(a)\, dx. \quad (4)$$

Substituting the soliton (3) in these relations we obtain, in first approximation,

$$N = 2A\left(\frac{2\varkappa}{\alpha}\right)^{1/2}, \quad P = qN,$$

$$\frac{dN}{dt} = N\left[2(\gamma_0 + \gamma_1 q + \gamma_2 q^2) + \frac{2}{3}\gamma_2\tilde{k}^2 - \frac{4}{3}(\rho_0 + \rho_1 q)A^2\right], \quad (5)$$

$$\frac{dP}{dt} = qN\left[2(\gamma_0 + \gamma_1 q + \gamma_2 q^2) + \frac{2}{3}\gamma_2\tilde{k}^2 - \frac{4}{3}(\rho_0 + \rho_1 q)A^2\right] + \frac{4}{3}\gamma_1\tilde{k}A^2 + \frac{16}{15}s\tilde{k}A^4.$$

It should be observed here that the characteristics of the envelope soliton is the complete wave number $k_s = \bar{k} + q$ of the high-frequency population. In Eqs. (5), the coefficients $\alpha, \varkappa, \gamma, \rho, s$ are functions of the variable k_s and are written in the form of an expansion in powers of the quantity $q = k_s - \bar{k}$. With the help of the change of variable

$$a = b \exp[iqx - i(vq + \varkappa q^2)t]$$

it is not difficult to show that the amplitude b of the wave with wave number k_s also obeys Eq. (2), in which the coefficients that are functions of the wave number are calculated at $k = k_s$. Thus, for the amplitude A of the envelope soliton and of the wave number k_s we have from (5), with accuracy to terms $\sim A^4$:

$$\frac{dk_s}{dt} = \frac{\alpha A^2}{3\varkappa}\left(\frac{d\gamma}{dk_s} + \frac{4}{5}sA^2\right),$$

$$\frac{dA}{dt} = A\left[2\gamma + \frac{A^2}{6}\frac{d(\alpha/\varkappa)}{dk_s}\frac{d\gamma}{dk_s} + \frac{\alpha A^2}{6\varkappa}\frac{d^2\gamma}{dk_s^2} - \frac{4}{3}\rho A^2\right]. \quad (6)$$

We emphasize that the system (6) does not contain the parameter $\bar{k}$ and, consequently, it preserves its shape under relatively large changes of k_s ($|k_s - \bar{k}| \gtrsim \bar{k}$), when the complex amplitude $a(x,t)$ becomes a rapidly oscillating function and it is impossible to use Eq. (2). Equations (6) determine the evolution of the one-dimensional envelope solitons in a medium with instability and dissipation. The adiabatic approximation considered here is applicable only in the case of sufficiently weak dissipation, when the establishment of the shape of the soliton, perturbed by the nonconservative corrections, takes place more rapidly than the change in its parameters, i.e., under the conditions

$$\gamma \ll \alpha A^2,\quad k_s d\gamma/dk_s \ll \alpha A^2,\quad k_s^2 d^2\gamma/dk_s^2 \ll \alpha A^2,$$
$$\rho \ll \alpha,\quad sk_s \ll \alpha. \tag{7}$$

We now consider the different variants of the evolution of the envelope solitons within the framework of the model (6). In a dissipative medium ($\gamma(k) = -\nu(k) < 0$) the solitons decrease in amplitude, simultaneously spreading out. Up to the time when the conditions (7) are violated, the damping proceeds in a nonlinear regime, which, at small amplitudes, is described by the simplified system

$$\frac{dA}{dt} = -2\nu A,\quad \frac{dk_s}{dt} = -\frac{\alpha A^2}{3\varkappa}\frac{d\nu}{dk_s}. \tag{8}$$

The damping decrement of the amplitude of the soliton exceeds by a factor of two the damping decrement $\nu(k)$ of a monochromatic wave, while the wave number changes, so that the spectrum of the wave packet is shifted towards smaller values of the decrement. In the damping of a soliton with initial parameters A_0 and k_0 to infinitely small amplitudes, the wave number $k_s \to k_\infty$ (see the drawing, case *a*), where

$$A_0^2 = -\int_{k_0}^{k_\infty} 12\varkappa\nu\left(\alpha\frac{d\nu}{dk}\right)^{-1} dk. \tag{9}$$

Taking it into account that $\alpha A_0^2/\varkappa k_0^2 \ll 1$, we have

$$\frac{k_\infty - k_0}{k_0} = -\frac{\alpha(k_0)A_0^2}{12\varkappa(k_0)k_0}\frac{[d\nu/dk]_{k=k_0}}{\nu(k_0)}. \tag{10}$$

As an example, we consider viscous damping of gravitational surface waves in a deep medium (the swell of the ocean). The nondimensional amplitude $a = k\zeta$ (ζ is the amplitude of the displacement of the surface) of weakly nonlinear two-dimensional gravitational waves in the conservative approximation obeys the nonlinear Schrödinger equation [Eq. (2) in the case $\widetilde{R} = 0$], where $\alpha = -2(gk)^{1/2}$, $\varkappa = -1/8(g/k^3)^{1/2}$. The development of a modulation instability leads to the formation of envelope solitons of type (3), which then decay in amplitude, simultaneously shifting along the wave-number spectrum. Substituting the viscous damping decrement $\nu(k) = 2\nu_w k^2$ in (10) ($\nu_w = 0.01$ cm^2·s^{-1} is the kinematic viscosity of water),[11] we obtain, for example in the case of an initial amplitude $A_0 = 0.2$, the relative shift in the wave number in the damping process:

$$(k_\infty - k_0)/k_0 = -{}^8/_3 A_0^2 = -0.11. \tag{11}$$

It should be emphasized that the nonlinear damping regime of the surface waves is continued to very small amplitudes. Actually, the conditions (7) are violated only at $A \lesssim A_\nu = (\nu_w^2 k^3/g)^{1/4}$. At a wavelength $\lambda = 1$, we have $A_\nu = 7\times 10^{-5}$, which corresponds to the maximum amplitude of the surface displacement $\zeta = 10^{-2}$ mm.

In an active medium ($\gamma > 0$), the amplitude of the solitons increases, while the wave number shifts toward a larger increment (see Fig. 1, case b). A stable equilibrium state with

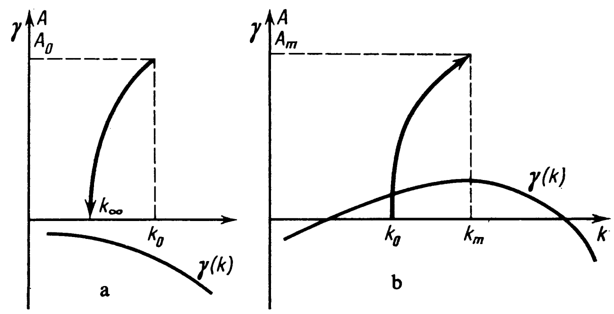

FIG. 1.

small amplitude is possible in the system (6) at $k_s = k_m$, when the increment $\gamma(k)$ takes on the maximum value

$$[d\gamma/dk]_{k=k_m} = 0,\quad h = -{}^1/_2[d^2\gamma/dk^2]_{k=k_m} > 0.$$

Near this state, we can use the approximation[2)] $\gamma(k) = \gamma(k_m) - h(k - k_m)^2$ and find the equilibrium value of the amplitude

$$A = A_m = [2\gamma(k_m)]^{1/2}[{}^4/_3\rho(k_m) + \mu\alpha(k_m)/3\varkappa(k_m)]^{-1/2}. \tag{12}$$

Thus, in a weakly nonequilibrium medium, solitons of small amplitude, which arise because of the initial perturbation, are flattened out in amplitude and wave number and therefore acquire the same velocity, which is equal to the group velocity of the most unstable harmonic (cf. Ref. 9). Further evolution is determined by the interaction of solitons having close parameters, as a result of which a periodic chain of solitons arises with alternating phases—a periodic envelope wave.[8]

Upon increase in the amplitude, a calculation of the interaction with the mean flow becomes necessary. This flow is determined by the coefficient $s(k)$ in the system (6). Here the wave number is shifted from the value $k_s = k_m$.

3. EVOLUTION OF LANGMUIR SOLITONS

The system (1) in the conservative approximation ($\widehat{D}_v = \widehat{D}_s = \widehat{N} = 0$) also has a solution in the form of solitons (3), where $\alpha = \alpha_0 + \beta\delta(c^2 - V^2)^{-1}$.

Such solitons arise, for example, as a result of the development of a modulation instability of the Langmuir waves in a nonisothermal plasma. The study of their properties has drawn a great deal of attention in recent years in connection with attempts at the construction of a theory of strong plasma turbulence.[5]

Here we consider their evolution under the action of the dissipation of the high-frequency and low-frequency modes. For this purpose, as in the preceding section, we calculate the rate of change of the number of quasiparticles

$$N = \int_{-\infty}^{+\infty} |a|^2 dx$$

and of the quasimomentum

$$P = \int_{-\infty}^{+\infty} [(2i)^{-1}(a^*\partial a/\partial x - a\partial a^*/\partial x) + \beta nu/\delta]\, dx,$$

where $\partial u/\partial x + \partial n/\partial t = 0$ which take for a solution the form

$$N=\frac{2A^2}{\tilde{k}}, \quad P=qN+\frac{4}{3}\frac{\beta\delta VA^4}{\tilde{k}(V^2-c^2)^2}. \tag{13}$$

Here $\tilde{k} = A(\alpha/2\varkappa)^{1/2}$. We write down the corresponding conservation laws:

$$\frac{dN}{dt}=2\,\mathrm{Re}\int_{-\infty}^{+\infty} a^*[\hat{D}_v(a)+\hat{N}(a)]\,dx,$$

$$\frac{dP}{dt}=2\,\mathrm{Im}\int_{-\infty}^{+\infty} a^*[\hat{D}_v(a)+\hat{N}(a)]\,dx+\frac{\beta}{\delta}\int_{-\infty}^{+\infty} n\hat{D}_s(u)\,dx. \tag{14}$$

The equations of the evolution of the parameters A and q are obtained in first approximation by substituting the soliton (3) in (14). Taking it into account here that for the stationary wave $u = Vn, n = \delta|a|^2(V^2 - c^2)^{-1}$, we obtain

$$\frac{dN}{dt}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\gamma(k)\,|a_k|^2\,dk+2\,\mathrm{Re}\int_{-\infty}^{+\infty} a^*\hat{N}(a)\,dx,$$

$$\frac{dP}{dt}=\frac{1}{\pi}\int_{-\infty}^{+\infty}k\gamma(k)\,|a_k|^2\,dk+\frac{\beta V}{2\pi\delta}\int_{-\infty}^{+\infty}\Gamma(k)\,|n_k|^2\,dk$$

$$+2\,\mathrm{Im}\int_{-\infty}^{+\infty} a^*\frac{\partial}{\partial x}\hat{N}(a)\,dx, \tag{15}$$

where

$$a_k=\int_{-\infty}^{+\infty} ae^{-ikx}\,dx, \quad n_k=\int_{-\infty}^{+\infty} ne^{-ikx}\,dx,$$

$$\hat{D}_v(a)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\gamma(k)\,a_ke^{ikx}\,dk, \quad \hat{D}_s(n)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Gamma(k)\,n_ke^{ikx}\,dk. \tag{16}$$

For the soliton we have

$$|a_k|=\frac{\pi A}{\tilde{k}\,\mathrm{ch}[\pi(k-q)/2\tilde{k}]}, \quad |n_k|=\frac{\pi\delta|k|A^2}{\tilde{k}^2|V^2-c^2|\,\mathrm{sh}(\pi|k|/2\tilde{k})}. \tag{17}$$

If the spectral dependence of the growth rates $\gamma(k)$ and $\Gamma(k)$ is slow in the scale $\tilde{k}$, we can use the expansions

$$\gamma(k)=\gamma_0+\gamma_1(k-q)+\gamma_2(k-q)^2, \quad \Gamma(k)=\Gamma_0+\Gamma_1k+\Gamma_2k^2 \tag{18}$$

and then Eqs. (15) are simplified significantly. At $\hat{N}(a) = 0$, we have

$$\frac{dN}{dt}=N\left(2\gamma_0+\frac{2}{3}\gamma_2\tilde{k}^2\right),$$

$$\frac{dP}{dt}=q\frac{dN}{dt}+\frac{4}{3}\gamma_1\tilde{k}A^2$$

$$+\frac{2}{3}N\frac{\beta\delta VA^2}{(V^2-c^2)^2}\left(\Gamma_0+\frac{18\zeta(3)}{\pi^3}\Gamma_1\tilde{k}+\frac{4}{5}\Gamma_2\tilde{k}^2\right), \tag{19}$$

where $\zeta(z)$ is the Riemann zeta function. Equations (5) for the Schrödinger solitons can be obtained from (19) under the conditions $\tilde{k}\ll k_s$, $V\ll c$. In such an approximation, the system (1) can be replaced by Eq. (2) if $\Gamma_0 = \Gamma_1 = 0$; here the coefficient $s = \beta\delta V\Gamma_2/c^4$.

We estimate here the effect of average-current dissipation characterized by s on the evolution of gravitational waves on the surface of a liquid of depth H. If the wavelength $\lambda \ll H$, then we can use the deep-water approximation, so that the frequency and the wave number are connected by the relation $\omega^2 = gk$ in the linear approximation. At the same time, the mean flow induced by the nonlinear wave packet represents a large-scale perturbation, which can be described in the shallow-water approximation if its characteristic length $L \sim \lambda/|a|^2 \gg H$ (where $a = k\zeta$). Thus, upon satisfaction of these conditions, the quasimonochromatic surface wave is described by the system (1) for nonlinearly coupled short and long waves, where $a(x,t)$ is the dimensionless amplitude of the short wave, $u = (\omega/2k)n$ is the horizontal component of the velocity in the long wave, $\beta = \omega/2$, $\delta = \omega^2/k^2$. The group velocity of the deep-water waves $V = \omega/2k \ll c = (gH)^{1/2}$; therefore, the envelope solitons of the amplitude of the short waves are described by the system (6). We make use of the expression $\nu(k) = 2\nu_w k$ of the viscous damping of surface waves,[11] and as the basic mechanism of damping of the waves in shallow water we consider the scattering from the two-dimensional large-scale roughnesses of the bottom.[12] If the mean square relative change of the bottom level is $\epsilon = \langle(\Delta H/H)^2\rangle^{1/2} \ll 1$, while the correlation length of the two-dimensional fluctuations is $l \gg L$, then the damping decrement is determined by the expression $\Gamma(k) = -c\epsilon^2 lk^2$, so that the coefficient in (6) $s = -\epsilon^2 l\sqrt{g}/4kH^{3/2}$. At $H = 1$, $\epsilon = 0.1$, $l = 5$ the change in wave number of waves of length $\lambda = 50$ cm, due to the effect of the roughnesses of the bottom on the mean flow, becomes comparable with the corresponding change due to the spectral derivative of the damping decrement $d\nu/dk \neq 0$ at an amplitude $A = 0.15$.

We now consider the application of the theory developed here to the study of the dynamics of Langmuir solitons in a collisionless plasma. The amplitude of the electrostatic field $a(x,t)$ and the perturbation of the density $n(x,t)$ in the one-dimensional model of a non-isothermal plasma obey the system (1), where

$$\varkappa=3v_{Te}^2/2\omega_{pe}, \quad \beta=\omega_{pe}/2n_0, \quad \alpha_0=0, \quad \hat{N}(a)\equiv 0,$$
$$\delta=(16\pi M)^{-1}, \quad c=(T_e/M)^{1/2}, \quad v_{Te}=(T_e/m)^{1/2}, \tag{20}$$

$\omega_{pe} = (4\pi e^2 n_0/m)^{1/2}$ is the Langmuir frequency of the electrons; m, T_e and M, $T_i \ll T_e$ are the mass and temperature of electrons and (singly-charged) ions, respectively; v_{Te} is the thermal velocity of the electrons; n_0 is the unperturbed concentration.[5] Solitons of the form (3) exist in this model at $V<c$ $(k_s < (\omega_{pe}/v_{Te})(m/M)^{1/2})$. Under the condition $A/(n_0T_e)^{1/2} \ll 1$ [when the model (1) is applicable], and at subsonic velocities ($|1 - V^2/c^2| \sim 1$) their spectral width is $\tilde{k} \ll \omega_{pe}/v_{Te}$. In this case, for the characteristic frequency of the ion-sound perturbations, the condition $\Omega \ll \omega_{pi}$ is satisfied, so that for a collisionless plasma we can use the expression (8), which characterizes the damping of the ion sound, setting $\Gamma_0 = \Gamma_2 = 0$.

$$\Gamma_1=-\left(\frac{\pi}{8}\frac{mT_e}{M^2}\right)^{1/2}\left[1+\left(\frac{M}{m}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}\right.$$
$$\left.\times\exp\left(-\frac{T_e}{2T_i}-\frac{3}{2}\right)\right]\left(\frac{V}{c}\right)^4. \tag{21}$$

Two terms in this expression are determined by the Landau damping of ion sound by electrons and ions. At the same time, the Landau damping of the Langmuir waves, which is determined by the decrement,[13]

$$\nu(k)=\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_{pe}^4}{k^3 v_{Te}^3}\exp\left(-\frac{\omega_{pe}^2}{2k^2 v_{Te}^2}-\frac{3}{2}\right), \tag{22}$$

is exponentially small. Neglecting the latter[3)] at $k_s, \tilde{k} \ll \omega_{pe}/v_{Te}$, we can use the system (19), which, with account of (13) and the relation $V=2\varkappa k_s$,[5] takes the form

$$N=2A\left[\frac{2\varkappa(c^2-V^2)}{\beta\delta}\right]^{1/2}=\text{const},$$
$$\frac{dV}{dt}\left[1+\frac{(\beta\delta N)^2}{6}\frac{c^2+5V^2}{(c^2-V^2)^4}\right]=\frac{(\beta\delta N)^3}{24\varkappa(c^2-V^2)^4}\frac{18\zeta(3)}{\pi^3}\Gamma_1 V. \tag{23}$$

Thus the Landau damping of the ion-sound mode leads to a slowing of the soliton, the velocity of which falls off exponentially. Here the wave number k_s of the high-frequency population of the soliton decreases. Such a result can be interpreted with the help of quantum considerations. Actually, the dissipation of the momentum of the soliton P in the case of conservation of the number of quasiparticles N leads obviously to a decrease in the momentum $\hbar k_s$ of each quasiparticle. Under the condition

$$A^2/n_0T_e \ll (m/M)(1-V^2/c^2)^3$$

the second term on the left side of (23) can be neglected (Schrödinger solitons) and then the characteristic inverse time of slowing of the soliton,

$$\tau^{-1}=\frac{3\sqrt{3}\zeta(3)}{128\pi^4}\omega_{pe}\left[\frac{A}{(n_0T_e)^{1/2}}\right]^3\left(\frac{V}{c}\right)^4 \tag{24}$$

is small in comparison with the damping decrement of the ion sound

$$|\Gamma_1\tilde{k}|=\frac{\omega_{pe}}{8\sqrt{6}}\frac{m}{M}\frac{A}{(n_0T_e)^{1/2}}\left(\frac{V}{c}\right)^4.$$

The relative slowing of the soliton within the time of traversal of its length,

$$(\tau^{-1}/V\tilde{k}) \lesssim (\tilde{k}/k_s)^2(m/M)^{1/2}$$

is small under the condition $\tilde{k} \lesssim k_s$.

4. CONCLUSION

The process of the nonlinear evolution of quasimonochromatic wave packets that was considered above includes decay into an envelope soliton with high-frequency population, the wave number of which then shifts adiabatically along the spectrum under the action of a weak dissipation or instability. It was shown above that in an active medium, the envelope solitons are expressed in terms of the amplitude and the wave number. Thus, if quasimonochromatic wave packets with different wave numbers are excited at the boundary of the amplifying medium, they are transformed in the propagation process into a series of identical solitons with $A=A_m$ and $k_s=k_m$.

Such a spectral shift of the soliton can be shown to be the essential element in the theory of the origin of self-oscillations in a medium with a nondecaying spectrum. Let the instability growth rate $\gamma(k)$ be positive in some spectral interval, with the increase of the parameter R characterizing the degree of nonequilibrium character of the medium (for example, in hydrodynamic flows, R is the Reynolds number). Near the instability threshold (in the "soft" boundary regime), the amplitude of the established quasimonochromatic perturbations is small. The envelope solitons that arise move in the direction of the maximum of the spectral dependence of the increment (see the drawing and also Refs. 3 and 10), transforming, in the final analysis, into a periodic wave of the envelope. Upon increase in the parameter R, the amplitude of the perturbations increases and now it is necessary to take into account the interaction of the wave packet s with the mean flow (low-frequency mode), the dissipation of which leads to an additional shift of the wave number and to the departure of the soliton from the region of maximum increment.

[1)]We note that Eq. (2) at $\hat{R}\equiv 0$ has an infinite number of first integrals. However, the evolution of the soliton is determined only by the change of the first two: N and P (Ref. 10). The change of the remaining first integrals in the soliton as it evolves is in general associated with the emission of oscillating perturbations, so that the soliton can preserve its shape.

[2)]We note that in this approximation Eq. (2) becomes a nonlinear Schrödinger equation with complex coefficients, the solutions of which have been investigated in Refs. 3 and 9.

[3)]The Landau damping of the Langmuir waves must be taken into account only at $|V-c| \ll c$, when the spectrum of the soliton is strongly spread out, so that $\tilde{k} \sim \omega_{pe}/v_{Te} \gg k_s$ (cf. Ref. 2). Such a possibility, however, requires special consideration.

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