Multiphonon singularity in the spectrum of an external particle interacting with phonons

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(Submitted 31 March 1983)
Zh. Eksp. Teor. Fiz. **85**, 2076–2086 (December 1983)

The asymptote is found for the imaginary part of a Green's function of a particle emitting a large number of acoustic phonons near its decay threshold. The quasiclassical tunneling trajectory, which gives the main contribution to the appropriate functional integral, corresponds to imaginary times and imaginary coordinates of the particle.

PACS numbers: 63.20.Mt

1. INTRODUCTION

As is seen from the work of Pitaevskiĭ,[1] an external particle, interacting with phonons in He II (or in a solid), has a threshold of decay to an acoustic phonon at that point of its spectrum $E(p)$ (p is the momentum of the particle, E, is its energy) where the velocity $v(p) = (\partial E/\partial p)_{p_c}$ become equal to the velocity of sound. At high momenta, where $\partial E/\partial p > c$, a damping appears ($c$ is the velocity of sound) and a finite imaginary part in the reciprocal of the Green's function $G^{-1}(p, \omega)|_{\omega = E(p)}$ of the particle. In this connection, we can raise the question of the threshold for the appearance of an imaginary part in the Green's function: $\operatorname{Im} G(p,\omega_c(p)) = 0$, $p > p_c$, similar to what was done in the work of Pitaevskiĭ and one of the authors[2] for the single-particle Green's function in He II.

We limit ourselves to the consideration of positive dispersion of the acoustic phonons so that their spectrum is

$$\omega(k) = ck(1+\gamma k^2), \quad \gamma > 0, \tag{1.1}$$

since the assumption $\gamma < 0$ leads to a decay threshold with emission of phonons having a finite momentum at a point preceding $p = p_c$, $\omega = E(p_c)$; $\partial E/\partial p < c$ on the curve of the spectrum of the particles. In this case, the condition of decay at any point of the plane ω, p with the emission of n phonons has the form

$$\omega = \sum_{i=1}^{n} \omega(\mathbf{k}_i) + E(\mathbf{p}-\mathbf{k}), \quad \mathbf{k} = \sum_{i=1}^{n} \mathbf{k}_i. \tag{1.2}$$

It is easy to show that there is an minimum number of phonons $n(\mathbf{p}, \omega)$ for each $\mathbf{p}$, ω corresponding to the emission of n identical phonons[2]: $\mathbf{k}_i = \mathbf{k}/n$, $\mathbf{p} \| \mathbf{k}$. Actually, expanding $E(\mathbf{p}-\mathbf{k})$ near $\mathbf{p}-\mathbf{k} = p_c$, we obtain

$$\omega = E(p_c) + c(p-k-p_c) + \frac{1}{2}\frac{\partial^2 E}{\partial p^2}\bigg|_{p_c} (p-k-p_c)^2 + ck + c\gamma\frac{k^3}{n^2},$$

whence we find

$$n^2 = c\gamma k^3 \left[\omega - E(p_c) - c(p-p_c) - \frac{1}{2}\frac{\partial^2 E}{\partial p^2}\bigg|_{p_c}(p-k-p_c)^2\right]^{-1}.$$

It is then seen that the choice $p - k = p_c$ corresponds to the minimum $n(p, \omega)$, namely,

$$n^2 = c\gamma(p-p_c)^3/\delta\omega, \quad \delta\omega = \omega - E(p_c) - c(p-p_c). \tag{1.3}$$

At small $\delta\omega$ the discarded terms in the expansions of $E(\mathbf{p}-\mathbf{k})$ and $\omega(k_i)$ are small and (1.3) makes the principal term in the expansion of n^2 in powers of $\delta\omega$.

Thus, the tangent to the curve of the particle spectrum at the threshold point of decay into acoustic phonons determines the threshold of appearance of the imaginary part of the Green's function of the particle $G(\mathbf{p}, \omega)$ at a momentum greater than the critical (see Fig. 1). At $\delta\omega < 0$, the decays are impossible, since we have assumed that below the threshold $p = p_c$ the function $G(p, \omega)$ has a pole at $\omega = E(p)$ and is real outside the spectrum curve, as is usually done in decay theory.[1]

The purpose of the present work is the calculation of the asymptote of the imaginary part $\operatorname{Im} G(p, \omega)$ near the sound line $\delta\omega = 0$, when a large number of phonons with small momentum are emitted. This quantity will determine the absorption of the external field with amplitude $\mathbf{p}$ and frequency ω if this latter interacts weakly only with the particle and does not interact with the phonons. Moreover, the problem is of methodological interest, since it allows us to perform more detailed calculations than in Ref. 2 if we assume that the phonons do not interact with one another.

A problem of similar form was considered in Ref. 3 for the probability of light absorption with formation of excitons in semiconductors. However, the law of conservation of momentum was not important in Ref. 3 and the analysis was carried out at finite temperatures. We shall consider only the case of the zero temperature.

In what follows, we shall assume that the phonons do not interact and that we have a set of units in which $\gamma = 1$, $\hbar = 1$, $c = 1$.

2. FORMULATION OF THE PROBLEM WITH THE HELP OF A FUNCTIONAL INTEGRAL

The interaction of a particle with acoustic phonons can be obtained by expansion of its energy in the small change of the density of the liquid ρ', in the form

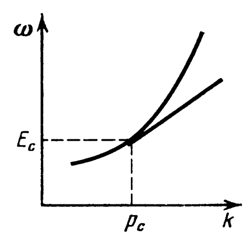

FIG. 1.

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$$H_{int}=\left(\frac{\delta E}{\delta\rho}\rho\right)\frac{\rho'}{\rho}+\mathbf{pv}=E'(\mathbf{p})\frac{\rho'}{\rho}+\mathbf{pv}, \qquad (2.1)$$

where we have introduced the notation $E'(\mathbf{p}) = (\partial E/\partial\rho)\rho$ and added the usual term $\mathbf{p}\cdot\mathbf{v}$ ($\mathbf{v}$ is the velocity of the liquid) connected with the Galilean invariance in the liquid (and absent in a solid). The interactions with the shortwave excitations enter into the energy spectrum of the particle $E(\mathbf{p})$ and will not be considered below. Along with this, we assume that the particle has the effective Hamiltonian

$$H(\mathbf{p},\mathbf{q})=E(\mathbf{p})+H_{int}(\mathbf{p},\rho'/\rho,\varphi), \qquad (2.2)$$

while the change in the density ρ'/ρ and the velocity potential φ, which is defined by the relation $\mathbf{v} = \nabla\varphi$, must be taken at the point of location of the particle, i.e., at $\mathbf{x} = \mathbf{q}(t)$.

The causal Green's function of the particle, in the limit of vanishing density of the particle, is identical with the retarded Green's function, and, according to Feynmann,[4] can be written in the form of the functional integral

$$g(\mathbf{q}_0t_0)=\begin{cases}-i\int\exp\{iS_p(\mathbf{q}_0t_0)\}DqDp, & t>0\\ 0, & t<0\end{cases}, \qquad (2.3)$$

where

$$S_p(\mathbf{q}_0t_0)=\int_0^{q_0}p\,dq-\int_0^{t_0}[E(p)+H_{int}]dt,$$

and we have used the Hamiltonian formulation. The Green's function $g(\mathbf{q}_0t_0)$ describes the process of propagation of the particle from the point (0,0) to the point $(\mathbf{q}_0,t_0)$ in the classical phonon field ρ', φ. If we average (2.3) over the zero-point vibrations of the phonon field with the help of functional integration with weight $\exp(iS_{ph})$, where the action for the phonons has the form (see, for example, Ref. 4),

$$S_{ph}=\int\varphi\frac{\partial\rho'}{\partial t}d^3x\,dt-\int\left\{\rho\frac{(\nabla\varphi)^2}{2}+\frac{c^2}{2\rho}[\rho'^2+2(\nabla\rho')^2]\right\}d^3x\,dt, \qquad (2.4)$$

then we obtain the Green's function of a particle interacting with the phonons. We shall be interested in the imaginary part of its Fourier transform

$$\operatorname{Im}G(k,\omega)=\frac{1}{2i}\int[\exp(i\omega t_0-i\mathbf{kq}_0)g(\mathbf{q}_0t_0)-\exp(-i\omega t_0+i\mathbf{kq}_0)g^*(\mathbf{q}_0t_0)]\times\exp(iS_{ph})D\rho'\,D\varphi\,dt_0\,dq_0\Big/\int\exp(iS_{ph})D\rho'\,D\varphi.$$

Using the invariance of S_{ph} under the substitutions $t\to -t$, $\mathbf{x}\to-\mathbf{x}$, we can transform this expression to the form

$$2\operatorname{Im}G(k,\omega)=\int\exp(i\omega t_0-i\mathbf{kq}_0)\exp[iS_p(q_0t_0)+iS_{ph}]\times Dp\,Dq\,D\rho'\,D\varphi\,dt_0\,dq_0\Big/\int\exp(iS_{ph})D\rho'\,D\varphi. \qquad (2.5)$$

Since the emission of a large number of phonons is difficult, the imaginary part of the Green's function should be small and we can use the saddle-point method for the calculation of (2.5) near the sound line. Since the phonon variables enter in the operation quadratically, we must directly substitute the saddle-point values $\tilde\rho'$, $\tilde\varphi$ in the total expression and omit the corresponding integration over the phonon variables; such an operation corresponds to the exact integration over the phonon variables.[5] The saddle-point values are determined also by the extremum conditions for the total action with respect to the variables ρ', φ, p, and q, which provide the equations for the liquid:

$$\begin{aligned}&\frac{\partial\rho'}{\partial t}+\rho\Delta\varphi=-\mathbf{p}\nabla_x\delta(\mathbf{x}-\mathbf{q}(t))\theta(t)\theta(t_0-t),\\&\frac{\partial\varphi}{\partial t}+\frac{1}{\rho}(\rho'-2\Delta\rho')=-E'\frac{1}{\rho}\delta(x-q(t))\theta(t)\theta(t_0-t),\end{aligned}\qquad \theta(t)=\begin{cases}1, & t>0\\0, & t<0\end{cases}, \qquad (2.6)$$

and the equations for the particles:

$$\begin{aligned}\frac{d\mathbf{p}}{dt}&=-\frac{\partial H_{int}}{\partial q}=-\left(E'(p)\frac{1}{\rho}\frac{\partial\rho'}{\partial x}+p_k\frac{\partial v_k}{\partial x}\right)_{\mathbf{x}=\mathbf{q}(t)},\\ \frac{d\mathbf{q}}{dt}&=\frac{\partial E}{\partial\mathbf{p}}+\frac{\partial H_{int}}{\partial\mathbf{p}}=\frac{\partial E}{\partial\mathbf{p}}+\left(\frac{dE'}{d\mathbf{p}}\frac{\rho'}{\rho}+\mathbf{v}\right)_{\mathbf{x}=\mathbf{q}(t)}.\end{aligned} \qquad (2.7)$$

Moreover, variation of the integrand of (2.5) with respect to t_0 and q_0 yields

$$\omega-(E(p)-H_{int})|_{t_0}=0,\quad \mathbf{k}_0-p(t_0)=0. \qquad (2.8)$$

Using the linear dependence of H_{int} on the phonon variables and Eqs. (2.6), it is not difficult to obtain the relation

$$S_{ph}(\tilde\rho',\tilde\varphi)=\frac{1}{2}\int_0^{t_0}\tilde H_{int}\,d^3x\,dt,$$

so that for the imaginary part, we obtain

$$2\operatorname{Im}G(k\omega)\sim\exp\left[i\omega t_0-ik_0q_0+i\int_0^{q_0}\tilde p\,dq-i\int_0^{t_0}\left(E(\tilde p)+\frac{1}{2}\tilde H_{int}\right)dt\right]. \qquad (2.9)$$

Here we have omitted the pre-exponential factor arising from integration over p, q, t_0, q_0 near the saddle-point values, $\tilde H_{\text{int}}$ is the value of H_{int} on the saddle-point trajectory $\tilde p$, $\tilde q$, $\tilde\rho'$, $\tilde\varphi$, given by (2.6)–(2.8).

3. DETERMINATION OF THE SADDLE-POINT VALUES

For the determination of the saddle-point values we must solve Eq. (2.6) by means of the corresponding Green's

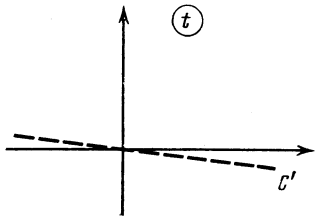

FIG. 2.

function for the equations of acoustics. By correspondence with the diagram technique, we should choose a causal function. It can also be assumed that the temporal contour for S_{ph} is slightly shifted to the left in the complex t plane in order to allow continuation from real positive t onto negative and conversely, from the negative real t to imaginary positive (see Fig. 2). Then the choice of the Green's function of the phonons $D(t-t', x-x')$ will be determined by the requirement of the vanishing of $\tilde{\rho}'$, $\tilde{\varphi}$ at the ends of the contour C'. Without adducing the corresponding standard calculations, we write down directly the equation for the phonon variables that yield the necessary solution of Eqs. (2.6) in the form of a Fourier integrals:

$$\tilde{\rho}'=\int\frac{d^3k}{(2\pi)^3}\left\{\int_0^t A_{\mathbf{k}}(t-t',\mathbf{x}-\mathbf{q}')\,dt'+\int_t^{t_0}B_{\mathbf{k}}(t-t',\mathbf{x}-\mathbf{q}')\,dt'\right\}, \tag{3.1}$$

$$\tilde{\varphi}=\int\frac{d^3k}{(2\pi)^3}\left\{\int_0^t A_{\mathbf{k}}(t-t',\mathbf{x}-\mathbf{q}')\,dt'-\int_t B_{\mathbf{k}}(t-t',\mathbf{x}-\mathbf{q}')\,dt'\right\}\times\frac{c_k^2}{i\omega_k\rho},$$

where $\mathbf{q}(t)=\mathbf{q}$, $\mathbf{q}(t')=\mathbf{q}'$, $\mathbf{p}(t)=\mathbf{p}$, $\mathbf{p}(t')=\mathbf{p}'$; $\omega_k=kc_k$, $c_k^2=1+2k^2$,

$$A_{\mathbf{k}}(t-t',x-q')=\frac{E'(p')+c_k\mathbf{k}\mathbf{p}'}{2ic_k}\exp\{-i\omega_k(t-t')\}\exp\{i\mathbf{k}(\mathbf{x}-\mathbf{q}')\}, \tag{3.2}$$

$$B_{\mathbf{k}}(t-t',x-q')=\frac{E'(p')k+c_k\mathbf{k}\mathbf{p}'}{2ic_k}\exp\{i\omega_k(t-t')\}\exp\{i\mathbf{k}(\mathbf{x}-\mathbf{q}')\}.$$

Substituting (3.1) and (3.2) in Eq. (2.1) for H_{int}, we obtain

$$\begin{aligned}H_{int}=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2i\rho\omega_k}\Big\{&\int_0^t dt'[E'(p)k+\mathbf{p}\mathbf{k}c_k][E'(p')k+\mathbf{p}'\mathbf{k}c_k]\\&\times\exp[-i\omega_k(t-t')+i\mathbf{k}(\mathbf{q}-\mathbf{q}')]\\&+\int_t^{t_0}dt'[E'(p)k+\mathbf{p}\mathbf{k}c_k][E'(p')k+\mathbf{p}'\mathbf{k}c_k]\\&\times\exp[-i\omega_k(t'-t)+i\mathbf{k}(\mathbf{q}'-\mathbf{q})]\Big\}.\end{aligned} \tag{3.3}$$

However, it is necessary to remember that in (3.3) and similar expressions the small $t'-t$ and the singularities associated with them are insignificant, since they correspond to emission of shortwave phonons, the effect of which is already taken into account in $E(p)$, which is the exact energy of the particle with account of such interactions. Therefore, the corresponding functions should be regularized at small $t'-t$.

According to (3.3), the quantity H_{int} is complex for real times and real p, and q, while the conditions (2.8) require that $H_{\text{int}}|_{t_0}=\omega-E(k_0)<0$, since ω is close to the sound line $\delta\omega=0$ (see Fig. 1). However, it is easy to see that at purely imaginary times and coordinates $\mathbf{q}(t)$, the interaction H_{int}, according to (3.3), becomes real.

In order to find the correspond continuation of (3.3), we assume that the motion of the particle takes place along a straight line parallel to the real vector $\mathbf{k}_0(\mathbf{p}\|\mathbf{k}_0, \mathbf{q}\|\mathbf{k})$ by virtue of the isotropy. Moreover, we shall assume that the velocity of the particle is everywhere positive and, correspondingly, $q(t)>q(t')$ at $t>t'$. In the case of continuation on imaginary q and t, this leads to the result that $\operatorname{Im} q(t)<\operatorname{Im} q(t')$ at $\operatorname{Im} t$ $\operatorname{Im} t'$, while, according to (2.8), the momentum and velocity of the particle will be assumed to be real. The integrals in (3.3) are already written down so that the continuation is easily carried out for each of them. We consider the first integral

$$H_1=\frac{1}{2\rho i}\int\frac{k^2\,dk\,dO}{(2\pi)^3c_kk}\int_0^{t_0}dt'[E'(p)k+\mathbf{p}\mathbf{k}c_k][E'(p')k+\mathbf{p}'\mathbf{k}c_k]\times\exp\{-i\omega_k(t-t')+ik(q-q')\cos\theta\},$$

where we have transformed to spherical coordinates in k space. Replacing $k_z=k\cos\theta$ by $-i\partial/\partial q(t)$ and carrying out the integration over the angles ($dO=d\varphi\sin\theta d\theta$), we obtain

$$H_1=\frac{1}{2\rho i}\int_0^{t_0}dt'\int\frac{dk}{c_k(2\pi)^2}\left[E'(p)k-ic_kp\frac{\partial}{\partial q}\right]\left[E'(p')k-ic_kp'\frac{\partial}{\partial q}\right]e^{-i\omega_k(t-t')}\frac{1}{i(q-q')}\{e^{ik(q-q')}-e^{-ik(q-q')}\}.$$

Upon continuation to purely imaginary q and q', the first term in the curly brackets becomes exponentially large in comparison with the second if $\operatorname{Im}(q'-q)$ is large. Therefore, only this term need be taken into account, since only large $q'-q$ and $t'-t$ are important in the problem. It is easy to see that the substitutions $t\to-i\tau$, $q\to-i\varkappa$, $t_0=-i\tau_0$ effect the necessary continuation both for H_1 and for the second integral H_2 which enters into (3.3). Shifting the origin of the imaginary time by $-\tau_0/2$, we obtain

$$\begin{aligned}H_{int}(\tau)=\left(-\frac{1}{2\rho}\right)\Big\{&\int_{-\tau_0/2}^{\tau}d\tau'\int\frac{dk}{(2\pi)^2}\left[E'(p(\tau))\frac{\partial}{\partial\xi}+p\frac{\partial}{\partial\varkappa}\right]\times\left[E'(p(\tau'))\frac{\partial}{\partial\xi}+p(\tau')\frac{\partial}{\partial\varkappa}\right]\frac{e^{-k^3(\tau-\tau')+k(\xi-\xi')}}{\varkappa-\varkappa'}\\&+\int_{\tau}^{\tau_0/2}d\tau'\int\frac{dk}{(2\pi)^2}\left[E'(p(\tau))\frac{\partial}{\partial\xi}+p\frac{\partial}{\partial\varkappa}\right]\left[E'(p(\tau'))\frac{\partial}{\partial\xi}+p(\tau')\frac{\partial}{\partial\varkappa}\right]\frac{e^{-k^3(\tau'-\tau)+k(\xi'-\xi)}}{\varkappa'-\varkappa}\Big\},\end{aligned}$$

$$\xi=\varkappa-\tau,\quad\xi'=\varkappa'-\tau', \tag{3.4}$$

where, using the fact that large value of $|\tau-\tau'|$, $|q-q'|$ and correspondingly, small values of k, are important, we set $c_k = c_0 = 1, \omega_k = c_0 k(1+k^2)$. Moreover, in differentiation with respect to $\varkappa$ it is necessary to differentiate also ξ whereas when differentiating with respect to ξ the variable $\varkappa$ can be regarded as constant. The quantity $H_{\rm int}$ does not change in the transformation $\tau \to -\tau$, $\varkappa \to -\varkappa(-\tau)$, $p(\tau) \to p(-\tau)$. It is easy to see that both the boundary conditions (2.8) and the equations of motion (2.7) are transformed into themselves in such a representation. Therefore, the saddle-point solution itself must satisfy the evenness condition

$$p(\tau)=p(-\tau), \quad \varkappa(\tau)=-\varkappa(-\tau).$$

Since the principal contribution to integrals determining $H_{\rm int}$ are made by the regions near $\pm\tau_0/2$, then it is seen that

$$H_{int}\left(\frac{\tau_0}{2}\right)=H_{int}\left(-\frac{\tau_0}{2}\right)$$

$$\approx \frac{-1}{2\rho}\int\limits_{-\tau_0/2}^{\tau_0/2} d\tau' \int\limits_0^\infty \frac{dk}{(2\pi)^2}\left[E'\left(p\left(\frac{\tau_0}{2}\right)\right)\frac{\partial}{\partial\xi}\right.$$

$$\left.+p\left(\frac{\tau_0}{2}\right)\frac{\partial}{\partial\varkappa}\right]^2 \exp\left[-k^3\left(\frac{\tau_0}{2}-\tau'\right)+k\left(\frac{\xi_0}{2}-\xi'\right)\right]\frac{1}{\varkappa_0}<0,$$

where

$$\xi_0=\xi(\tau_0/2)-\xi(-\tau_0/2), \quad \varkappa_0=\varkappa(\tau_0/2)-\varkappa(-\tau_0/2),$$

and consequently we chose the required continuation. We now introduce the function

$$D(\tau,\xi)=\int\limits_0^\infty \frac{dk}{(2\pi)^2}e^{-k^3\tau+k\xi},$$

that enters into $H_{\rm int}$. The important quantity in it is the asymptote at large ξ, which we can find by the saddle-point method:

$$D\approx\frac{\pi^{1/2}}{(2\pi)^2}\exp\left[\frac{2}{3}\left(\frac{\xi}{3\tau}\right)^{3/2}\tau\right]\frac{1}{(3\tau\xi)^{1/4}}\left(1+O\left(\frac{\tau^{1/2}}{\xi}\right)\right). \quad (3.5)$$

The function $D(\xi,\tau)$ has a singularity at small τ, corresponding to the interaction with the shortwave phonons (large k in the integral). This interaction is presumed to have already been taken into account in the form of the function $E(p)$, and the function D is assumed to be regularized at small τ; thus, at $\xi=\xi'=0$ we have $H_{\rm int}=0$.

It is seen from Eqs. (3.4) and (3.5) that $H_{\rm int}$ becomes small at small $\xi(\tau) = =\varkappa-\tau$ and consequently the solution of Eq. (2.7) should have the form pictured in Fig. 3, where the inner region corresponds to $\xi\approx0, p\approx p_c, H_{\rm int}\approx0$. Only near the ends does the interaction become significant, since $H_{\rm int}$ increases and accelerates the particle up to $p=k_0$ and emission of the phonons takes place during acceleration.

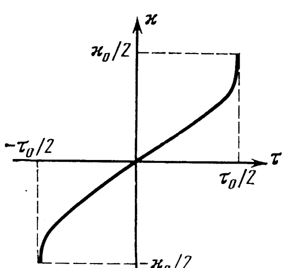

FIG. 3.

In the following, we shall see that the straight-line portions are large, so that $\varkappa(\tau_0/2)\approx c(\tau_0/2)\gg\xi(\tau_0/2)$. By virtue of this fact, it suffices when differentiating in Eq. (3.4) to differentiate the exponential in D and, if we want to find the next higher-order term, differentiate the factor $(\xi\tau)^{1/4}$. We shall regard the remaining factors as constants, since their differentiation yields small corrections to the exponential that determines Im $G(k\omega)$.

For this same reason, we can assume that the fundamental contribution to the integral over τ' for $H_{\rm int}$ is made by the region in which ξ-ξ' is maximal, i.e., the vicinity $\pm\tau_0/2$, if ξ increases monotonically with τ; the latter condition is satisfied in the case of the spectrum shown in Fig. 1 (i.e., when $\partial^2E/\partial p^2>0$ at $p>p_c$). If $\partial E/\partial p$ is nonmonotonic at $k_0>p>p_c$, then $\xi-\xi'$ is maximal at certain interior points and the integral must be taken by the saddle-point method. In what follows, we shall limit ourselves to the case of monotonic $\partial E/\partial p$, which is true in every case in a sufficiently small vicinity of p_c.

We now calculate $H_{\rm int}$ near $\tau_0/2$. In this case, only the quantity H_1 is important, as it contains a large exponential. In the calculation of the integral over τ' we shall assume that the values

$$\delta\xi'=\xi(\tau')+\xi_0/2\ll\xi_0/2,$$

are significant and carry out an expansion of the exponential in D in powers of $\delta\xi'$ and $\delta\xi=-\xi(\tau)+\xi_0/2$, i.e.,

$$\exp\left\{2\left[\frac{\xi(\tau)-\xi(\tau')}{3(\tau-\tau')}\right]^{3/2}(\tau-\tau')\right\}\approx\exp\left[2\left(\frac{\xi_0}{3\tau_0}\right)^{3/2}\tau_0\right.$$

$$\left.-\left(\frac{\xi_0}{3\tau_0}\right)^{1/2}(\delta\xi+\delta\xi')+\frac{1}{2}\left(\frac{\xi_0}{3\tau_0}\right)^{1/2}\frac{(\delta\xi+\delta\xi')^2}{\xi_0}+\ldots\right].$$

It will be seen from what follows that $\delta\xi\sim(\xi_0/3\tau_0)^{-1/2}\ll\xi_0$ is significant; hence the third term in the exponential gives a small contribution. Moreover, account of the departure of τ' from $-\tau/2$ gives a correction of order $(\tau'+\tau_0/2)(\tau_0/2)^{-1}$ which is significantly smaller because of the large τ_0 and will not be taken into account below. Terms of order $\delta\xi/\xi_0$ also arise in the calculation of the dependence of τ' of the factors that arise in the differentiation of D in Eq. (3.4), which in principal order can be set equal to their values at $\tau'=-\tau_0/2$. We shall return to this question later since a contribution to the exponential in Im $G(k,\omega)$, which we shall also take into account, arises from similar corrections.

Using the expansion of the exponential in Eq. (3.4), we obtain in principal order,

$$H_{int}=-IAf(p)\exp[-\lambda^{-1}\delta\xi(\tau)], \quad (3.5')$$

where

$$A=\frac{\pi^{1/2}}{2\rho(2\pi)^2}\lambda^{-2}\lambda^{1/4}\left(1-\frac{1}{4}\lambda\frac{1}{\xi_0}\right)\frac{1}{3^{1/4}\tau_0^{1/2}}\exp(2\lambda^{-3}\tau_0)$$

$$f(p)=E'(p)+pc, \quad \lambda=(3\tau_0/\xi_0)^{1/2}; \quad (3.6)$$

$$I=\int_{-\tau_0/2}^{0} d\tau' f(p')\exp[-\lambda^{-1}\delta\xi(\tau')]. \tag{3.7}$$

In obtaining these equations, we have assumed that the integral (3.7) is accumulated from a small vicinity of $-\tau_0/2$, and we have set the unimportant upper limit equal to zero.

Using the boundary conditions (2.8), we obtain

$$IA=-\frac{H_{int}(\tau_0/2)}{f(k_0)}=\frac{E(k_0)-\omega}{f(k_0)}. \tag{3.8}$$

The equations of motion (2.7) near $\tau_0/2$ take the form

$$\frac{dp}{d\tau}=-\frac{\partial H_{int}}{\partial\delta\xi}=\frac{1}{\lambda}IAf(p)\exp(-\lambda^{-1}\delta\xi),$$
$$\frac{d\delta\xi}{d\tau}=\frac{\partial H_{int}}{\partial p}+\left(\frac{\partial E}{\partial p}-c\right). \tag{3.8'}$$

They can be integrated in view of the energy conservation law

$$E(p)-cp+H_{int}=E(k_0)-ck_0+H_{int}(\tau_0/2),$$

so that

$$\delta\xi=-\lambda\ln\frac{E(p)-\omega+c(k_0-p)}{E(k_0)-\omega}\frac{f(k_0)}{f(p)}, \tag{3.9}$$
$$H_{int}=\omega-c(k_0-p)-E(p). \tag{3.10}$$

The integral I is easily calculated if we transform to integration over p and use the first of Eqs. (3.8):

$$I=\frac{k_0-p_c}{E(k_0)-\omega}f(k_0)\lambda, \tag{3.11}$$

from which it is seen that the effective region of integration over τ' is of the order of λ. Since we have found $I(\lambda)$ and, in accord with (3.6) $A=A(\lambda,\tau_0)$, Eq. (3.8) gives the equation that connects λ and τ_0. We also need a relation to express λ and τ_0 in terms of ω and k_0. This can be done by solving the equations of motion (2.7) with sufficient accuracy, using the conditions $\xi(0)=0, p(0)=p_c$. The required accuracy is connected with the fact that we need to take into account terms of order $\delta\omega$. In addition, it is simpler and clearer to use the laws of conservation of energy and momentum for the complete set of equations (2.6)–(2.7) with exclusion of the phonon variables:

$$E(p)+H_{int}+H_{ph}=\omega,\quad p+P_{ph}=k_0, \tag{3.12}$$

where the energy and momentum of the phonons are

$$H_{ph}=\int\left\{\rho\frac{(\nabla\varphi)^2}{2}+\frac{1}{2}\frac{c^2}{\rho}(\rho'^2+2(\nabla\rho')^2)\right\}d^3x,$$
$$P_{ph}=\int\rho'\nabla\varphi d^3x.$$

Using Eq. (3.1) for $\tilde{\rho}'$, $\tilde{\varphi}$, we can easily obtain the formulas

$$H_{ph}=\int\frac{d^3k}{(2\pi)^3}\frac{2c_k^2}{\rho}\int_0^t A_{\mathbf{k}}(t;t')dt'\int_t^{t_0}B_{-\mathbf{k}}(t;t'')dt'',$$

$$H_{ph}-cP_{ph}$$
$$=\int\frac{d^3k}{(2\pi)^3}\frac{2c_k^2}{\rho\omega_k}(\omega_k-c_0k_z)\int_0^t A_{\mathbf{k}}(t;t')dt'\int_t^{t_0}B_{-\mathbf{k}}(t;t'')dt'',$$

where we have introduced the small quantity $H_{ph}-c_0P_{ph}\propto\delta\omega$, in order to carry out the calculation with necessary accuracy.

Obtaining the analytic continuation to imaginary q and t is completely analogous to obtaining Eq. (3.4), and we write out the result directly

$$H_{ph}=\frac{1}{2\rho}\int_\tau^{\tau_0/2}d\tau''\int_{-\tau_0/2}^{\tau}d\tau' f(p')f(p'')\frac{1}{\varkappa''-\varkappa'}\frac{\partial^3}{\partial\xi''^3}D$$
$$\times(\xi''-\xi',\tau''-\tau'), \tag{3.13a}$$

$$H_{ph}-cP_{ph}=\frac{1}{2\rho}\int_\tau^{\tau_0/2}d\tau''$$
$$\times\int_{-\tau_0/2}^{\tau}d\tau' f(p')f(p'')\frac{1}{\varkappa''-\varkappa'}\frac{\partial^5}{\partial\xi''^5}D(\xi''-\xi',\tau''-\tau'), \tag{3.13b}$$

where D is given by Eq. (3.5)

The basic difference of these formulas from Eq. (3.4) for H_{int} lies in the double integration over τ' and τ''. Here, because of the properties of the D function, the principal contribution for values of τ in the interior region will arise near the ends $\pm\tau_0/2$ for each of the variables, so that $H_{ph}(\tau)\ll H_{int}(\tau)$ if τ is not too close to the ends of the interval and the corresponding exponential in D (3.5) is small for H_{int} and large for H_{ph}. As is seen from the answer, the condition $H_{int}\ll H_{ph}-c_0P_{ph}$ turns out to be satisfied for the interior region and we can neglect the quantity H_{int} in the conservation laws (3.12), so that

$$H_{ph}=\omega-E(p_c), \tag{3.14}$$
$$H_{ph}-c_0P_{ph}=\omega-E(p_c)-c_0(k_0-p_c)=\delta\omega. \tag{3.15}$$

The missing connection between τ_0 and λ can be obtained by dividing (3.15) by (3.14), for which it is necessary to calculate (3.13) for the case $\tau=0$. The calculations are similar to those for H_{int}. Here, however, we require somewhat greater accuracy if we want to take into account terms of relative order $\delta\xi/\xi_0$. The only difference between (3.13a) and (3.13b) lies in the different number of differentiations of the function D. Therefore it is necessary to take into account only the difference in the corresponding factors, since the other corrections are canceled out in the division. We therefore obtain

$$H_{ph}=AI^2\left(1-\frac{3}{4}\frac{\lambda}{\xi_0}-3\alpha\frac{\lambda}{\xi_0}\right)\lambda^{-3}, \tag{3.16}$$

$$H_{ph}-c_0P_{ph}=AI^2\left(1-\frac{5}{4}\frac{\lambda}{\xi_0}-5\alpha\frac{\lambda}{\xi_0}\right)\lambda^{-5}, \tag{3.17}$$

where

$$\alpha=\int_{p_c}^{k}\ln\left[\frac{E(p)-\omega+c(k_0-p)}{E(k_0)-\omega}\right]\frac{dp}{p_c-k}; \tag{3.18}$$

the last term in the brackets here is connected with the calculation of the corrections that arise in the expansion of the factors $(\xi''-\xi')/(\tau''-\tau')$ near the ends of the interval. Dividing (3.17) by (3.16) and using (3.14) and (3.15), we obtain

$$\lambda^2=\frac{\omega-E(p_c)}{\delta\omega}\left[1-\frac{\lambda^3}{3\tau_0}\left(\frac{1}{2}+2\alpha\right)\right]. \tag{3.19}$$

The second relation for the determination of λ and τ_0 is obtained from (3.6), (3.8) and (3.11) and has the form

$$\exp\left(\frac{2\tau_0}{\lambda^3}\right)=\tau_0^{3/2}\lambda^{1/2}B,$$

$$B=\left[\frac{E(k_0)-\omega}{f(k_0)}\right]^2\frac{1}{k_0-p_c}\left(\frac{3}{\pi}\right)^{1/2}2\rho(2\pi)^2. \qquad (3.20)$$

It is then seen hence that the corrections in the connection of λ with $\delta\omega$, contained in (3.19), are important only in the exponential. Finally, we obtain the following value for the tunneling time τ_0:

$$\tau_0=\frac{5}{4}\delta\tilde{\omega}^{-3/2}\ln\delta\tilde{\omega}^{-1}+\frac{3}{4}\delta\tilde{\omega}^{3/2}\ln\ln\delta\tilde{\omega}^{-3/2}$$
$$+\frac{1}{2}\delta\tilde{\omega}^{-3/2}\ln\frac{B}{e^{(1+4\alpha)/2}2^{3/2}}, \qquad (3.21)$$
$$\delta\tilde{\omega}=\delta\omega/(\omega-E(p_c)).$$

We have omitted here terms of the relative order $1/\ln\delta\tilde{\omega}^{-1}$.

The quantity $\xi_0=(3\tau_0/\lambda^2)\ll\tau_0$; moreover, using Eq. (3.4), we can easily show that $H_{\rm int}(0)\ll\delta\tilde{\omega}$, if we use the relation (3.20), which validates the assumptions that have been made.

4. CONCLUSION

The results we have obtained permit us to calculate the principal terms in the exponential for the imaginary part of the Green's function. For this purpose, we rewrite Eq. (2.9) with account of the fact that for most of the tunneling time the particle has a momentum $p=p_c$ and a velocity v equal to the velocity of sound c_0:

$$\operatorname{Im}G(k,\omega)\propto\exp\Bigg[(\omega-k_0c_0+p_cc_0-E(p_c))\tau_0-\frac{3\tau_0}{\lambda^2}(k_0-p_c)-\int_{-\tau_0/2}^{\tau_0/2}\left\{(p-p_c)\frac{d\varkappa}{d\tau}-E(p)+E(p_c)-\frac{1}{2}H_{int}\right\}d\tau\Bigg].$$

Making use of the fact that the principal contribution to the rest of the integral is made by regions near the ends of the interval, it is easy to make the calculation, using Eqs. (3.8)–(3.10), so that we obtain, finally,

$$\operatorname{Im}G(k,\omega)\propto\exp\Bigg\{-2\tau_0\delta\omega-\delta\tilde{\omega}^{-1/2}(\omega-E(p_c))\left(\frac{1}{2}+2\alpha\right)+2\delta\tilde{\omega}^{-1/2}\int_{p_c}^{k_0}\left[\frac{v(p)(p-p_c)+E(p_c)-E(p)}{\omega-c(k_0-p)-E(p)}+(p-p_c)\frac{f'(p)}{f(p)}-\frac{1}{2}\right]dp\Bigg\},$$

which also gives the final result for the imaginary part of the Green's function. If we limit ourselves to the principal term in the exponential, then

$$\operatorname{Im}G(k\omega)\propto\exp(-5n\ln n),$$

where the minimum number of phonons n is given by Eq. (1.3). This result is identical with the result of Ref. 2 for decays due to anharmonisms of the phonons. Thus, the contribution of the anharmonism of the phonons to the imaginary part of the Green's function is identical, in its principal order, with the harmonic part. This same result can be obtained from an estimate of the orders of the diagrams, as has been done, for example, in Ref. 3. The remaining terms in the exponential (the contribution of $n\ln\ln n$ and so forth) cannot be obtained in practice by such a method.

As a result of the calculations that have been carried out, the following physical picture emerges for the tunneling process that leads to a finite imaginary part of the Green's function of the particle. The tunnel trajectory corresponds to imaginary coordinates and times and to real momenta of the particle, which is somewhat unusual for quasiclassical problems. One most of the tunnel trajectory, the particle moves freely with the velocity of sound c_0, not emitting any phonons. Near the start of the trajectory, the particle has a velocity greater than the velocity of sound, and begins to accelerate because of the self-action associated with exchange of phonons. At this same moment, emission of phonons takes place. The emitted phonons move with a different velocity than the particle and move away from it; as a result, the interaction is weakened, the particle reaches the velocity of sound and from then on the emitted phonons and the particle move independently. Thereafter, however, because of the weak dispersion, the phonons gradually overtake the particle, an intensive process of interaction begins, and the particle is again accelerated, taking energy away from the emitted phonons. The results of the research can, without significant changes, be transferred to the case of the motion of a particle in a solid.

We note that the asymptote that has been found for the imaginary part of the Green's function of the particle describes the process of absorption of a certain field that can directly create such particles. If we are speaking of the scattering of an external field by these particles (for example, the scattering of neutrons), then it is necessary to find the corresponding expression for the two-particle Green's function, which reduces to the calculation of similar but somewhat more complicated functional integrals. However, it can be assumed that the result does not change essentially in the principal order of the number of created phonons. In the limit of low density of the particles, their statistics do not play a role, since only transitions into distant states are important. If we assume that the particles are Bose particles and produce a Bose condensate at zero temperature, then the found imaginary part describes, accurate to a factor, scattering with a transition from the condensate to an excited state near the threshold.

The authors thank L. P. Pitaevskiĭ for numerous discussions.

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Translated by R. T. Beyer