Collision integral in a Friedmann universe

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The introduction of a random function makes it possible to obtain a hierarchy of equations for the relativistic distribution functions of charged particles in an isotropic spatially flat Friedmann universe. The hierarchy of equations is used to derive a relativistic kinetic equation taking into account the influence of the gravitational field on collisions. In the limit of nonrelativistic velocities, the collision integral agrees with the Landau collision integral only if the parameter $t_{\rm coll}/t$ ($t_{\rm coll}$ is the collision time, and t is the cosmological time) is set equal to zero. It is also shown that for a relativistic plasma in the limit $t_{\rm coll}/t \to 0$ the collision integral agrees with the well-known Belyaev-Budker integral. The collision integral does not contain divergences at large values of the impact parameter, which makes it possible, after replacement of e^2 by Gm^2, to use the collision integral for a nonrelativistic plasma for a neutral gas of gravitating particles.

INTRODUCTION

The Coulomb collision integral for a nonrelativistic plasma was obtained by Landau,[1] and for a relativistic plasma by Belyaev and Budker.[2] For detailed investigation of the processes that takes place in the radiation-dominated stage in the expansion of the universe, it is necessary to know collision integrals, in particular the Coulomb collision integral on the background of the Friedmann metric. A relativistic kinetic equation in an arbitrary gravitational field for point collisions was obtained by Chernikov.[3] This equation does not take into account the influence of the gravitational field on particle collisions, and it is therefore simply a covariant generalization of the kinetic equation in the special theory of relativity.

In the present paper, we obtain the Coulomb collision integral in a Friedmann universe with allowance for the influence of the gravitational field on the collision process. In the limit $t_{\rm coll}/t \to 0$, where $t_{\rm coll}$ is the time during which the collision takes place and t is the cosmological time, this integral goes over into Landau's collision integral (for a nonrelativistic plasma) or the Belyaev-Budker integral (for a relativistic plasma).

For $t_{\rm coll}/t \neq 0$, the collision integral for a nonrelativistic plasma is not the Landau collision integral. The Maxwell distribution function does not make the obtained integral vanish.

Allowance for the expansion of the universe eliminates the logarithmic divergence of the collision integral at large values of the impact parameters. For a gas of nonrelativistic gravitating particles, this fact is established in Ref. 4.

DERIVATION OF THE KINETIC EQUATION

To obtain a collision integral in a Friedmann universe, we use the method of Klimontovich, who in Refs. 5 and 6 obtained the collision integral (of Belyaev and Budker) for a plasma in Minkowski space.

We introduce a random function of eight variables for the particles of species a:

$$N_a(q^i, p_j) = \sum_{l=1}^{n_a} \int ds\, \delta^4(q^i - q^i_{(l)}(s))\, \delta^4(p_i - p_i^{(l)}(s)). \quad (1)$$

Here, q^i are the coordinates, and p_i are the covariant momenta of the particles ($i, j, k = 1,2,3,4$). The expression (1) is a scalar. By means of (1), the current vector of the particles of species a can be calculated in accordance with the expression

$$J_a^i = e_a c \int N_a(q,p)\, u^i \frac{d^4p}{\sqrt{-g}}, \quad (2)$$

$$u^i = p^i/m_a c, \quad d^4p = dp_1 dp_2 dp_3 dp_4, \quad p^i = g^{ij} p_j.$$

The functions $q^i_{(l)}(s)$ and $p_i^{(l)}(s)$ in (l) determine the trajectory of the lth particle of species a in an eight-dimensional phase space and satisfy the equations

$$dq^i_{(l)}/ds = p^i_{(l)}/m_a c, \quad (3a)$$

$$dp_i^{(l)}/ds = {}^1/_2 \partial_i g_{jk} p^j_{(l)} u^k_{(l)} + (e_a/c) F_{ik} u^k_{(l)}. \quad (3b)$$

An equation for the function $N_a(q,p)$ follows from the definition (1) of the function $N_a(q,p)$ and Eqs. (3):

$$u^i \frac{\partial N_a}{\partial q^i} + \left(\frac{1}{2}\partial_i g_{jk} p^j + \frac{e_a}{c} F_{ik}\right) u^k \frac{\partial N_a}{\partial p_i} = 0. \quad (4)$$

Here, F_{ik} is the electromagnetic field tensor in a plasma; it satisfies the system of Maxwell equations

$$F^{ik}{}_{;k} = -\frac{4\pi}{c} \sum_a J_a^i, \quad F_{(ik;j)} = 0. \quad (5)$$

In the spatially flat Friedmann metric ($q^4 \equiv \eta$)

$$ds^2 = a^2(\eta)\,(d\eta^2 - (dq^1)^2 - (dq^2)^2 - (dq^3)^2).$$

Eqs. (4) and (5) become ($a' = da/d\eta$)

$$\frac{\partial^2 A_i}{\partial \eta^2} - \Delta A_i = \frac{4\pi}{a^2} \sum_a e_a \int u_i N_a(q,p)\, d^3p, \quad (5')$$

$$u^i \frac{\partial N_a}{\partial q^i} + \frac{a'}{a} u^k p_k \frac{\partial N_a}{\partial p_4} + \frac{e_a}{c} F_{ik} u^k \frac{\partial N_a}{\partial p_i} = 0. \quad (4')$$

Here, A_i is the vector potential of the electromagnetic field ($F_{ik} = \partial_i A_k - \partial_k A_i$), and it satisfies the noncovariant gauge condition

$$\eta^{km} \partial_k A_m = 0 \quad (6)$$

(η^{km} is the Minkowski tensor).

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Equation (5) together with (6) gives Maxwell's equations in the special theory of relativity (see Ref. 7). The solution with retarded potentials for A_i has the form

$$A_i(\mathbf{q},\eta)=\sum_a e_a \frac{u_i' N_a(\mathbf{q}',\eta-|\mathbf{q}-\mathbf{q}'|,p_i')}{a^2(\eta-|\mathbf{q}-\mathbf{q}'|)|\mathbf{q}-\mathbf{q}'|}\,d^4p'\,d^3q'.$$

We represent this solution in the form ($k=|\mathbf{k}|$)

$$A_i(\mathbf{q},\eta)=\sum_a \frac{e_a}{2\pi^2}\int_{\eta_0}^{\eta} d\eta' \int d^3q'\,d^4p' \frac{u_i' N_a(q',p')}{a^2(\eta')} \times\int\frac{d^3k}{k}\exp(-i\mathbf{k}(\mathbf{q}-\mathbf{q}'))\sin k(\eta-\eta'). \quad (7)$$

We substitute (7) in (4):

$$u^i\frac{\partial N_a}{\partial q^i}+\frac{a'}{a}u^k p_k\frac{\partial N_a}{\partial p_i}+\sum_b\int_{\eta_0}^{\eta}d\eta'\int d^3q'\,d^4p'\frac{1}{a^2(\eta')} \times L_{ik}^{ab}(q-q',p')N_b(q',p')u^k\frac{\partial N_a(q,p)}{\partial p_i}=0. \quad (8)$$

Here,

$$L_{ik}^{ab}(q-q',p') =\frac{e_a e_b}{2\pi c}\int\frac{d^3k}{k}\left\{\frac{\partial}{\partial q^i}(u_k'\exp(-i\mathbf{k}(\mathbf{q}-\mathbf{q}'))\sin k(\eta-\eta')) -\frac{\partial}{\partial q^k}(u_i'\exp(-i\mathbf{k}(\mathbf{q}-\mathbf{q}'))\sin k(\eta-\eta'))\right\}. \quad (9)$$

Averaging (8) over the particle trajectories, we obtain

$$\frac{u^i\partial\langle N_a\rangle}{\partial q^i}+\frac{a'}{a}u^k p_k\frac{\partial\langle N_a\rangle}{\partial p_i}+\sum_b\int_{\eta_0}^{\eta}d\eta'\int d^3q'\,d^4p'\frac{1}{a^2(\eta')} \times L_{ik}^{ab}(q-q',p')u^k\frac{\partial}{\partial p_i}\langle N_a(x)N_b(x')\rangle=0. \quad (10)$$

Here, x denotes the set of all eight variables q^i, p_j.

Multiplying (8) by $N_b(x')$ and averaging, we obtain

$$u^i\frac{\partial}{\partial q^i}\langle N_a(x)N_b(x')\rangle+\frac{a'}{a}u^k p_k\frac{\partial}{\partial p_i}\langle N_a(x)N_b(x')\rangle +\sum_c\int_{\eta_0}^{\eta}d\eta''\int d^3q''\int d^4p''\frac{1}{a^2(\eta'')}L_{ik}^{ac}(q-q'',p'')\frac{\partial}{\partial p_i} \times\langle N_a(x)N_b(x')N_c(x'')\rangle=0. \quad (11)$$

The other equation for the second moment is obtained from (11) by the substitution $a\leftrightarrow b$, $x\leftrightarrow x'$.

We introduce the single-particle, two-particle, three-particle, etc., distribution functions:

$$\left\langle\int\delta(x-x_a(s))\,ds\right\rangle=f_a(x),$$

$$\left\langle\int ds\,\delta(x-x_a(s))\int ds'\,\delta(x'-x_b(s'))\right\rangle=f_{ab}(x,x'),$$

$$\left\langle\int ds\,\delta(x-x_a(s))\int ds'\,\delta(x'-x_b(s'))\int ds''\,\delta(x''-x_c(s''))\right\rangle =f_{abc}(x,x',x'').$$

We have[5]

$$\langle N_a(x)\rangle=n_a f_a(x), \quad (12a)$$

$$\langle N_a(x)N_b(x')\rangle=(n_a n_b-n_a\delta_{ab})f_{ab}(x,x') +n_a\delta_{ab}\int ds'\,\delta(x'-x_a(s|x))f_a(x) \quad (12b)$$

and similar relations for the higher moments. Here, $x_a(s|x)$ denotes the phase-space trajectory of the particles of species a that passes through the phase space point x, and n_a is the number of particles of species a.

Substituting (12) in (10) and (11) and noting that $L_{ik}^{ab}(q-q',p')$ vanishes for $q=q'$, $a=b$, we obtain a system of equations for the relativistic distribution functions ($n_a\gg 1$):

$$u^i\frac{\partial f_a}{\partial q^i}+\frac{a'}{a}u^k p_k\frac{\partial f_a}{\partial p_i}+\sum_b n_b\int_{\eta_0}^{\eta}d\eta'\int d^3q'\,d^4p'\frac{1}{a^2(\eta')} \times L_{ik}^{ab}(q-q',p')u^k\frac{\partial}{\partial p_i}f_{ab}(x,x')=0, \quad (13)$$

$$u^i\frac{\partial}{\partial q^i}f_{ab}(x,x')+\frac{a'}{a}u^k p_k\frac{\partial}{\partial p_i}f_{ab}(x,x') +\sum_c n_c\int_{\eta_0}^{\eta}d\eta''\int d^3q''\int d^4p'' L_{ik}^{ac}(q-q'',p'')\frac{1}{a^2(\eta'')} \times u^k\frac{\partial}{\partial p_i}f_{abc}(x,x',x'')+\int_{\eta_0}^{\eta}d\eta''\int d^3q''\int d^4p''\frac{1}{a^2(\eta'')} \times L_{ik}^{ab}(q-q'',p'')u^k\frac{\partial}{\partial p_i}f_{ab}(x,x')\int ds''\,\delta(x''-x_b(s''|x'))=0. \quad (14)$$

In obtaining (14), we assumed that x' is not on the trajectory of the particles of species a that passes through the point x of the phase space: $x'\neq x_a(s|x)$.

To obtain the kinetic equation to terms of second order in the interaction, we set

$$f_{ab}(x,x')=f_a(x)f_b(x')+g_{ab}(x,x'),$$

$$f_{abc}(x,x',x'')\approx f_a(x)f_b(x')f_c(x'').$$

The approximate system for $f_a(x)$ and $g_{ab}(x,x')$ is

$$u^i\frac{\partial f_a}{\partial q^i}+\frac{a'}{a}u^k p_k\frac{\partial f_a}{\partial p_i}+\sum_b n_b\int_{\eta_0}^{\eta}d\eta'\int d^3q'\,d^4p'\frac{1}{a^2(\eta')} \times L_{ik}^{ab}(q-q',p')u^k f_b(x')\frac{\partial f_a(x)}{\partial p_i} +\sum_b n_b\int_{\eta_0}^{\eta}d\eta'\int d^3q'\,d^4p'\frac{1}{a^2(\eta')}L_{ik}^{ab}(q-q',p') \times u^k\frac{\partial}{\partial p_i}(g_{ab}(x,x')+g_{ba}(x',x))=0, \quad (15)$$

$$u^i\frac{\partial}{\partial q^i}g_{ab}(x,x')+\frac{a'}{a}u^k p_k\frac{\partial g_{ab}(x,x')}{\partial p_i} +\int_{\eta_0}^{\eta}d\eta''\int d^3q''\int d^4p''\frac{1}{a^2(\eta'')}L_{ik}^{ab}(q-q'',p'') \times u^k f_b(x')\frac{\partial f_a(x)}{\partial p_i}\int ds''\,\delta(x''-x_b(s''|x'))=0. \quad (16)$$

As in Ref. 6, the right-hand side of Eq. (15) contains a symmetric combination of solutions of Eqs. (16) for $g_{ab}(x, x')$ and the solution of the equation for $g_{ba}(x', x)$ obtained from (16) by the substitution $a \leftrightarrow b$, $x \leftrightarrow x'$.

Because the interaction is weak, the trajectory of the second particle in the integrand in (16) can be assumed to be a geodesic in the Friedmann universe. We then obtain a kinetic equation to terms of second order in the interaction. For the geodesic trajectory in the spatially flat Friedmann metric we have for a particle of species b ($p^2 = p_1^2 + p_2^2 + p_3^2$)

$$p_\alpha^{(b)}(\eta''|x') = p_\alpha' = \text{const},$$

$$p_4^{(b)}(\eta''|x') = (\varepsilon_b a^2(\eta'') + p'^2)^{1/2} = p_4'(\eta''),$$

where

$$\varepsilon_b = g^{ij} p_i^{(b)} p_j^{(b)} = \text{const},$$

$$q^\alpha(\eta''|x') = q^{\alpha'} - p_\alpha' \int_{\eta'}^{\eta''} \frac{d\tau}{p_4'(\tau)}, \quad q^4(\eta''|x') = \eta''.$$

Integrating in (16) over s'', $q^{\alpha''}$, and p'', we obtain [$\mathbf{p} = (p_1, p_2, p_3)$]

$$u^i \frac{\partial g_{ab}(x, x')}{\partial q^i} + \frac{a'}{a} u^k p_k \frac{\partial g_{ab}(x, x')}{\partial p_4}$$

$$+ \int_{\eta_0}^{\eta} d\eta'' \frac{m_b c}{p_4'(\eta'')} L_{ik}^{ab}\left(\mathbf{q} - \mathbf{q}' + \mathbf{p}' \int_{\eta'}^{\eta''} \frac{d\tau}{p_4'(\tau)}, \eta\right.$$

$$\left. -\eta'', p_\alpha', p_4'(\eta'')\right) u^k f_b(x') \frac{\partial f_a(x)}{\partial p_i} = 0. \quad (17)$$

In (17) we substitute the explicit form for L_{ik}^{ab} from (9) and write down the general solution of Eq. (17):

$$g_{ab}(x, x') = -\frac{e_a e_b}{2\pi^2 c} \int_{\eta_0}^{\eta} d\tau v^k(\tau) f_b(x')$$

$$\times \left(\frac{\partial f_a(x)}{\partial p_i}\right)_\tau \int_{\eta_0}^{\tau} d\tau' \frac{d^3 k}{k} \left\{ v_k'(\tau') \frac{\partial}{\partial q_\tau^i}\right.$$

$$\times (\exp(-i\mathbf{k}(\mathbf{q} - \mathbf{q}')) \sin k(\tau - \tau'))$$

$$- v_i'(\tau') \frac{\partial}{\partial q_\tau^k} (\exp(-i\mathbf{k}(\mathbf{q} - \mathbf{q}'))$$

$$\left.\times \sin k(\tau - \tau'))\right\} \exp\left(-ik \int_\tau^\eta v(\eta'') d\eta'' - i\mathbf{k} \int_{\eta'}^{\tau'} \mathbf{v}'(\eta'') d\eta''\right). \quad (18)$$

In this expression, we have introduced the following notation: $\mathbf{v}$ is the three-dimensional vector with coordinates p_α / p_4, $v_k = (\mathbf{v}, 1)$, $v^i = \eta^{ik} v_k$. The primes identify particles of species b, while the unprimed quantities correspond to the particles of species a; $q_\tau^i = (q^\alpha, \tau)$; $(\partial f_a(x)/\partial p_i)_\tau$ denotes the derivative in which the arguments η, $\mathbf{q}$, p_α, p_4 are replaced, respectively, by the arguments τ, and

$$\mathbf{q} + \int_\tau^\eta \mathbf{v}(\eta'') d\eta'', p_\alpha, p_4(\tau).$$

Substituting (18) in (15) and using (9), we obtain after integrating in (15) over $d^3 q'$ the kinetic equation

$$u^i \frac{\partial f_a}{\partial q^i} + \frac{a'}{a} u^k p_k \frac{\partial f_a}{\partial p_4} + \frac{e_a}{c} \langle F_{ik} \rangle u^k \frac{\partial f_a}{\partial p_i} = \frac{\partial}{\partial p_j} \sum_b J_j^{ab}, \quad (19)$$

where

$$J_j^{ab} = \frac{2 e_a^2 e_b^2 n_b}{\pi c^2} \int_{\eta_0}^{\eta} \frac{d\eta'}{a^2(\eta')} \int d^4 p' \int \frac{d^3 k}{k^2}$$

$$\times \left\{ \left[u_l' \frac{\partial}{\partial q^j} (\exp(-i\mathbf{k}\mathbf{q})\right.\right.$$

$$\left. \times \sin k(\eta - \eta')) - u_j' \frac{\partial}{\partial q^l} (\exp(-i\mathbf{k}\mathbf{q}) \sin k(\eta - \eta'))\right] u^l$$

$$\times \left[\int_{\eta_0}^{\eta} d\tau v^k(\tau) \left(\frac{\partial f_a}{\partial p_i}\right)_\tau f_b'(x')\right.$$

$$\times \int_{\eta_0}^{\tau} d\tau' \exp\left(i\mathbf{k} \int_\tau^\eta \mathbf{v}(\eta'') d\eta'' + i\mathbf{k} \int_{\eta'}^{\tau'} \mathbf{v}'(\eta'') d\eta''\right)$$

$$\times \left[v_k'(\tau') \frac{\partial}{\partial q_\tau^i} (\exp(i\mathbf{k}\mathbf{q}) \sin k(\tau - \tau')) - v_i'(\tau') \frac{\partial}{\partial q_\tau^k}\right.$$

$$\left. \times (\exp(i\mathbf{k}\mathbf{q}) \times \sin k(\tau - \tau'))\right]$$

$$+ \int_{\eta_0}^{\eta} d\tau' v^{k'}(\tau') \left(\frac{\partial f_b}{\partial p_i'}\right)_{\tau'} f_a(x) \int_{\eta_0}^{\tau'} d\tau \exp\left(i\mathbf{k} \int_\tau^\eta \mathbf{v}(\eta'') d\eta''\right.$$

$$\left. + i\mathbf{k} \int_{\eta'}^{\tau'} \mathbf{v}'(\eta'') d\eta''\right) \left[v_k(\tau) \frac{\partial}{\partial q_\tau^i} (\exp(i\mathbf{k}\mathbf{q}) \sin k(\tau - \tau'))\right.$$

$$\left.\left.\left. - v_i(\tau) \frac{\partial}{\partial q_\tau^k} (\exp(i\mathbf{k}\mathbf{q}) \sin k(\tau - \tau'))\right]\right]\right\}, \quad (20)$$

and $\langle F_{ik} \rangle$ is the self-consistent electromagnetic field produced by the particles.

It is more convenient to make the remaining calculations by going over to the seven-dimensional distribution function $F_a(q, \mathbf{p})$,[3] which depends on the coordinates and spatial components p_α of the momenta:

$$n_a f_a(q, p) = F_a(q, \mathbf{p}) \delta((g^{ij} p_i p_j)^{1/2} - m_a c).$$

The equation for F_a is obtained from (19) after integration of both sides of Eq. (19) over dp_4. The integration is also to be performed over dp_4' in (20).

As a result, we obtain

$$p^i \frac{\partial F_a}{\partial q^i} + \frac{e_a}{c} \langle F_{\alpha k} \rangle p^k \frac{\partial F_a}{\partial p_\alpha} = \frac{\partial}{\partial p_\alpha} \sum_b J_\alpha^{ab}, \quad (21)$$

where

$$J_\alpha^{ab} = \frac{2 e_a^2 e_b^2}{\pi c^2} \int_{\eta_0}^{\eta} d\eta' \int d^3 p' \int \frac{d^3 k}{k^2}$$

$$\times \left\{ \left[v_l' \frac{\partial}{\partial q^\alpha} (\exp(-i\mathbf{k}\mathbf{q}) \sin k(\eta - \eta'))\right.\right.$$

$$\left. - v_\alpha' \frac{\partial}{\partial q^l} (\exp(-i\mathbf{k}\mathbf{q}) \sin k(\eta - \eta'))\right]$$

$$\times p^l \left[\int_{\eta_0}^{\eta} d\tau\, v^k(\tau) \left(\frac{\partial F_a(x)}{\partial p_\beta}\right)_\tau \right.$$

$$\times F_b(x') \int_{\eta_0}^{\tau} d\tau' \exp\left(i\mathbf{k}\int_{\tau}^{\eta} \mathbf{v}(\eta'')\,d\eta'' + i\mathbf{k}\int_{\eta'}^{\tau'} \mathbf{v}'(\eta'')\,d\eta''\right)$$

$$\times \left[v_k'(\tau') \frac{\partial}{\partial q^\beta} (\exp(i\mathbf{kq}) \sin k(\tau-\tau'))\right.$$

$$- v_\beta'(\tau') \frac{\partial}{\partial q_\tau^k} (\exp(i\mathbf{kq})$$

$$\left.\times \sin k(\tau-\tau'))\right] + \int_{\eta_0}^{\eta} d\tau' v^{k'}(\tau') \left(\frac{\partial F_b(x')}{\partial p_\beta'}\right)_{\tau'} F_a(x) \int_{\eta_0}^{\tau'} d\tau \exp$$

$$\times \left(i\mathbf{k}\int_{\tau}^{\eta} \mathbf{v}(\eta'')\,d\eta'' + i\mathbf{k}\int_{\eta'}^{\tau'} \mathbf{v}'(\eta'')\,d\eta''\right)$$

$$\times \left[v_k(\tau) \frac{\partial}{\partial q^\beta} (\exp(i\mathbf{kq}) \sin k(\tau-\tau'))\right.$$

$$\left.\left.\left. - v_\beta(\tau) \frac{\partial}{\partial q_\tau^k} (\exp(i\mathbf{kq}) \sin k(\tau-\tau'))\right]\right]\right\}. \qquad (22)$$

Here, the primed quantities referring to the particles of species b taken without argument are calculated at the time η'. The unprimed quantities are calculated at the time η. If an argument is indicated, the function is taken at the time indicated in the argument:

$$v_k(\tau) = \left(\frac{p_\alpha}{(m_a^2c^2a^2(\tau)+p^2)^{1/2}},\ 1\right),$$

$$v_k'(\tau') = \left(\frac{p_\alpha'}{(m_b^2c^2a^2(\tau')+p'^2)^{1/2}},\ 1\right),$$

$(\partial F_a(x)/\partial p_\beta)_\tau$ denotes the derivative with respect to p_β of F_a, with the arguments η and q replaced by the arguments τ and

$$\mathbf{q} + \int_\tau^\eta \mathbf{v}(\eta'')\,d\eta''.$$

Similarly, $(\partial F_b(x')/\partial p_\beta')_{\tau'}$ means that the arguments η' and q' are replaced after differentiation with respect to p_β' by τ' and

$$\mathbf{q}' + \int_{\tau'}^{\eta'} \mathbf{v}'(\eta'')\,d\eta''.$$

Further simplifications of the expression (22) can be achieved by assuming the parameter $k\eta$ to be large: $k\eta \gg 1$. This approximation corresponds to collisions occurring over distances much less than the horizon distance:

$$1/k\eta \sim r/ct \ll 1. \qquad (23)$$

To estimate the integrals in the approximation (23), we use the method of stationary phase.[8] By this method we can readily estimate integrals of the form

$$\int_{\eta_0}^{\tau} d\tau' \exp\left(i\mathbf{k}\int_{\eta'}^{\tau'} \mathbf{v}'(\eta'')\,d\eta''\right) \varphi(\tau') \begin{Bmatrix} \sin k(\tau-\tau') \\ \cos k(\tau-\tau') \end{Bmatrix}$$

$$\approx \begin{Bmatrix} k \\ i\mathbf{kv}(\tau) \end{Bmatrix} \frac{\varphi(\tau)}{k^2-(\mathbf{kv}(\tau))^2} \exp\left(i\mathbf{k}\int_{\eta'}^{\tau} \mathbf{v}'(\eta'')\,d\eta''\right) \qquad (24)$$

and others like them. Here, $\varphi(\tau')$ is a function that depends weakly on $k\tau'$.

Integrating twice in (22) by means of (24), we obtain

$$J_\alpha^{ab} = \frac{2e_a^2e_b^2p^4}{\pi c^2} \int d^3p' \int d^3k\, [k_\alpha (v^l v_l') + v_\alpha' ((\mathbf{kv}) - (\mathbf{kv}'))]$$

$$\times \int_{\eta_0}^{\eta} d\tau \exp\left(i\mathbf{k}\int_{\eta}^{\tau} (\mathbf{v}'(\eta'') - \mathbf{v}(\eta''))\,d\eta''\right)$$

$$\times \left\{ \frac{1}{[k^2-(\mathbf{kv}')^2]} \frac{1}{[k^2-(\mathbf{kv}'(\tau))^2]} \right.$$

$$\times \left(\frac{\partial F_a}{\partial p_\beta}\right)_\tau F_b(x') [v^k(\tau) v_k'(\tau) k_\beta$$

$$+ v_\beta' (\mathbf{kv}(\tau) - \mathbf{kv}'(\tau)) - \frac{1}{[k^2-(\mathbf{kv})^2]}$$

$$\times \frac{1}{[k^2-(\mathbf{kv}(\tau)^2]} \left(\frac{\partial F_b'}{\partial p_\beta'}\right)_\tau$$

$$\left. \times F_a(x) [v^k(\tau) v_k'(\tau) k_\beta + v_\beta(\tau) (\mathbf{kv}'(\tau) - \mathbf{kv}(\tau))]\right\}. \qquad (25)$$

Here, the quantities for which the time argument is not indicated are calculated at the time η.

For a nonrelativistic plasma, when $|v_\alpha| \ll 1$, we obtain from (25)

$$J_\alpha^{ab} = p^4 \Xi_\alpha^{ab} = \frac{2e_a^2e_b^2p^4}{\pi c^2} \int d^3p' \int \frac{d^3k}{k^4} k_\alpha k_\beta \int_{\eta_0}^{\eta} d\tau$$

$$\times \exp\left(i\mathbf{k}\int_{\eta}^{\tau} (\mathbf{v}'-\mathbf{v})\,d\eta''\right) \left[\left(\frac{\partial F_a}{\partial p_\beta}\right)_\tau F_b(x') \right.$$

$$\left. - \left(\frac{\partial F_b}{\partial p_\beta'}\right)_\tau F_a(x)\right]. \qquad (26)$$

Here $v_\alpha = p_\alpha/m_a ca$, $v_\alpha' = p_\alpha'/m_b ca$.

The integral over $d\tau$ is most readily calculated in the case when the scale factor a varies in accordance with the law $a = a_1\eta$, i.e., for a plasma in the radiation-dominated stage of expansion of the universe.[9] In this case $\mathbf{v} = \mathbf{v}_0/\eta$, $\mathbf{v}' = \mathbf{v}_0'/\eta$, where $\mathbf{v}_0$ and $\mathbf{v}_0'$ do not depend on the time, and the expression (26) takes the form (if we ignore the dependence of the distribution function on the time and the coordinates)

$$\Xi_\alpha^{ab} = \frac{2e_a^2e_b^2}{\pi c^2} \int d^3p' \int \frac{d^3k}{k^4} \frac{k_\alpha k_\beta \eta}{i\mathbf{k}(\mathbf{v}'-\mathbf{v})\eta+1} \left(\frac{\partial F_a}{\partial p_\beta} F_b' - \frac{\partial F_b'}{\partial p_\beta'} F_a\right). \qquad (27)$$

We calculate the integral ($\mathbf{u} = \mathbf{v}' - \mathbf{v}$)

$$\int \frac{d^3k}{k^4} \frac{k_\alpha k_\beta \eta}{(i\mathbf{ku}\eta+1)} = A\left(\delta_{\alpha\beta} - \frac{u_\alpha u_\beta}{u^2}\right) + B\delta_{\alpha\beta}. \qquad (28)$$

Here

$$2A+3B=\frac{4\pi}{u}\int_{k_D}^{k}\frac{dk}{k}\tan^{-1}ku\eta,$$

$$B=\frac{1}{u}\left[\frac{\pi}{2}+\frac{1}{k_Du\eta}-\tan^{-1}k_Du\eta-\frac{1}{(k_Du\eta)^2}\tan^{-1}k_Du\eta\right].$$

In the calculation of the integral for B the upper limit for k is set equal to $+\infty$, since this integral converges at infinity. The lower limit for k is set equal to some nonvanishing value of k_D despite the fact that the integral converges as $k\to 0$. In this way, we take into account the screening of the electromagnetic interactions at distances greater than the Debye radius l_D:

$$l_D=(\Theta/4\pi e^2N)^{1/2}=a/k_D\,. \tag{29}$$

For $2A+3B$, we obtain an integral that diverges logarithmically as $k\to\infty$ but converges as $k\to 0$. As k_∞, we must take[1]

$$k_\infty=a/r_{min}=a\Theta/e^2. \tag{30}$$

The scale factor a in (29) and (30) is introduced because the electromagnetic field is expanded with respect to harmonics: $\exp(i\mathbf{k}\cdot\mathbf{q})$. Since the spatial distances are measured by aq, the wave vector is not k but k/a.

We note that the kernel in the collision integral (27) differs from the Landau kernel. This difference has the consequence that the Maxwell distribution function

$$F=\text{const}\exp(-\beta p^2)$$

does not make the collision integral vanish.

If the parameter $k_Du\eta$ tends to infinity, then the kernel in the collision integral (27) takes Landau's form, since in this case $A=\pi^2L/2u$ and $B=0$, where

$$L=2\int_{k_D}^{k_\infty}\frac{dk}{k}$$

is the Coulomb logarithm.

In reality, k_Duc has the following form when the mean thermal velocity is substituted in place of uc:

$$k_Du\eta=a\left(\frac{4\pi e^2N}{\Theta}\right)^{1/2}\frac{1}{c}\left(\frac{\Theta}{m}\right)^{1/2}\eta=2\left(\frac{4\pi e^2N}{m}\right)^{1/2}t.$$

Substituting in this expression the time dependence of the electron density from Ref. 9, $N\sim 10^{-8}\times 10^{31}\,t^{-3/2}$(sec), we obtain

$$t/t_{\text{coll}}\sim k_Du\eta\sim 10^{15}\,t^{1/4}\gg 1. \tag{31}$$

Taking into account (31), we obtain the collision integral for a nonrelativistic plasma for arbitrary variation of the scale factor a. For this, we apply to the integral over $d\tau$ in (26) the method of stationary phase, assuming the following parameter to be large:

$$t/t_{\text{coll}}\sim\mathbf{k}(\mathbf{v}'-\mathbf{v})\eta. \tag{31a}$$

If at the same time we do not ignore a possible time dependence of the distribution function, we obtain after calculation of the integrals over d^3k

$$\Xi_\alpha^{ab}=\frac{e_a^2e_b^2}{\pi c^2}\int d^3p'\left\{\frac{\pi^2L}{u}\left(\delta_{\alpha\beta}-\frac{u_\alpha u_\beta}{u^2}\right)\left(F_b'\frac{\partial F_a}{\partial p_\beta}-F_a\frac{\partial F_b'}{\partial p_\beta'}\right)-\frac{8\pi}{k_Du^2}\left(\delta_{\alpha\beta}-2\frac{u_\alpha u_\beta}{u^2}\right)\right.$$
$$\left.\times\left[F_b'\left(\frac{a'}{a}+\frac{\partial}{\partial\eta}\right)\frac{\partial F_a}{\partial p_\beta}-F_a\left(\frac{a'}{a}+\frac{\partial}{\partial\eta}\right)\frac{\partial F_b'}{\partial p_\beta'}\right]\right\}. \tag{32}$$

Finally, the kinetic equation for the nonrelativistic plasma in the Friedmann universe has the form

$$\frac{\partial F_a}{\partial\eta}+v^\alpha\frac{\partial F_a}{\partial q^\alpha}+\frac{e_a}{c}\langle F_{\alpha k}\rangle v^k\frac{\partial F_a}{\partial p_\alpha}=\frac{\partial}{\partial p_\alpha}\sum_b\Xi_\alpha^{ab}, \tag{33}$$

where Ξ_α^{ab} has the form (27) in the case of the dependence $a=a_1\eta$ of the scale factor when the dependence of the distribution function on the time is ignored but the parameter t_{coll}/t has arbitrary value. In the general case, the expression (32) gives the expression for the collision integral in the limit $t_{\text{coll}}\ll t$.

The collision integral in the form (26) also holds for a nonrelativistic gas of gravitating particles if in (26) we replace e^2 by Gm^2, where G is the gravitational constant. This is due to the fact that nonrelativistic particles in a plasma interact in accordance with the Coulomb law, while in a gravitating gas they interact through Newton's law. These laws are identical if e^2 is replaced by Gm^2. After such replacement, the collision integral (26) agrees with the expression obtained in Ref. 4 [see Eq. (15)] if the dependence of the scale factor a is taken in the form $a=1/2a_0\eta^2$ (nonrelativistic stage of the expansion). In the case $a=a_1\eta$, the collision integral can be obtained from (27) and (28) after the substitution $e^2\to Gm^2$ and the passage to the limit $k_D\to 0$ (gravitational interactions are not screened).

In the case of a relativistic plasma, we can obtain from the collision integral (25) the following result: In the limit $t_{\text{coll}}/t\to 0$, the collision integral is given by the Belyaev-Budker expression.[2]

For this, we apply to the integral over $d\tau$ in (25) the method of stationary phase, assuming that the parameter (31a) is large. We obtain

$$J_\alpha^{ab}=\frac{2e_a^2e_b^2p^4}{c^2}\int d^3p'\int d^3k\frac{k_\alpha k_\beta(v^lv_l')^2}{[k^2-(\mathbf{kv})^2]^2}$$
$$\times\delta(\mathbf{k}(\mathbf{v}'-\mathbf{v}))\left(\frac{\partial F_a}{\partial p_\beta}F_b'-\frac{\partial F_b'}{\partial p_\beta'}F_a\right). \tag{34}$$

Equation (19) together with (34) gives the Belyaev-Budker equation[2] written in Klimontovich's form.[6]

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