Ponderomotive force of a high-frequency field in media with temporal and spatial dispersion

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Using a macroscopic approach based upon the general principles of the electrodynamics of continuous media we obtain an expression for the ponderomotive force of a high-frequency quasi-monochromatic electromagnetic field in media with temporal and spatial dispersion, including dissipative terms. We show that this force corresponds to a symmetric stress tensor which contains time-derivatives of the field amplitudes and the anti-Hermitean part of the dielectric tensor. We study the conditions for the equilibrium of a boundary between two media when there is a high-frequency field present. We show that under certain conditions equilibrium may be impossible in the presence of dissipation and a change with time of the field amplitude. We obtain incidentally some relations of the electrodynamics of non-uniformly moving dispersive media.

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1. INTRODUCTION

There are a large number of papers devoted to the problem of finding the ponderomotive force of a high-frequency (HF) field in dispersive media. There are two approaches to the calculation of this force. The first of them is microscopic and is usually applied to relatively simple models of plasmas and gases (see, for instance, Refs. 1 to 10 and the literature cited there). The other approach is macroscopic, uses the methods of the electrodynamics of continuous media and was considered in Refs. 11 to 13. Although this approach also has its limitations it has on the whole led to very general results applicable to a wide class of condensed media, gases and plasmas. At the same time a number of aspects of the theory of the ponderomotive force in dispersive media, in particular some terms containing time-derivatives of the slowly varying amplitude, are up to now the subject of discussion which to an appreciable extent is connected with the problem of the symmetry of the stress tensor which is non-trivial even in the simpler case of non-dispersive media (see, e.g., Refs. 14, 15). Of great interest is also the calculation of the contribution of dissipative effects to the ponderomotive force. Up to now even the possibility of finding such a contribution in the framework of the macroscopic approach remained unclear even when dissipation is weak (see §81 of Ref. 13). Taking spatial dispersion into account is also of interest.

Using a unified macroscopic approach, we obtain in the present paper expressions for the ponderomotive force and stress tensor of a (temporally) quasi-monochromatic HF field, taking into account time-derivatives of the amplitude and also dissipation and spatial dispersion (sufficiently far from resonances). It is important here that the stress tensor automatically turns out to be symmetric. The results are applicable to any spatially-non-uniform fluids and, in particular, to isotropic and anisotropic plasmas. In the appropriate particular cases they are the same as expressions obtained using microscopic approaches. As our stress tensor contains time-derivatives, the ponderomotive force also contains terms with mixed derivatives with respect to r_α and t. Some of those are the same as those proposed in Ref. 16.

An important feature of the method applied is the use of energy relations in moving media, which lead to an expression for the work done by the ponderomotive force **f** per unit time: $(\mathbf{f}\cdot\mathbf{v})$. The functional derivative of this expression with respect to the velocity **v** gives the ponderomotive force. To obtain the symmetric stress tensor we must consider a velocity field with curl $\mathbf{v}\neq 0$. After variation we put $\mathbf{v}=0$. The velocity **v** can thus be considered to be a small parameter. To realize this program we need a number of relations from the electrodynamics of non-uniformly moving dispersive media (Sec. 3). As far as we know, a number of them have not been considered before. Using these results and also auxiliary relations from Secs. 2 and 3 we obtain in Sec. 4 expressions for the ponderomotive force and the stress tensor for a medium with temporal dispersion (including dissipative terms). In Sec. 5 we consider the contribution to the ponderomotive force and the stress tensor from spatial dispersion in inhomogeneous media.

2. STARTING RELATIONS

The relation between the induction and the electric field strength can be written as

$$\tilde{D}_\alpha(\mathbf{r},t)=\int_{-\infty}^{t}dt'\int d\mathbf{r}'\,\varphi_{\alpha\beta}\left(t-t',\frac{t+t'}{2};\ \mathbf{r}-\mathbf{r}',\frac{\mathbf{r}+\mathbf{r}'}{2}\right)\tilde{E}_\beta(t',\mathbf{r}'). \quad (2.1)$$

For a (temporally) quasi-monochromatic wave

$$\tilde{\mathbf{E}}(t,\ \mathbf{r})={}^1/_2\mathbf{E}(\omega,\ t;\ \mathbf{r})\exp(-i\omega t)+\text{c.c.}, \quad (2.2)$$

where $\mathbf{E}(\omega, t;\mathbf{r})$ depends slowly on g (so far the **r**-dependence is arbitrary), Eq. (2.1) becomes

$$\tilde{\mathbf{D}}(t)=\frac{1}{2}e^{-i\omega t}\int_0^\infty d\tau\,\hat{\varphi}\left(\tau,t-\frac{\tau}{2}\right)\mathbf{E}(\omega,t-\tau)e^{i\omega\tau}+\text{c.c.} \quad (2.3)$$

The dependence of all quantities on the spatial coordinates is not written out explicitly and the quantity $\hat{\varphi}(\tau, t-\tau/2)$ is an

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operator with kernel $\varphi_{\alpha\beta}[\tau, t-\tau/2;\mathbf{r}-\mathbf{r}',(\mathbf{r}+\mathbf{r}')/2]$, the action of which upon $E_\beta(\omega, t-\tau,\mathbf{r}')$ includes summation over the discrete index β and integration over the spatial coordinate $\mathbf{r}'$ [see (2.1)].

By using other combinations of independent variables we can write the response function contained in Eq. (2.1), for instance, in the form $\widetilde{\varphi}_{\alpha\beta}(t, t';\mathbf{r},\mathbf{r}')$. The form of writing it which we chose in (2.1) is convenient, for in the limit of weak non-stationarity and inhomogeneity of the medium it is necessary to separate the variables $t-t'$ and $\mathbf{r}-\mathbf{r}'$. Moreover, when writing it thus we obtain in a natural way corrections to the permittivity in a non-stationary medium under conditions where the adiabatic invariant is conserved (see footnotes 1 and 2 below). At the same time, writing (2.1) in this form does not yet assume that the adiabaticity condition is satisfied and it is general in nature until the dependence of the permittivity on the changing parameters of the medium is given a concrete form [see below Eqs. (2.10) and (2.17) and their discussion].

In a stationary medium, $\hat{\varphi}(\tau, t)$ is independent of t. We shall consider a weakly non-stationary medium assuming that $\hat{\varphi}(\tau, t)$ depends slowly on t. Expanding the slow functions in the integrand in (2.3) in powers of τ up to second derivatives in the slow time we find

$$\widetilde{\mathbf{D}}={}^1/_2\mathbf{D}\exp(-i\omega t)+\text{c.c.},$$

$$\mathbf{D}(\omega,t)=\left[\hat{\varepsilon}(\omega,t)+\frac{i}{2}\frac{\partial^2\varepsilon(\omega,t)}{\partial\omega\,\partial t}-\frac{1}{8}\frac{\partial^4\varepsilon(\omega,t)}{\partial\omega^2\,\partial t^2}\right]\mathbf{E}(\omega,t)$$
$$+\left[i\frac{\partial\hat{\varepsilon}(\omega,t)}{\partial\omega}-\frac{1}{2}\frac{\partial^3\hat{\varepsilon}(\omega,t)}{\partial\omega^2\,\partial t}\right]$$
$$\times\frac{\partial\mathbf{E}(\omega,t)}{\partial t}-\frac{1}{2}\frac{\partial^2\hat{\varepsilon}(\omega,t)}{\partial\omega^2}\frac{\partial^2\mathbf{E}(\omega,t)}{\partial t^2}, \tag{2.4}$$

where

$$\hat{\varepsilon}(\omega,t)=\int_0^\infty\hat{\varphi}(\tau,t)\exp(i\omega\tau)\,d\tau. \tag{2.5}$$

Below we consider only the case of a weakly dissipative medium, i.e., we assume that the anti-Hermitean term in the expansion of the permittivity operator in a Hermitean and an anti-Hermitean part $\hat{\varepsilon}=\hat{\varepsilon}'+i\hat{\varepsilon}''$ ($\hat{\varepsilon}'$ and $\hat{\varepsilon}''$ are Hermitean operators both in the indexes α,β and in the spatial variables $\mathbf{r}$, $\mathbf{r}'$) is so small that we can neglect terms containing time-derivatives of $\hat{\varepsilon}''$ and also products of the quantity $\hat{\varepsilon}''$ with small time-derivatives of the amplitude.[1)] Substituting (2.4) into the relation

$$-c\,\mathrm{div}[\widetilde{\mathbf{E}}\times\widetilde{\mathbf{B}}]=\widetilde{\mathbf{E}}\frac{\partial\widetilde{\mathbf{D}}}{\partial t}+\widetilde{\mathbf{B}}\frac{\partial\widetilde{\mathbf{B}}}{\partial t} \tag{2.6}$$

and then integrating term by term over the whole volume occupied by the medium, we then find (after averaging over a period of the HF field)

$$-\frac{c}{4\pi}\oint\overline{[\widetilde{\mathbf{E}}\times\widetilde{\mathbf{B}}]_n}\,d\sigma=\frac{\omega}{8\pi}\int d\mathbf{r}\langle\mathbf{E}^*(\omega,t)\hat{\varepsilon}''(\omega,t)\mathbf{E}(\omega,t)\rangle$$
$$+\frac{\partial W}{\partial t}+\frac{1}{16\pi}\int d\mathbf{r}\left\{\left\langle\mathbf{E}^*(\omega,t)\frac{\partial\hat{\varepsilon}'(\omega,t)}{\partial t}\mathbf{E}(\omega,t)\right\rangle\right.$$
$$\left.-\mathrm{Im}\left\langle\mathbf{E}^*(\omega,t)\frac{\partial^2\hat{\varepsilon}'(\omega,t)}{\partial\omega\,\partial t}\frac{\partial\mathbf{E}(\omega,t)}{\partial t}\right\rangle\right\}. \tag{2.7}$$

Here and henceforth the angle brackets indicate the following operation:

$$\langle\mathbf{X}^*\hat{T}\mathbf{Y}\rangle=\int d\mathbf{s}\,X_\alpha^*(\mathbf{r}+\mathbf{s}/2)\,T_{\alpha\beta}(\mathbf{s},\mathbf{r})\,Y_\beta(\mathbf{r}-\mathbf{s}/2) \tag{2.8}$$

for arbitrary vectors $\boldsymbol{\chi}(\omega, t;\mathbf{r})$, $\mathbf{Y}(\omega, t;\mathbf{r})$ and operator $\hat{T}(\omega, t)$ with kernel $T_{\alpha\beta}(\omega, t;\mathbf{s},\mathbf{r})$ while

$$W=\frac{1}{16\pi}\int d\mathbf{r}\left\{\left\langle\mathbf{E}^*\frac{\partial(\omega\hat{\varepsilon}')}{\partial\omega}\mathbf{E}\right\rangle\right.$$
$$\left.-\mathrm{Im}\left\langle\mathbf{E}^*\frac{\partial^2(\omega\hat{\varepsilon}')}{\partial\omega^2}\frac{\partial\mathbf{E}}{\partial t}\right\rangle+|\mathbf{B}(\mathbf{r},t)|^2\right\}. \tag{2.9}$$

It is important to note that in Eq. (2.7) the term containing $\langle\mathbf{E}^*(\partial\hat{\varepsilon}''/\partial t)\mathbf{E}\rangle$ was canceled, while taking into account the term of the form $\langle\mathbf{E}^*(\partial^2\hat{\varepsilon}''/\partial t\partial\omega)\partial\mathbf{E}/\partial t\rangle$ goes beyond the limits of the approximation used here.

When $\hat{\varepsilon}''=0$, $\partial\hat{\varepsilon}/\partial t=0$, the quantity W is the electromagnetic energy in the medium up to terms with $\partial\mathbf{E}/\partial t$ inclusive (see, e.g., Ref. 17). In order that W have the same meaning when $\hat{\varepsilon}''\neq 0$ one must require not only that $\varepsilon''/\varepsilon'\ll 1$ but, in general, also that the change in amplitude is sufficiently fast.[2] The corresponding quantitative conditions may be different depending on the dissipation mechanism. For instance, for a uniform collisional plasma considered in Ref. 2 we must have $\nu\ll T^{-1}$, where T is the characteristic time for changes in the amplitude E and ν the collision frequency.

When $\partial\hat{\varepsilon}/\partial t\neq 0$ the quantity (2.9) has the meaning of an energy density, generally speaking, only when one can neglect deviations from adiabaticity. In this connection one must bear in mind that the operator $\hat{\varepsilon}(\omega, t)$ in the general case is not the same as $\varepsilon_s[\omega,\Lambda(t)]$, where $\hat{\varepsilon}_s(\omega,\Lambda)$ is the permittivity under stationary conditions while $\Lambda=\Lambda(t)$ is the total set of independent parameters which at time t determine the state of the body. Up to terms of first order in $\partial\Lambda/\partial t$ we can write

$$\hat{\varepsilon}(\omega,t)=\hat{\varepsilon}_s(\omega,\Lambda(t))+\sum_i\hat{\Psi}_{\Lambda_i}\frac{\partial\Lambda_i}{\partial t}, \tag{2.10}$$

where the last term takes into account the non-adiabatic correction. Generally speaking, it can occur in the general expressions for the energy and dissipation densities in a non-stationary medium. Below we assume that the contribution from non-adiabatic effects is sufficiently small that we can neglect terms containing Ψ_Λ. The appropriate criterion is given at the end of the paper. We note also that it follows from Pitaevskiĭ's result mentioned in footnote 1 that $\Psi''_\Lambda=0$ if in the stationary state the medium is transparent.

We now consider in more detail the formulae given above in the case of spatial dispersion. To do this we write

$$\{\hat{\varepsilon}(\omega,t)\mathbf{E}(\omega,t)\}_\alpha=\int d\mathbf{s}\,\varepsilon_{\alpha\beta}(\omega,t;\mathbf{s}\ \mathbf{r}-\mathbf{s}/2)E_\beta(\omega,t;\mathbf{r}-\mathbf{s}). \tag{2.11}$$

In a weakly non-uniform medium

$$\varepsilon_{\alpha\beta}\left(\omega,t;\mathbf{s},\mathbf{r}-\frac{\mathbf{s}}{2}\right)\approx\varepsilon_{\alpha\beta}(\omega,t;\mathbf{s},\mathbf{r})-\frac{\mathbf{s}}{2}\frac{\partial\varepsilon_{\alpha\beta}(\omega,t;\mathbf{s},\mathbf{r})}{\partial\mathbf{r}}. \tag{2.12}$$

In the case of a quasi-planar wave

$$E_\beta(\omega, t; \mathbf{r}) = \mathscr{E}_\beta(\omega, t; \mathbf{k}, \mathbf{r}) \exp(i\mathbf{kr}), \qquad (2.13)$$

where $\mathscr{E}_\beta$ weakly depends on $\mathbf{r}$. Substituting (2.12) and (2.13) into (2.11) we find up to terms of first order[2)]

$$\{\hat{\varepsilon}(\omega, t)\mathbf{E}(\omega, t)\}_\alpha = \{[\varepsilon_{\alpha\beta} - (i/2)\partial^2\varepsilon_{\alpha\beta}/\partial k_\gamma \partial r_\gamma]\mathscr{E}_\beta - i(\partial\varepsilon_{\alpha\beta}/\partial k_\gamma)\nabla_\gamma \mathscr{E}_\beta\}\exp(i\mathbf{kr}); \qquad (2.14)$$

here

$$\mathscr{E}_\beta = \mathscr{E}_\beta(\omega, t; \mathbf{k}, \mathbf{r}), \qquad \nabla_\gamma = \partial/\partial r_\gamma,$$

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(\omega, t; \mathbf{k}, \mathbf{r}) \equiv \int ds \exp(-i\mathbf{ks})\varepsilon_{\alpha\beta}(\omega, t; \mathbf{s}, \mathbf{r}). \qquad (2.15)$$

Expression (2.14) gives in detail the first term in Eq. (2.4) for the electrical induction. The remaining terms in (2.4) can be written according to (2.14) by appropriate differentiations with respect to ω and t.

Similarly, using (2.13), (2.15), and (2.8) and expanding the amplitude $\mathscr{E}_\beta$ we get

$$\langle \mathbf{E}^*\varepsilon\mathbf{E}\rangle = \mathscr{E}_\alpha^*(\omega, t; \mathbf{k}, \mathbf{r})\varepsilon_{\alpha\beta}(\omega, t; \mathbf{k}, \mathbf{r})\mathscr{E}_\beta(\omega, t; \mathbf{k}, \mathbf{r}) + \mathrm{Im}\left\{\mathscr{E}_\alpha^*(\omega, t; \mathbf{k}, \mathbf{r})\frac{\partial\varepsilon_{\alpha\beta}(\omega, t; \mathbf{k}, \mathbf{r})}{\partial k_\gamma}\nabla_\gamma\mathscr{E}_\beta(\omega, t; \mathbf{k}, \mathbf{r})\right\}. \qquad (2.16)$$

One must bear in mind that the kernel $\varepsilon_{\alpha\beta}(\omega, t; \mathbf{s}, \mathbf{r})$ is, in general, not the same as the corresponding quantity $\varepsilon^0_{\alpha\beta}(\omega, t; s, \Lambda)$ for a uniform medium with $\Lambda = \Lambda(r, t)$ even if we neglect in (2.10) the terms with $\partial\Lambda/\partial t$. Let, for instance, Λ be a set of three parameters–the density ρ, the temperature, and the average magnetic field $\mathbf{B}^0$. For a non-isothermal medium we then have to first order in $\nabla_\mu \Lambda$

$$\varepsilon_{\alpha\beta}(\omega; \mathbf{s}, \mathbf{r}) = \varepsilon_{\alpha\beta}^0(\omega; \mathbf{s}, \Lambda(\mathbf{r})) + X_{\alpha\beta}^{\nu}\nabla_\nu\rho + Y_{\alpha\beta}^{\nu\mu}\nabla_\nu B_\mu^0. \qquad (2.17)$$

In this case X and Y depend, in general, on the same parameters as $\varepsilon^0_{\alpha\beta}$. For example, for a non-uniform plasma one can write down the coefficients X, Y using the general expressions from Ref. 20. The gradient terms (taking into account non-adiabaticity corrections) may turn out to be important ony when spatial dispersion is taken into account. One can therefore omit them when the latter is neglected.

As one application of the relations written above we evaluate the magnetic moment density $\mathbf{M}(\mathbf{r}, t)$ induced by the *HF* field. Let the external magnetic field $\mathbf{B}^0$ and hence also $\hat{\varepsilon}$ be variable but let the medium remain fixed. We can then use Eq. (2.7), where on the right-hand side we must substitute

$$\frac{\partial\hat{\varepsilon}'}{\partial t} = \frac{\partial\hat{\varepsilon}'}{\partial B_\gamma^0}\frac{\partial B_\gamma^0}{\partial t} + \frac{\partial\hat{\varepsilon}'}{\partial(\nabla_\mu B_\nu^0)}\frac{\partial(\nabla_\mu B_\nu^0)}{\partial t}. \qquad (2.18)$$

On the other hand, the energy conservation law can in this case be written in the form

$$-\frac{c}{4\pi}\oint[\widetilde{\mathbf{E}}\times\widetilde{\mathbf{B}}]_n d\sigma = \frac{\omega}{8\pi}\int d\mathbf{r}\,\langle\mathbf{E}^*\hat{\varepsilon}''\mathbf{E}\rangle + \frac{\partial W}{\partial t} + \int d\mathbf{r}\mathbf{M}\frac{\partial\mathbf{B}^0}{\partial t}, \qquad (2.19)$$

where the last term with opposite sign is equal to the increase in energy due to the change in $\mathbf{B}^0$.[13] Comparing (2.19) and (2.7) we get

$$\mathbf{M}(\mathbf{r}) = \frac{1}{16\pi}\frac{\delta}{\delta(\partial\mathbf{B}^0/\partial t)}\int d\mathbf{r}'\left\{\left\langle\mathbf{E}^*\frac{\partial\hat{\varepsilon}'}{\partial t}\mathbf{E}\right\rangle - \mathrm{Im}\left\langle\mathbf{E}^*\frac{\partial^2\varepsilon'}{\partial t\partial\omega}\frac{\partial\mathbf{E}}{\partial t}\right\rangle\right\}, \qquad (2.20)$$

where $\delta/\delta(\partial\mathbf{B}^0/\partial t)$ indicates the functional derivative of $\partial\mathbf{B}^0(\mathbf{r}, t)/\partial t$ at the point $\mathbf{r}$ (t is here a fixed parameter and for the sake of simplicity is not written down in the arguments). Substituting (2.18) and noting that after evaluating the functional derivative we must put $\partial\mathbf{B}^0/\partial t = 0$ we find

$$\mathbf{M}(\mathbf{r}) = \frac{1}{16\pi}\int d\mathbf{r}'\left\{\left\langle\mathbf{E}^*\frac{\delta\varepsilon'}{\delta\mathbf{B}^0(\mathbf{r})}\mathbf{E}\right\rangle - \mathrm{Im}\left\langle\mathbf{E}^*\frac{\partial}{\partial\omega}\left(\frac{\delta\hat{\varepsilon}'}{\delta\mathbf{B}^0(\mathbf{r})}\right)\frac{\partial\mathbf{E}}{\partial t}\right\rangle\right\}, \qquad (2.21)$$

where $\hat{\varepsilon}$ as a functional of $\mathbf{B}^0$ is given in (2.17) up to the first derivatives. Nelgecting spatial dispersion [in which case we can also neglect the gradient terms in (2.17)] we get from (2.21)

$$\mathbf{M}(\mathbf{r}, t) = \frac{1}{16\pi}\left\{E_\alpha^*(\omega, t; \mathbf{r})\frac{\partial\varepsilon_{\alpha\beta}'(\omega, \mathbf{B}^0)}{\partial\mathbf{B}^0}E_\beta(\omega, t; \mathbf{r}) - \mathrm{Im}\left[E_\alpha^*(\omega, t; \mathbf{r})\frac{\partial^2\varepsilon_{\alpha\beta}'}{\partial\mathbf{B}_0\partial\omega}\frac{\partial E_\beta(\omega, t; \mathbf{r})}{\partial t}\right]\right\} \qquad (2.22)$$

(we remember that the $\mathbf{r}$-dependence of the field $\mathbf{E}$ here is arbitrary). The second term on the right-hand side is a correction to Pitaevskiĭ's well known formula[11] and is due to the time-dependence of $\mathbf{E}$.

If we take spatial dispersion into account for the quasi-planar wave (2.13), we get, using (2.21), (2.16) and restricting ourselves to terms of first order in the derivatives

$$\mathbf{M} = \frac{1}{16\pi}\left\{\mathscr{E}_\alpha^*\frac{\partial\varepsilon_{\alpha\beta}'}{\partial\mathbf{B}^0}\mathscr{E}_\beta - \nabla_\gamma\left(\mathscr{E}_\alpha^*\frac{\partial\varepsilon_{\alpha\beta}'}{\partial(\nabla_\gamma\mathbf{B}^0)}\mathscr{E}_\beta\right) - \mathrm{Im}\left[\mathscr{E}_\alpha^*\frac{\partial^2\varepsilon_{\alpha\beta}'}{\partial\omega\partial\mathbf{B}^0}\frac{\partial\mathscr{E}_\beta}{\partial t} - \mathscr{E}_\alpha^*\frac{\partial^2\varepsilon_{\alpha\beta}'}{\partial k_\gamma\partial\mathbf{B}^0}\nabla_\gamma\mathscr{E}_\beta\right]\right\}. \qquad (2.23)$$

3. GENERALIZATION OF THE BASIC RELATIONS FOR THE CASE OF MOVING MEDIA

Our next problem is to find the ponderomotive force, taking into account time-derivatives of the field amplitudes, dissipation, and spatial dispersion. To do this, however, it is necessary to generalize the formulae of Sec. 2 to the case of a moving medium with an arbitrary velocity field $\mathbf{v}(\mathbf{r})$. We shall express the ponderomotive force as a functional derivative of a functional of $\mathbf{v}(\mathbf{r})$ and after differentiation we put $\mathbf{v} \equiv 0$. We can thus assume that $\mathbf{v}$ is small and that it is sufficient to take into account only terms linear in the velocity and its spatial derivatives. As to terms with $\partial\mathbf{v}/\partial t$ we assume here that they are small of higher order and that we therefore can neglect them. Indeed, $\partial\mathbf{v}/\partial t \propto \nabla|\mathbf{E}|^2$ when there are no external forces. In the present section we assume that in the comoving reference frame in which the medium is at rest we can neglect spatial dispersion. Let the field have the form (2.2) in the laboratory frame. As $\mathbf{v}$ is sufficiently small, the function $\mathbf{E}[\omega, t, \mathbf{r}(\mathbf{r}', t, \mathbf{v})]$, where $\mathbf{r}'$ is the radius vector in the comoving reference frame, depends slowly on t although it is possible that $(\partial E/\partial t)_{r'} \gg (\partial E/\partial t)_r$. This fact will be used in what follows in an essential way.

We shall now start from a generalization of Minkowski's formulae. For small constant $\mathbf{v}$ and $\partial\hat{\varepsilon}/\partial t$ the first of them has the form[21]

$$\widetilde{\mathbf{D}} = \hat{\varepsilon}(\hat{\omega}')\widetilde{\mathbf{E}} + c^{-1}(\hat{\varepsilon} - \hat{I})[\mathbf{v}\times\widetilde{\mathbf{B}}], \tag{3.1}$$

where $\hat{I}$ is the unit tensor, $\hat{\omega} = i(\partial/\partial t - \mathbf{v}\cdot\nabla)$ is the frequency operator in the comoving reference frame [in the second term in (3.1) we can neglect the difference between $\hat{\omega}'$ and $\hat{\omega} = i\partial/\partial t$]. When $\Delta_\alpha v_\beta \neq 0$, $\partial\hat{\varepsilon}/\partial t \neq 0$ the substitution $\hat{\omega} \to \hat{\omega}'$ is insufficient as it does not take into account local rotation of the medium and its non-stationarity. When these factors are taken into account one must replace the first term in (3.1) by Eq. (2.4) where instead of $\partial\mathbf{E}/\partial t$ and $\partial\hat{\varepsilon}/\partial t$ derivatives in the comoving frame of reference occur which we shall denote by $(\partial\mathbf{E}/\partial t)_0$, $(\partial\hat{\varepsilon}/\partial t)_0$, and so on. For small $\mathbf{v}$ they can be expressed in terms of the derivatives in the laboratory frame $\partial\mathbf{E}/\partial t$, $\partial\hat{\varepsilon}/\partial t$ through the formulae

$$\left(\frac{\partial\mathbf{E}}{\partial t}\right)_0 = \frac{\partial\mathbf{E}}{\partial t} + (\mathbf{v}\nabla)\mathbf{E} - \frac{1}{2}[\operatorname{rot}\mathbf{v}\times\mathbf{E}], \tag{3.2}$$

$$\left(\frac{\partial\varepsilon_{\alpha\beta}}{\partial t}\right)_0 = \frac{\partial\varepsilon_{\alpha\beta}}{\partial t} + (\mathbf{v}\nabla)\varepsilon_{\alpha\beta} - \frac{1}{2}(\nabla_\gamma v_\alpha - \nabla_\alpha v_\gamma)\varepsilon_{\gamma\beta} - \frac{1}{2}\varepsilon_{\alpha\gamma}(\nabla_\gamma v_\beta - \nabla_\beta v_\gamma). \tag{3.3}$$

Terms containing $(\mathbf{v}\nabla)$ are due to the translational motion of a volume element with its center at the point $\mathbf{r}$ with velocity $\mathbf{v}$. The third term in (3.2) gives a contribution to $(\partial\mathbf{E}/\partial t)_0$ due to the rotation of the vector $\mathbf{E}$ with a local angular velocity $-\mathbf{\Omega}$ (where $\mathbf{\Omega} = \frac{1}{2}\operatorname{curl}\mathbf{v}$) relative to the axes of the comoving frame. Of course, a formula of the form (3.2) does hold not only for $\mathbf{E}$ but also for any other vector. Similarly, a relation such as (3.3) holds for any tensor as one can easily verify by considering a tensor of the form $A_\alpha A_\beta$ and taking into account that

$$[\operatorname{rot}\mathbf{v}, \times\mathbf{A}]_\alpha = (\nabla_\beta v_\alpha - \nabla_\alpha v_\beta)A_\beta.$$

Now substituting $(\partial\mathbf{E}/\partial t)_0$ and $(\partial\varepsilon_{\alpha\beta}/\partial t)_0$ from (3.2) and (3.3) into (2.4) we get the first two corrections to the induction in the form

$$\delta_1 D_\alpha = v_\beta\left[i\frac{\partial\varepsilon_{\alpha\gamma}}{\partial\omega}\nabla_\beta E_\gamma - \frac{\partial^2\varepsilon_{\alpha\gamma}}{\partial\omega^2}\nabla_\beta\frac{\partial E_\gamma}{\partial t} + \frac{i}{2}\left(\nabla_\beta\frac{\partial\varepsilon_{\alpha\gamma}}{\partial\omega}\right)E_\gamma - \frac{1}{4}\left(\frac{\partial}{\partial t}\nabla_\beta\frac{\partial^2\varepsilon_{\alpha\gamma}}{\partial\omega^2}\right)E_\gamma - \frac{1}{2}\left(\nabla_\beta\frac{\partial^2\varepsilon_{\alpha\gamma}}{\partial\omega^2}\right)\times\frac{\partial E_\gamma}{\partial t} - \frac{1}{2}\frac{\partial^3\varepsilon_{\alpha\gamma}}{\partial t\partial\omega^2}\nabla_\beta E_\gamma\right], \tag{3.4}$$

$$\delta_2 D_\alpha = \delta\varepsilon_{\alpha\beta}E_\beta + i\frac{\partial\delta\varepsilon_{\alpha\beta}}{\partial\omega}\frac{\partial E_\beta}{\partial t}, \tag{3.5}$$

where

$$\delta\varepsilon_{\alpha\beta} = \frac{1}{4i}\left\{\frac{\partial\varepsilon_{\alpha\gamma}}{\partial\omega}(\nabla_\beta v_\gamma - \nabla_\gamma v_\beta) + \frac{\partial\varepsilon_{\gamma\beta}}{\partial\omega}(\nabla_\gamma v_\alpha - \nabla_\alpha v_\gamma)\right\}. \tag{3.6}$$

The correction $\delta_1 D_\alpha$ is here due to the terms with $(\mathbf{v}\cdot\nabla)$ and $\delta_2 D_\alpha$ occurred due to the rotational terms in Eqs. (3.2), (3.3). One checks easily that all terms in $\delta_1 D_\alpha$ can be obtained from the first term in (3.1) as corrections describing the spatial dispersion appearing due to the motion of the medium.

However, we neglect here yet another correction due to the occurrence of spatial dispersion. To find it we assume to begin with that the wave is quasi-planar with wavevector $\mathbf{k}$. In that case $\varepsilon_{\alpha\beta}(\hat{\omega}') \to \varepsilon_{\alpha\beta}[\omega - \mathbf{k}\cdot\mathbf{v}(\mathbf{r})]$. As there appears in $\varepsilon_{\alpha\beta}$ an additional $\mathbf{r}$-dependence due to $\mathbf{v}(\mathbf{r})$ it is necessary to take into account the second term in (2.14). This leads to the fact that we must add to $\varepsilon_{\alpha\beta}$

$$-(i/2)\,\partial^2\varepsilon_{\alpha\beta}/\partial k_\gamma\partial r_\gamma.$$

Substituting here $\varepsilon_{\alpha\beta}[\omega - \mathbf{k}\cdot\mathbf{v}(\mathbf{r})]$ and differentiating we get an additional term of the form

$$(i/2)\,(\partial\varepsilon_{\alpha\beta}(\omega)/\partial\omega)\operatorname{div}\mathbf{v}. \tag{3.7}$$

As it does not contain $\mathbf{k}$ it can be applied to fields with any $\mathbf{r}$-dependence. From (3.7) it follows that in the expression for $\mathbf{D}$ there must in a moving medium be a term

$$\delta_3 D_\alpha = \left(\frac{i}{2}\frac{\partial\varepsilon_{\alpha\beta}}{\partial\omega}E_\beta - \frac{1}{2}\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\omega^2}\frac{\partial E_\beta}{\partial t}\right)\operatorname{div}\mathbf{v}. \tag{3.8}$$

Together with $\delta_2 D_\alpha$ this expression determines the corrections to $\mathbf{D}$ connected with the non-uniformity of the velocity field. Taking these corrections into account the effective permittivity has at $v = 0$ and $\nabla_\alpha v_\beta \neq 0$ the form

$$\varepsilon_{\alpha\beta}^{eff} = \varepsilon_{\alpha\beta} - \frac{i}{4}\left[\frac{\partial\varepsilon_{\alpha\gamma}}{\partial\omega}(\nabla_\beta v_\gamma - \nabla_\gamma v_\beta) + \frac{\partial\varepsilon_{\gamma\beta}}{\partial\omega}(\nabla_\gamma v_\alpha - \nabla_\alpha v_\gamma)\right] + \frac{i}{2}\frac{\partial\varepsilon_{\alpha\beta}}{\partial\omega}\operatorname{div}\mathbf{v}. \tag{3.9}$$

The second term is in the particular case of an isotropic medium the same as those given in §102 of Ref. 13, while the third one, containing $\operatorname{div}\mathbf{v}$ is a kinematic correction which, as far as we know, has not been discussed before.[3)]

Finally, we consider the fourth correction to the induction arising from the second term in (3.1). For a quasi-monochromatic wave we can write it in the form

$$\delta_4 D_\alpha = \frac{1}{c}(\varepsilon_{\alpha\beta} - \delta_{\alpha\beta})[\mathbf{v}\times\mathbf{B}]_\beta + \frac{i}{c}\frac{\partial\varepsilon_{\alpha\beta}}{\partial\omega}\left[\mathbf{v}\times\frac{\partial\mathbf{B}}{\partial t}\right]_\beta. \tag{3.10}$$

The complete expression for the amplitude of the electrical induction in a moving medium thus has the form

$$D_\alpha = \left(\varepsilon_{\alpha\beta}^{eff} + \frac{i}{2}\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial t\partial\omega} - \frac{1}{8}\frac{\partial^4\varepsilon_{\alpha\beta}}{\partial t^2\partial\omega^2}\right)E_\beta + \left(i\frac{\partial\varepsilon_{\alpha\beta}^{eff}}{\partial\omega} - \frac{1}{2}\frac{\partial^3\varepsilon_{\alpha\beta}}{\partial\omega^2\partial t}\right)\frac{\partial E_\beta}{\partial t} - \frac{1}{2}\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\omega^2}\frac{\partial^2 E_\beta}{\partial t^2} + \delta_1 D_\alpha + \delta_4 D_\alpha. \tag{3.11}$$

The quantity $\varepsilon_{\alpha\beta}^{\text{eff}}$ is given in (3.9) and we can write $\partial\varepsilon_{\alpha\beta}/\partial t$ in the form

$$\frac{\partial \varepsilon_{\alpha\beta}}{\partial t}=\frac{\partial \varepsilon_{\alpha\beta}}{\partial \Lambda}\frac{\partial \Lambda}{\partial t}-\frac{1}{2}\varepsilon_{\alpha\gamma}(\nabla_\beta v_\gamma-\nabla_\gamma v_\beta)-\frac{1}{2}\varepsilon_{\gamma\beta}(\nabla_\alpha v_\gamma-\nabla_\gamma v_\alpha)$$
$$-\frac{1}{2}\frac{\partial \varepsilon_{\alpha\beta}}{\partial \mathbf{B}^0}[\mathrm{rot}\,\mathbf{v}\times\mathbf{B}^0]. \tag{3.12}$$

In the first term on the right-hand side we assume summation over all parameters Λ on which $\varepsilon_{\alpha\beta}$ depends, while the derivative $\partial\Lambda/\partial t$ is taken in the laboratory frame. The second and third terms describe the contribution from the local rotation of axes of the tensor $\varepsilon_{\alpha\beta}$ relative to the laboratory frame with angular velocity $\mathbf{\Omega}=\frac{1}{2}$ curl $\mathbf{v}$ as if it were rigidly fixed to the medium. However, the change in the magnetic field $\mathbf{B}^0$ which also contributes to the anisotropy does not reduce to a rotation with angular velocity $\mathbf{\Omega}$. This fact is taken into account by the last term in (3.12) which cancels the contribution of the rotation of the vector $\mathbf{B}^0$ with angular velocity $\mathbf{\Omega}$. However, the contribution from the complete change in the magnetic field is contained in the first term on the right-hand side of (3.12) and is considered in detail in the next section.

If the anisotropy of the medium is caused solely by the field $\mathbf{B}^0$, i.e.,

$$\varepsilon_{\alpha\beta}=\varepsilon_1(\omega,\ |B^0|^2)\delta_{\alpha\beta}+\varepsilon_2(\omega,|B^0|^2)B_\alpha{}^0B_\beta{}^0+i\varepsilon_3(\omega,\ |B^0|^2)\delta_{\alpha\beta\gamma}B_\gamma{}^0, \tag{3.13}$$

where $\delta_{\alpha\beta\gamma}$ is the totally antisymmetric tensor, one verifies easily that the last three terms in (3.12) cancel completely.

We must supplement the formula for the electrical induction with an expression for the magnetic field strength in a moving medium which has the form

$$\tilde{\mathbf{H}}=\tilde{\mathbf{B}}+c^{-1}[\mathbf{v}(\tilde{\mathbf{D}}-\tilde{\mathbf{E}})]. \tag{3.14}$$

We now consider the electromagnetic energy density in a moving medium. This is connected with the energy density w' in a medium at rest through the relation

$$w=w'+(\mathbf{v}/4\pi c)\ \mathrm{Re}\,[\mathbf{E}\times\mathbf{B}^*], \tag{3.15}$$

which is obtained through a Lorentz transformation using only terms linear in $\mathbf{v}$. By w' we understand here the adiabatic expression for the energy density given by Eq. (2.9) where we must replace $\mathbf{E}$ and $\mathbf{B}$ by $\mathbf{E}'$ and $\mathbf{B}'$ which are connected with the field strengths in the laboratory frame through the relations

$$\mathbf{E}'=\mathbf{E}+c^{-1}[\mathbf{v}\times\mathbf{B}],\quad \mathbf{B}'=\mathbf{B}-c^{-1}[\mathbf{v}\times\mathbf{E}]. \tag{3.16}$$

We must also recognize that the derivative $(\partial\mathbf{E}'/\partial t)_0$ occurs in Eq. (2.9). As a result we get from (2.9), (3.15), (3.16), and (3.2) the following expression for the energy density in the laboratory frame of reference:

$$w=\frac{1}{16\pi}\left\{E_\alpha{}^*\frac{\partial(\omega\varepsilon_{\alpha\beta}{}')}{\partial\omega}E_\beta+|\mathbf{B}|^2-\mathrm{Im}\left(E_\alpha{}^*\frac{\partial^2(\omega\varepsilon_{\alpha\beta}{}')}{\partial\omega^2}\frac{\partial E_\beta}{\partial t}\right)\right\}$$
$$+\frac{1}{8\pi c}\mathrm{Re}\left\{E_\alpha{}^*\frac{\partial(\omega\varepsilon_{\alpha\beta}{}')}{\partial\omega}[\mathbf{v}\times\mathbf{B}]_\beta-\mathbf{E}[\mathbf{v}\times\mathbf{B}^*]\right\}$$
$$-\frac{1}{16\pi}\mathrm{Im}\left\{E_\alpha{}^*\frac{\partial^2(\omega\varepsilon_{\alpha\beta}{}')}{\partial\omega^2}(\mathbf{v}\nabla)E_\beta\right.$$
$$+\frac{1}{c}E_\alpha{}^*\frac{\partial^2(\omega\varepsilon_{\alpha\beta}{}')}{\partial\omega^2}\left[\mathbf{v}\times\frac{\partial\mathbf{B}}{\partial t}\right]_\beta-\frac{1}{c}\frac{\partial E_\alpha{}^*}{\partial t}\frac{\partial^2(\omega\varepsilon_{\alpha\beta}{}')}{\partial\omega^2}[\mathbf{v}\times\mathbf{B}]_\beta$$
$$\left.+\frac{1}{2}E_\alpha{}^*\frac{\partial^2(\omega\varepsilon_{\alpha\beta}{}')}{\partial\omega^2}E_\gamma(\nabla_\beta v_\gamma-\nabla_\gamma v_\beta)\right\}. \tag{3.17}$$

We find now an expression for the density of dissipation q per unit time in a moving medium. Here $q=q'(1-v^2/c^2)^{1/2}\approx q'$, where q' is the volume density of dissipation in the comoving frame.[14] The adiabatic expression for q' is given by the formula

$$q'=(1/4\pi)\ \mathrm{Re}\ \overline{\{\mathbf{E}'^*(\partial\mathbf{D}'/\partial t)_0\}}_{\varepsilon'=0}.$$

Using Eqs. (3.2), (3.3) and (3.16) to express $(\partial\tilde{\mathbf{D}}'/\partial t)_0\tilde{\mathbf{E}}'^*$ in terms of $(\partial\mathbf{D}/\partial t)\mathbf{E}^*$ we find (neglecting high-order terms)

$$q=\frac{\omega}{8\pi}E_\alpha{}^*\varepsilon_{\alpha\beta}{}''E_\beta-\frac{\mathbf{v}\omega}{4\pi c}\mathrm{Re}[\hat{\varepsilon}''\mathbf{E}\times\mathbf{B}^*]$$
$$-\frac{1}{8\pi}\mathrm{Im}\left\{E_\alpha{}^*(\mathbf{v}\cdot\nabla)(\varepsilon_{\alpha\beta}{}''E_\beta)\right.$$
$$\left.+E_\alpha{}^*\omega\frac{\partial\varepsilon_{\alpha\beta}{}''}{\partial\omega}(\mathbf{v}\nabla)E_\beta+\frac{1}{2}\mathrm{rot}\,\mathbf{v}\left[\mathbf{E}^*\times\frac{\partial(\omega\hat{\varepsilon}'')}{\partial\omega}\mathbf{E}\right]\right\}. \tag{3.18}$$

Substituting now into Eq. (2.6) (which is valid for a moving medium after replacing $\tilde{\mathbf{B}}\cdot\partial\tilde{\mathbf{B}}/\partial t$ by $\tilde{\mathbf{H}}\cdot\partial\tilde{\mathbf{B}}/\partial t$ the expressions for $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{H}}$ and using (3.11), (3.12), (3.17), and (3.18) we get

$$-\frac{c}{8\pi}\mathrm{div}\,\mathrm{Re}\left\{[\mathbf{E}^*\times\mathbf{H}]+\frac{\mathbf{v}\omega}{2c}\left(E_\alpha{}^*\frac{\partial\varepsilon_{\alpha\beta}{}'}{\partial\omega}E_\beta\right.\right.$$
$$\left.\left.+iE_\alpha{}^*\frac{\partial^2\varepsilon_{\alpha\beta}{}'}{\partial\omega^2}\frac{\partial E_\beta}{\partial t}\right)\right\}$$
$$=\frac{\partial w}{\partial t}+q+\frac{1}{16\pi}\mathrm{Re}\left\{E_\alpha{}^*\frac{\partial\varepsilon_{\alpha\beta}{}'}{\partial t}E_\beta\right.$$
$$+iE_\alpha{}^*\left[\left(\frac{\partial}{\partial t}+\mathbf{v}\nabla\right)\frac{\partial\varepsilon_{\alpha\beta}{}'}{\partial\omega}\right]\frac{\partial E_\beta}{\partial t}$$
$$+2\frac{\mathbf{v}}{c}\frac{\partial}{\partial t}[(\hat{\varepsilon}'-\hat{I})\mathbf{E}\times\mathbf{B}^*]$$
$$+2iv_\gamma(\nabla_\gamma E_\alpha{}^*)\frac{\partial\varepsilon_{\alpha\beta}{}'}{\partial\omega}\frac{\partial E_\beta}{\partial t}-\frac{i}{2}(\nabla_\alpha v_\gamma-\nabla_\gamma v_\alpha)$$
$$\left.\times\frac{\partial}{\partial t}\left(E_\alpha{}^*\frac{\partial\varepsilon_{\gamma\beta}{}'}{\partial\omega}E_\beta\right)+iE_\alpha{}^*\frac{\partial\varepsilon_{\alpha\beta}{}'}{\partial\omega}\frac{\partial E_\beta}{\partial t}\mathrm{div}\,\mathbf{v}\right\}$$
$$+\frac{1}{8\pi}\mathrm{Re}\left\{\frac{\omega}{c}[\varepsilon''\mathbf{E}\times\mathbf{B}^*]\mathbf{v}\right.$$
$$\left.+i(\nabla_\alpha E_\gamma{}^*)\varepsilon_{\alpha\beta}{}''E_\beta v_\gamma+(i/2)[\hat{\varepsilon}''\mathbf{E}\times\mathbf{E}^*]\mathrm{rot}\,\mathbf{v}\right\}. \tag{3.19}$$

It is here important that the quantity $\partial\hat{\varepsilon}/\partial t$ itself contains, according to (3.12), derivatives with respect to the velocity. Equation (3.19) which expresses the energy conservation law in a moving medium will be the starting point for finding the ponderomotive force.

4. THE PONDEROMOTIVE FORCE IN A MEDIUM WITH TEMPORAL DISPERSION

One can find the ponderomotive force from the energy conservation law

$$\partial w/\partial t+\operatorname{div}\mathbf{S}+q+\mathbf{fv}=0, \tag{4.1}$$

where q is the dissipation density in the laboratory coordinate system, $\mathbf{S}$ the energy flux density, and $\mathbf{f}$ the volume density of the ponderomotive force. It follows from (4.1) that in a medium at rest

$$\mathbf{f}(\mathbf{r})=-\frac{\delta}{\delta\mathbf{v}(\mathbf{r})}\int\left(\frac{\partial w}{\partial t}+q\right)d\mathbf{r}\Big|_{\mathbf{v}=0}, \tag{4.2}$$

where $\delta/\delta\mathbf{v}(\mathbf{r})$ is the functional derivative. We assume here that the non-stationarity of the medium is caused solely by its motion so that the permittivity of a medium at rest is time-independent. Determining $\partial w/\partial t+q$ from (3.19) and using (3.12), (2.2) we find from (4.2)

$$f_\alpha=\frac{\delta}{\delta v_\alpha(\mathbf{r})}\int\frac{d\mathbf{r}}{16\pi}\operatorname{Re}\left\{E_\beta^*\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\Lambda}\frac{\partial\Lambda}{\partial t}E_\gamma+iE_\beta^*\left[\left(\frac{\partial\Lambda}{\partial t}\frac{\partial}{\partial\Lambda}+\mathbf{v}\nabla\right)\times\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\omega}\right]\frac{\partial E_\gamma}{\partial t}\right\}_{v=0}+\frac{1}{8\pi}\operatorname{Re}\left\{\frac{1}{c}\frac{\partial}{\partial t}\left[(\hat{\varepsilon}'-\hat{I})\mathbf{E}\times\mathbf{B}^*\right]_\alpha+i(\nabla_\alpha E_\beta^*)\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\omega}\frac{\partial E_\gamma}{\partial t}-\frac{i}{2}\nabla_\alpha\left(E_\beta^*\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\omega}\frac{\partial E_\gamma}{\partial t}\right]\right\}+f_\alpha^{(a)}+f_\alpha^{(d)}, \tag{4.3}$$

where

$$f_\alpha^{(a)}=\nabla_\beta\left\{\frac{1}{16\pi}\operatorname{Re}(E_\beta^*D_\alpha^0-E_\alpha^*D_\beta^0)+\frac{1}{2}(M_\alpha B_\beta^0-M_\beta B_\alpha^0)\right\}, \tag{4.4}$$

$$f_\alpha^{(d)}=-\frac{1}{8\pi}\operatorname{Im}\left\{(\nabla_\alpha E_\beta^*)\varepsilon_{\beta\gamma}''E_\gamma+\frac{1}{2}\nabla_\beta(E_\beta^*\varepsilon_{\alpha\gamma}''E_\gamma-E_\alpha^*\varepsilon_{\beta\gamma}''E_\gamma)\right\}, \tag{4.5}$$

where we have varied terms containing $\mathbf{v}$ explicitly. We wrote in (4.4)

$$\mathbf{D}^0=\hat{\varepsilon}'\mathbf{E}+i\left(\frac{\partial\varepsilon'}{\partial\omega}\right)\frac{\partial\mathbf{E}}{\partial t},$$

while the vector $\mathbf{M}$ is given in (2.22). When obtaining (4.5) we used the fact that

$$[\hat{\varepsilon}''\mathbf{E}\times\mathbf{B}^*]_\alpha=i\frac{c}{\omega}(\nabla_\alpha E_\beta^*-\nabla_\beta E_\alpha^*)\varepsilon_{\beta\gamma}''E_\gamma.$$

To evaluate the functional derivative in (4.3) we must find the time-derivatives of the parameters on which $\varepsilon_{\alpha\beta}$ depends. Assuming for the sake of simplicity that these parameters are the density ρ and the average magnetic field $\mathbf{B}^0$ we have

$$\frac{\partial\hat{\varepsilon}'}{\partial\Lambda}\frac{\partial\Lambda}{\partial t}=\frac{\partial\hat{\varepsilon}'}{\partial\rho}\frac{\partial\rho}{\partial t}+\frac{\partial\hat{\varepsilon}'}{\partial\mathbf{B}^0}\frac{\partial\mathbf{B}^0}{\partial t}. \tag{4.6}$$

The quantity $\partial\rho/\partial t$ is determined from the continuity equation $\partial\rho/\partial t=-\operatorname{div}(\rho\mathbf{v})$. As far as $\partial\mathbf{B}^0/\partial t$ is concerned, we consider here two cases: a non-conducting and a highly conducting medium. In the first case $\partial\mathbf{B}^0/\partial t$ in the laboratory coordinate system when the medium is moving. In the second case we use the well known equation from magnetohydrodynamics[13]

$$\frac{\partial\mathbf{B}^0}{\partial t}=\operatorname{rot}[\mathbf{v}\times\mathbf{B}^0]-\frac{c^2}{4\pi}\operatorname{rot}\left(\frac{\operatorname{rot}\mathbf{B}^0}{\sigma}\right), \tag{4.7}$$

where σ is the static conductivity; as before we consider a medium with $\mu=1$. We neglect the dissipative term in (4.7) as it leads to higher spatial derivatives in the ponderomotive force and the conductivity is assumed to be high.

Taking what we have said into account and substituting (4.6) into the integral in (4.3) we find

$$\frac{\delta}{\delta v_\alpha(\mathbf{r})}\int d\mathbf{r}\operatorname{Re}\{\ldots\}_{\mathbf{v}=0}=\operatorname{Re}\left\{\rho\nabla_\alpha\left(E_\beta^*\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\rho}E_\gamma\right)+i\nabla_\alpha\left(E_\beta^*\rho\frac{\partial^2\varepsilon_{\beta\gamma}'}{\partial\rho\partial\omega}\frac{\partial E_\gamma}{\partial t}\right)+16\pi g(\sigma)[\mathbf{B}^0\times\operatorname{rot}\dot{\mathbf{M}}]_\alpha+iE_\beta^*\frac{\partial^2\varepsilon_{\beta\gamma}'}{\partial B_\delta^0\partial\omega}\frac{\partial E_\gamma}{\partial t}\nabla_\alpha B_\delta^0\right\}, \tag{4.8a}$$

where $\mathbf{M}$ is given by (2.22) and

$$g(\sigma)=0\text{ when }\sigma=0,\quad g(\sigma)=1\text{ when }\sigma\to\infty. \tag{4.8b}$$

Substituting (4.8a) into (4.3) we find finally, after simple transformations

$$f_\alpha=\frac{1}{16\pi}\nabla_\alpha\left\{E_\beta^*\rho\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\rho}E_\gamma-\operatorname{Im}\left(E_\beta^*\frac{\partial}{\partial\omega}\left[\rho\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\rho}-\varepsilon_{\beta\gamma}'\right]\frac{\partial E_\gamma}{\partial t}\right)\right\}-\frac{1}{16\pi}E_\beta^*(\nabla_\alpha\varepsilon_{\beta\gamma}')E_\gamma+g(\sigma)[\mathbf{B}^0\times\operatorname{rot}\mathbf{M}]_\alpha+M_\beta\nabla_\alpha B_\beta^0+\frac{1}{8\pi c}\operatorname{Re}\left\{\frac{\partial}{\partial t}[(\hat{\varepsilon}'-\hat{I})\mathbf{E}\times\mathbf{B}^*]_\alpha+\omega\left[\frac{\partial\varepsilon'}{\partial\omega}\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}^*\right]_\alpha\right\}-\frac{1}{8\pi}\operatorname{Im}\left\{(\nabla_\beta E_\alpha^*)\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\omega}\frac{\partial E_\gamma}{\partial t}\right\}+f_\alpha^{(a)}+f_\alpha^{(d)}, \tag{4.9}$$

where $\mathbf{f}^{(a)}$ and $\mathbf{f}^{(d)}$ are given by (4.4), (4.5). The term $f_\alpha^{(a)}$ is the divergence of an antisymmetric tensor. Below we see that just this term [together with the last term in (4.5)] guarantees in the general case the symmetry of the stress tensor. The term $f_\alpha^{(d)}$ determines the contribution from dissipative effects. In obtaining (4.9) we used the relation

$$\operatorname{Im}\left\{(\nabla_\alpha E_\beta^*)\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\omega}\frac{\partial E_\gamma}{\partial t}\right\}=-\operatorname{Re}\left\{\frac{\omega}{c}\left[\frac{\partial\hat{\varepsilon}'}{\partial\omega}\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}^*\right]_\alpha+i(\nabla_\beta E_\alpha^*)\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\omega}\frac{\partial E_\gamma}{\partial t}\right\}. \tag{4.10}$$

We discuss the expression for the ponderomotive force obtained here. We obtain Pitaevskiĭ's formula[11] without the term with $f_\alpha^{(a)}$ for non-conducting media with $\partial\mathbf{E}/\partial t=0$ and $\varepsilon_{\alpha\beta}''=0$. This is connected with the fact that in Ref. 11 apart from $\sigma=0$ it was assumed that the anisotropy of the medium was caused solely by the magnetic field $\mathbf{B}^0$, i.e., Eq. (3.13) holds. As we saw earlier, in that case the last three terms on the right-hand side of (3.12) which just lead to $\mathbf{f}^{(a)}$ cancel one another. The term $\mathbf{B}^0\times\operatorname{curl}\mathbf{M}$ was found in Ref. 3 for a cold plasma and in Ref. 12 for any well conducting medium.

We turn now to the terms in (4.9) containing time-derivatives. The terms

$$\frac{1}{8\pi c}\operatorname{Re}\left\{\frac{\partial}{\partial t}[(\hat{\varepsilon}'-\hat{I})\mathbf{E}\times\mathbf{B}^*]+\omega\left[\frac{\partial\varepsilon'}{\partial\omega}\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}^*\right]\right\}$$

were found in Ref. 12 for media with temporal dispersion while the term

$$-(1/8\pi)\operatorname{Im}\left\{(\nabla_\beta E_\alpha^*)\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\omega}\frac{\partial E_\gamma}{\partial t}\right\}$$

up to now has been obtained only for microscopic models of a plasma.[4,8] It vanishes in an isotropic medium for transverse waves. In the general case, however, and, in particular, for longitudinal waves this is not the case (this term was omitted by mistake in Refs. 9 and 12).

To estimate the relative role played by the terms with time-derivatives we assume that the wave is quasi-planar with wavevector $\mathbf{k}$ and characteristic ranges of change in amplitude L and T. The ratio of the terms with $\partial\mathbf{E}/\partial t$ to the first term in (4.9) is then of the order of magnitude $kL/\omega_* T$. For the first of the terms considered (the Abraham term in a non-dispersive medium) $\omega_* = \omega$; for the others, containing $\partial\hat{\varepsilon}/\partial t$, the quantity ω_* is a characteristic frequency scale of the dispersion (generally speaking, $\omega_* \neq \omega$ but one should always assume that $\omega_* T \gg 1$). When $kL \gg 1$ the parameter $kL/\omega_* T$ may not be small; in that case it is impossible to neglect the terms discussed here. Of course, taking polarization effects into account may introduce important corrections in these estimates.

Estimating similarly the terms containing mixed derivatives with respect to $\mathbf{r}$ and t [the second term in (4.9) is one of them, as are the corrections caused by taking into account terms with $\partial\mathbf{E}/\partial t$ contained in $\mathbf{f}^{(a)}$ and the magnetic moment) one can check that they are of order $(\omega_* T)^{-1} \ll 1$ relative to the first term in (4.9). Some of the terms with mixed derivatives, namely those which are contained in the first term in (4.4) were considered in Refs. 6 and 16.

We consider finally the contribution to the ponderomotive force from the dissipative terms $\mathbf{f}^{(d)}$. It is given by Eq. (4.5). For a quasi-planar wave when $kL \gg 1$ the last term in (4.5) can be neglected. In that case

$$\mathbf{f}^{(d)}\approx-(1/8\pi)\operatorname{Im}[(\nabla E_\beta^*)\varepsilon_{\beta\gamma}''E_\gamma]\approx(1/8\pi)\mathbf{k}E_\beta^*\varepsilon_{\beta\gamma}''E_\gamma. \quad (4.11)$$

In an isotropic medium this last expression is the same as those obtained in Refs. 7 and 8 for a collisionless plasma. In that particular case one can obtain it also by the method of Ref. 5. Its ratio to the first term in (4.9) is of order $kL\varepsilon''/\varepsilon'$. The magnitude of that parameter is limited by the possibility to interpret q and w as densities of dissipation and of electromagnetic energy which is the basis of our approach. It is difficult to write down the condition for such an interpretation in general form. For a uniform collisional isotropic plasma with characteristic collision frequency ν the condition $\nu \ll T^{-1}$ was obtained in Ref. 2. In that case the order of the ratio of $f^{(d)}$ to the first term in (4.9) in a collisional plasma is limited by the condition

$$kL\varepsilon''/\varepsilon'\sim kL\nu/\omega\ll kL/\omega T. \quad (4.12)$$

For a collisionless plasma the quantity $kL\varepsilon''/\varepsilon'$ may be comparable with $kL/\omega T$.

If $\varepsilon_{\alpha\beta}'$ depends linearly on the density ρ as is the case for gases and plasmas one can simplify Eq. (4.9) since

$$\rho\partial\varepsilon_{\alpha\beta}'/\partial\rho=\varepsilon_{\alpha\beta}'-\delta_{\alpha\beta}.$$

The second term in (4.9) then vanishes.

We now consider the stress tensor corresponding to the force (4.9). Substituting in (4.9) the relations

$$E_\beta^*(\nabla_\alpha\varepsilon_{\beta\gamma}')E_\gamma=\nabla_\alpha(E_\beta^*\varepsilon_{\beta\gamma}'E_\gamma)$$
$$-2\operatorname{Re}\{(\nabla_\alpha E_\beta^*)[D_\beta-i(\partial\varepsilon_{\beta\gamma}'/\partial\omega)\partial E_\gamma/\partial t-i\varepsilon_{\beta\gamma}''E_\gamma]\}, \quad (4.13)$$

$$\operatorname{Re}(D_\beta\nabla_\alpha E_\beta^*)=\operatorname{Re}\left\{\nabla_\beta\left(D_\beta E_\alpha^*+B_\beta B_\alpha^*-\frac{1}{2}\delta_{\alpha\beta}|B|^2\right)-\frac{1}{c}\frac{\partial}{\partial t}[\mathbf{D}\times\mathbf{B}^*]_\alpha\right\}, \quad (4.14)$$

$$[\mathbf{B}^0\times\operatorname{rot}\mathbf{M}]_\alpha+M_\beta\nabla_\alpha B_\beta^0=\nabla_\beta(\delta_{\alpha\beta}\mathbf{B}^0\mathbf{M}-B_\beta^0M_\alpha); \quad (4.15)$$

(the first of them is obtained using (2.4) and the second using (2.4) and the Maxwell equations) and using (4.4), (4.5) we get

$$f_\alpha=\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}-\frac{1}{8\pi c}\frac{\partial}{\partial t}\operatorname{Re}[\mathbf{E}\times\mathbf{B}^*]_\alpha, \quad (4.16)$$

where $\sigma_{\alpha\beta}$ is the desired stress tensor:

$$\sigma_{\alpha\beta}=\frac{\delta_{\alpha\beta}}{16\pi}\left\{E_\mu^*\left(\rho\frac{\partial\varepsilon_{\mu\nu}'}{\partial\rho}-\varepsilon_{\mu\nu}'\right)E_\nu-\operatorname{Im}\left[E_\mu^*\frac{\partial}{\partial\omega}\left(\rho\frac{\partial\varepsilon_{\mu\nu}'}{\partial\rho}-\varepsilon_{\mu\nu}'\right)\frac{\partial E_\nu}{\partial t}\right]\right\}$$
$$+\frac{1}{16\pi}\operatorname{Re}(D_\alpha E_\beta^*+D_\beta E_\alpha^*)+\frac{1}{16\pi}(B_\alpha B_\beta^*+B_\beta B_\alpha^*-\delta_{\alpha\beta}|\mathbf{B}|^2)+\frac{1}{8\pi}(B_\alpha^0H_\beta^0+B_\beta^0H_\alpha^0-(\mathbf{H}^0)^2\delta_{\alpha\beta})$$
$$-\frac{g(\sigma)}{4\pi}\left[B_\alpha^0B_\beta^0-\frac{1}{2}(\mathbf{B}^0)^2\delta_{\alpha\beta}\right]. \quad (4.17)$$

Here $\mathbf{H}^0 = \mathbf{B}^0 - 4\pi\mathbf{M}$ is the average magnetic field strength. In obtaining (4.17) we used the fact that curl $\mathbf{H}^0 = 0$ when $\sigma = 0$.

We see that the tensor $\sigma_{\alpha\beta}$ turns out automatically to be symmetric. If $\varepsilon_{\alpha\beta}$ is given by Eq. (3.13) and $\mathbf{B} = 0$, $\varepsilon_{\alpha\beta}'' = 0$, $\partial\mathbf{E}/\partial t = 0$, $\sigma = 0$, (4.17) is the same as Pitaevskiĭ's formula.[11]

We now consider the tensor (4.17) on the boundary between two media (1 and 2). The force through which a HF field acts per unit area equals $F_\alpha = \sigma_{\alpha\beta}n_\beta$ where $\mathbf{n}$ is the unit vector along the external normal to the surface (directed from region 1 to region 2). At the boundary of two media the HF field, and with it the force $\mathbf{F}$, is, generally speaking discontinuous: $\{\mathbf{F}\}_{12} = \mathbf{F}(1) - \mathbf{F}(2)$. In an equilibrium state the magnitude of that discontinuity must equal, with opposite sign, the sum of the jumps in the other forces (for instance, of hydrodynamic origin). For instance, in isotropic media the quantity $\{F_n\}_{12} = \{\sigma_{\alpha\beta}n_\alpha n_\beta\}_{12}$ is equal with opposite sign to the jump in pressure $\{p(\rho)\}_{12}$. This condition determines the jump in the density $\{\rho\}_{12}$ on the boundary caused by the action of an HF field (see, i.e., §15 of Ref. 13). As regards the jump in the tangential component of the force $\{\mathbf{F}_\tau\}_{12} = \{\mathbf{F}\times\mathbf{n}\}_{12}$ in an isotropic medium this jump

$\{\mathbf{F}_\tau\}_{12}=0$ when $\varepsilon''=0$ and $\partial\mathbf{E}/\partial t=0$. This follows from (4.17) and the relations $\{D_n\}_{12}=0$, $\{\mathbf{E}_\tau\}_{12}=0$.

In the general case, however, $\{\mathbf{F}_\tau\}_{12}\neq 0$. Indeed, from (4.17) we obtain

$$\{\mathbf{F}_\tau\}_{12}=\frac{1}{16\pi}\operatorname{Re}\{\mathbf{D}_\tau E_n^*\}_{12}+\frac{1}{8\pi}\{\mathbf{B}_\tau^0 H_n^0\}_{12}-\frac{g(\sigma)}{4\pi}\{\mathbf{B}_\tau^0 B_n^0\}_{12}. \tag{4.18}$$

If $\varepsilon''_{\alpha\beta}=0$, $\partial\mathbf{E}/\partial t=0$ and $\varepsilon_{\alpha\beta}$ is given by Eq. (3.13), i.e., the anisotropy of the medium is determined solely by the field $\mathbf{B}^0$, one checks easily that

$$\{\mathbf{F}_\tau\}_{12}+\frac{g(\sigma)}{4\pi}\{\mathbf{B}_\tau^0 B_n^0\}_{12}=0. \tag{4.19}$$

For a non-conducting medium $g(\sigma)=0$ and the tangential component of the force $\mathbf{F}$ is continuous, in agreement with Ref. 11. As $\sigma\to\infty$, when $g(\sigma)=1$, the jump $\{\mathbf{F}_\tau\}_{12}$ is compensated by a jump in the tangential component of the force caused by the average magnetic field $\mathbf{B}^0$. In that case

$$\{\mathbf{B}_\tau^0\}_{12}=\{\mathbf{H}_\tau^0\}_{12}+4\pi\{\mathbf{M}_\tau\}_{12}=4\pi\{\mathbf{M}_\tau\}_{12}.$$

In the general case, however, the jump $\{\mathbf{F}_\tau\}_{12}$ may not be compensated by jumps in other forces and this means the impossibility of equilibrium under the action of the HF field. As a simple example we consider isotropic media without an external magnetic field (in that case the average field $B^0\propto|\mathbf{E}|^2$). We can then write (4.18) in the form

$$\{\mathbf{F}_\tau\}_{12}=\frac{1}{16\pi}\operatorname{Re}\{\mathbf{D}_\tau E_n^*-D_n\mathbf{E}_\tau^*\}_{12}=\frac{1}{16\pi i}\{\varepsilon''(E_n\mathbf{E}_\tau^*-E_n^*\mathbf{E}_\tau)\}_{12}$$

$$+\frac{1}{32\pi i}\left\{\frac{\partial\varepsilon'}{\partial\omega}\frac{\partial}{\partial t}(E_n\mathbf{E}_\tau^*-E_n^*\mathbf{E}_\tau)\right\}_{12}. \tag{4.20}$$

From this we see that if the wave is elliptically polarized the jump $\{\mathbf{F}_\tau\}_{12}$ is, in general, non-vanishing and can be compensated only when the fluid moves (when there appear off-diagonal terms in the stress tensor due to viscosity or due to a dynamical momentum transfer) or due to the generation of a magnetic field.

This result is of interest, for instance, for the heating of a plasma by an HF field as it shows that even if $\partial\mathbf{E}/\partial t=0$ the plasma may start to move when it is heated by a wave with elliptical (or circular) polarization because $\hat{\varepsilon}''\neq 0$.

5. THE PONDEROMOTIVE FORCE IN A MEDIUM WITH SPATIAL DISPERSION

In this section we assume everywhere that the field of the wave has the form (2.13) and we restrict ourselves merely to terms containing the first derivatives of $\mathscr{E}_\beta$ with respect to $\mathbf{r}$ and t (we also neglect their products).

To obtain the corresponding corrections to the induction, generalizing Eq. (3.1) to the case of spatial dispersion we change in (2.13) to a spatial variable $\mathbf{r}'$ defined in the local comoving frame of reference in the vicinity of the point $\mathbf{r}_0$. Let at time t_0 the comoving and the laboratory frames be the same in the vicinity of $\mathbf{r}_0$. We then have at time t for small $\Delta t=t-t_0$ and $\Delta\mathbf{r}'=\mathbf{r}'-\mathbf{r}_0$

$$\mathbf{r}=\mathbf{r}_0+\mathbf{v}(\mathbf{r}')\Delta t+\Delta\mathbf{r}'+{}^1/{}_2[\operatorname{rot}\mathbf{v}(\mathbf{r}_0)\times\Delta\mathbf{r}']\Delta t. \tag{5.1}$$

Substituting (5.1) into (2.13) we get

$$\mathbf{E}(\mathbf{r}',t)=\mathbf{A}(\mathbf{r}',t)\exp(i\mathbf{k}\mathbf{r}'), \tag{5.2}$$

$$\mathbf{A}=\vec{\mathscr{E}}(\mathbf{r}',t)\exp\left\{i\mathbf{k}\mathbf{v}(\mathbf{r}')\Delta t+\frac{i}{2}([\mathbf{k}\times\operatorname{rot}\mathbf{v}(\mathbf{r}_0)]\Delta\mathbf{r}')\Delta t\right\}, \tag{5.3}$$

where we have used the fact that $\mathbf{k}[\operatorname{curl}\mathbf{v}\times\Delta\mathbf{r}']=[\mathbf{k}\times\operatorname{curl}\mathbf{v}]\Delta\mathbf{r}'$. It follows from (5.2) and (5.3) that when we use the variables $\mathbf{r}'$, t we must replace the amplitude $\vec{\mathscr{E}}(\mathbf{r}',t)$ by the "effective" amplitude $\mathbf{A}(\mathbf{r}',t)$ which differs from $\mathscr{E}(r',t)$ by the additional phase factor occurring in (5.3).

We now verify easily that taking spatial dispersion into account in the approximation considered leads to the following correction to the amplitude of the induction $\mathbf{D}$, which supplements (3.11):

$$e^{-i\mathbf{k}\mathbf{r}'}\delta_5 D_\alpha=-i\frac{\partial\varepsilon_{\alpha\beta}}{\partial k_\gamma}\frac{\partial A_\beta}{\partial r_\gamma'}+\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial\omega\,\partial k_\gamma}\frac{\partial^2 A_\beta}{\partial t\,\partial r_\gamma'}$$

$$-\frac{i}{2}\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial k_\gamma\,\partial r_\gamma'}A_\beta+\frac{1}{2}\frac{\partial^3\varepsilon_{\alpha\beta}}{\partial\omega\,\partial k_\gamma\,\partial r_\gamma'}\left(\frac{\partial A_\beta}{\partial t}\right)_0$$

$$-\frac{i}{2}\left(\frac{\partial\varepsilon_{\alpha\beta}}{\partial k_\gamma}\frac{\partial}{\partial r_\gamma'}[\mathbf{v}\times\mathbf{B}]_\beta+\frac{1}{2}\frac{\partial^2\varepsilon_{\alpha\beta}}{\partial k_\gamma\,\partial r_\gamma'}[\mathbf{v}\times\mathbf{B}]_\beta\right). \tag{5.4}$$

Where the origin of all terms is clear from Eqs. (2.4), (2.14), and (3.1), taking into account that the operator $\hat{\varepsilon}$ acts upon functions expressed in terms of $\mathbf{r}'$ and t. In (5.4) and in what follows we drop terms containing $\partial\hat{\varepsilon}/\partial t$ as they lead to terms of higher order of smallness (v^2, $\partial^2\mathscr{E}/\partial r^2$, and so on).

When $v\equiv 0$, $\partial\hat{\varepsilon}/\partial t=0$, it follows from (5.4) that

$$\frac{1}{4\pi}\left\{\tilde{\mathbf{E}}\frac{\partial(\delta_5\tilde{\mathbf{D}})}{\partial t}\right\}=\nabla_\gamma(\delta S_\gamma)+\frac{\partial\delta w}{\partial t}, \tag{5.5}$$

$$\partial S_\gamma=\frac{1}{16\pi}\left\{-\omega\mathscr{E}_\alpha^*\frac{\partial\varepsilon_{\alpha\beta}'}{\partial k_\gamma}\mathscr{E}_\beta+\operatorname{Im}\left[\mathscr{E}_\alpha^*\frac{\partial^2(\omega\varepsilon_{\alpha\beta}')}{\partial\omega\,\partial k_\gamma}\frac{\partial\mathscr{E}_\beta}{\partial t}\right.\right.$$

$$\left.\left.-\omega\mathscr{E}_\alpha^*\frac{\partial^2\varepsilon_{\alpha\beta}'}{\partial k_\gamma\,\partial k_\zeta}\frac{\partial\mathscr{E}_\beta}{\partial r_\zeta}\right]\right\}, \tag{5.6}$$

$$\delta w=\frac{1}{16\pi}\operatorname{Im}\left(\mathscr{E}_\alpha^*\frac{\partial^2(\omega\varepsilon_{\alpha\beta}')}{\partial\omega\,\partial k_\gamma}\nabla_\gamma\mathscr{E}_\beta\right), \tag{5.7}$$

where δw and δS_γ are corrections to the energy density and its flux caused by the spatial dispersion in a non-moving medium.

When $v\neq 0$ we get instead of (5.5) after some calculations

$$\frac{1}{4\pi}\left\{\tilde{\mathbf{E}}\frac{\partial\delta_5\tilde{\mathbf{D}}}{\partial t}+\delta_5\tilde{\mathbf{H}}\frac{\partial\tilde{\mathbf{B}}}{\partial t}\right\}$$

$$=\nabla_\gamma\delta S_\gamma+\frac{\partial\delta w}{\partial t}-\frac{k_\alpha v_\alpha}{16\pi}\nabla_\beta\left(\mathscr{E}_\mu^*\frac{\partial\varepsilon_{\mu\lambda}'}{\partial k_\beta}\mathscr{E}_\lambda\right), \tag{5.8}$$

where $\delta_5\mathbf{H}=c^{-1}[\mathbf{v}(\delta_5\mathbf{D})]$ [compare (3.14)] while $\delta\mathbf{S}$ and δw retain the same meaning as above but now have the following form [compare (5.6), (5.7)]:

$$\delta\mathbf{S}\approx\frac{1}{16\pi}\left\{-\omega A_\alpha'^*\frac{\partial\varepsilon_{\alpha\beta}'}{\partial\mathbf{k}}A_\beta'\right.$$

$$\left.+\operatorname{Im}\left[A_\alpha'^*\frac{\partial^2(\omega\varepsilon_{\alpha\beta}')}{\partial\omega\,\partial\mathbf{k}}\left(\frac{\partial A_\beta'}{\partial t}\right)_0\right]\right\}, \tag{5.9}$$

$$\delta w=\frac{1}{16\pi}\,\mathrm{Im}\left(A_\alpha'^{*}\frac{\partial^2(\omega\varepsilon_{\alpha\beta}')}{\partial\omega\,\partial k_\gamma}\frac{\partial A_\beta'}{\partial r_\gamma'}\right). \tag{5.10}$$

Here $\mathbf{A}'$ is the amplitude of the electrical field in the comoving frame of reference which is connected with $\mathbf{A}$ through a relation similar to (3.16).

Now adding (5.8) to the right-hand side of (3.19) and using the general formula (4.2) we get in the expression for the ponderomotive force the following additional term due to the spatial dispersion:

$$f_\alpha^{(s)}=-\frac{1}{16\pi}k_\alpha\nabla_\beta\left\{\mathscr{E}_\mu^{*}\frac{\partial\varepsilon_{\mu\nu}'(\omega,\mathbf{k})}{\partial k_\beta}\mathscr{E}_\nu\right\}. \tag{5.11}$$

Expression (5.11) has the form of a divergence, but it does not lead to a correction to the stress tensor. This is connected with the fact that when spatial dispersion is taken into account Eq. (4.13) is changed and takes the form

$$E_\beta^{*}(\nabla_\alpha\varepsilon_{\beta\gamma}')E_\gamma=\nabla_\alpha(E_\beta^{*}\varepsilon_{\beta\gamma}'E_\gamma)-2\,\mathrm{Re}\left\{(\nabla_\alpha E_\beta^{*})\left[D_\beta-i\frac{\partial\varepsilon_{\beta\gamma}}{\partial\omega}\frac{\partial E_\gamma}{\partial t}-i\varepsilon_{\beta\gamma}''E_\gamma\right]\right\}-k_\alpha\nabla_\beta\left(E_\gamma^{*}\frac{\partial\varepsilon_{\gamma\delta}'}{\partial k_\beta}E_\delta\right), \tag{5.12}$$

where

$$D_\beta=\left(\varepsilon_{\beta\gamma}'\mathscr{E}_\gamma+i\varepsilon_{\beta\gamma}''\mathscr{E}_\gamma+i\frac{\partial\varepsilon_{\beta\gamma}'}{\partial\omega}\frac{\partial\mathscr{E}_\gamma}{\partial t}-i\frac{\partial\varepsilon_{\beta\gamma}'}{\partial k_\delta}\nabla_\delta\mathscr{E}_\gamma-\frac{i}{2}\frac{\partial^2\varepsilon_{\beta\gamma}'}{\partial k_\delta\,\partial r_\delta}\mathscr{E}_\gamma\right)e^{i\mathbf{kr}}.$$

It then follows from Eq. (4.9), supplemented by the term (5.11) and taking (5.12) and (4.14) into account, that the expression for the stress tensor retains the form (4.17) also when spatial dispersion is taken into account.

The contribution of the spatial dispersion to the ponderomotive force was recently considered in Ref. 8 for an isotropic collisionless plasma. Using a microscopic approach the authors arrived at a term which differed from (5.11) in its sign.

Finally we consider the problem of corrections to the ponderomotive force and the magnetic moment arising due to non-adiabatic corrections to $\hat{\varepsilon}$. Together with terms containing $\partial A_i/\partial t$ in (2.10) we must also take into account the possibility of the appearance of additional terms in (2.17) which contain gradients of the velocity. An example of the latter is a term of the form

$$\hat{X}_v\,\mathrm{div}\,\mathbf{v}. \tag{5.13}$$

Expression (5.13) may be caused by adiabatic effects in convective spatial dispersion which is not taken into account when we obtained the term (3.7). In contrast to other additional terms in (2.17) the term (5.13) may be non-vanishing also when one can neglect spatial dispersion in a medium at rest.

In a number of cases, for instance, for a collisionless plasma the Hermitean part of the additional terms considered completely enters into the expression for w and the anti-Hermitean part into q. The additional terms in $\hat{\varepsilon}$ in that case do not lead to a change in the expressions found above for the ponderomotive force $\mathbf{f}$ and the magnetic moment $\mathbf{M}$.[4)] If, however, the corrections to $\hat{\varepsilon}$ discussed here do not or only partly enter into w and q we must add to $\mathbf{f}$ and $\mathbf{M}$ additional terms which one can easily estimate using the conservation laws (2.19) and (4.1). As to order of magnitude these terms are equal to

$$\delta M_\gamma\lesssim\frac{1}{16\pi}\left\{\frac{\partial}{\partial t}\left\langle\mathbf{E}^{*}\frac{\partial}{\partial\omega}(\omega\hat{\Psi}_{B_\gamma^0})\mathbf{E}\right\rangle+2\omega\langle\mathbf{E}^{*}\hat{\Psi}_{B_\gamma^0}''\mathbf{E}\rangle\right\}, \tag{5.14}$$

$$\delta f_\gamma\lesssim\frac{\nabla_\gamma}{16\pi}\left\{\frac{\partial}{\partial t}\left\langle\mathbf{E}^{*}\frac{\partial}{\partial\omega}[\omega(\rho\hat{\Psi}_\rho'-\hat{X}_v')]\mathbf{E}\right\rangle+2\omega\langle\mathbf{E}^{*}(\rho\hat{\Psi}_\rho''-\hat{X}_v'')\mathbf{E}\rangle\right\}, \tag{5.15}$$

where $\hat{\Psi}_\Lambda'$, $\hat{X}_v'$ and $i\hat{\Psi}_\Lambda''$, $i\hat{X}_v''$ are the Hermitean and anti-Hermitean parts of the operators $\hat{\Psi}_\Lambda$ and $\hat{X}_v$.

The first terms in (5.14) and (5.15) lead in the expression of the ponderomotive force (4.9) to additional terms containing mixed derivatives with respect to $\mathbf{r}$ and t. Earlier we verified that such terms are smaller by a factor $\omega_* T$ than the main terms in the ponderomotive force. As $\omega_* T\gg 1$ these terms are small, provided $\omega\Lambda\hat{\Psi}_\Lambda'$ and $\omega\hat{X}_v'$ are not too large.

The contribution from the second terms in (5.14) and (5.15) to the ponderomotive force are comparable with $\mathbf{f}^{(d)}$ from (4.5). As a result we find that we can neglect those corrections provided

$$|\omega(\rho\hat{\Psi}_\rho''-\hat{X}_v'')|\ll kL\hat{\varepsilon}'',\quad \omega B^0\hat{\Psi}_{B^0}''\ll kL\hat{\varepsilon}''. \tag{5.16}$$

These conditions are certainly satisfied for sufficiently large values of kL.

One obtains easily similar criteria also for the magnetic moment, comparing (5.14) with (2.22).

We emphasize that these conditions are sufficient but not necessary as $\delta\mathbf{M}$ and $\delta\mathbf{f}$ may turn out to be appreciably smaller than the right-hand sides of (5.14) and (5.15); in particular, as mentioned above, $\delta\mathbf{M}$ and $\delta\mathbf{f}$ may vanish completely.

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1) If the medium is transparent in the stationary state, one obtains the anti-Hermitean term $(i/2)\partial^2\hat{\varepsilon}/\partial\omega\partial t$ from the condition that the adiabatic invariant be conserved, as was shown first of all by Pitaevskiĭ.[11]

2) The term with $\frac{1}{2}i(\partial^2\varepsilon_{\alpha\beta}/\partial k_\gamma\partial r_\gamma)$ in (2.14) corresponds to the adiabatic correction to $\mathrm{Im}\,\varepsilon_{\alpha\beta}$ due to the inhomogeneity of the medium.[18–20]

3) In this case the quantity $\varepsilon_{\alpha\beta}$ in (3.9) may also contain terms with $\nabla_\alpha v_\beta$ of a dynamic origin (see §102 of Ref. 13).

4) As an example we can also cite terms with derivatives with respect to the velocity in the operator $\hat{\varepsilon}$ which are dynamic of origin; they describe the Maxwell effect and Coriolis effects (§102 of Ref. 13) and do not change the expressions found above for $\mathbf{f}$ and $\mathbf{M}$.

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