Connection between quantum chromodynamics and a free string on a lattice

V. A. Kazakov

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR
Submitted 26 April 1983
Zh. Eksp. Teor. Fiz. **85**, 1887–1898 (December 1983)

The strong-coupling expansion in the Wilson $U(\infty)$ lattice gauge theory in a space of an arbitrary number of dimensions is formulated in terms of the sum over free lattice surfaces with local intrinsic structure. This proves the equivalence of multicolored quantum chromodynamics and a certain free-string theory. The $1/N$ expansion then proves to be equivalent to a topological expansion in terms of these lattice surfaces. A generalization of this procedure that is suitable for describing the weak-coupling phase in the $U(\infty)$ gauge theory is proposed.

PACS numbers: 11.15.Ha, 12.35.Cn

1. INTRODUCTION

The possible equivalence between multicolored quantum chromodynamics (QCD) and a certain free-string theory is a relatively old idea in the theory of gauge fields. Such an equivalance could play an important role in the development of the theory of strong interactions. The theory of the chromodynamic string not only provides a physical description[1] but may also reveal new ways of solving the QCD problem.

The existence of sums over surfaces in the $U(N)$ gauge theory was first pointed out in the well-known paper by Wilson.[2] However, his strong-coupling expansion was more like an expansion over surface-type clusters than a sum over free (noninteracting on crossings) surfaces.

The second step was made by 't Hooft in his paper on planar diagrams in QCD.[3] However, the topological classification of the diagrams is still not equivalent to the development of a theory of a free chromodynamic string.

The first steps in this direction were made in Ref. 4, where an attempt was undertaken to establish the correspondence between multicolored QCD and the Nambu-Goto string. These papers threw some light on the structure of the possible correspondence, but the mathematical methods used were not sufficiently reliable.

An interesting attempt to sum the planar diagrams in the form of a string model was undertaken by Migdal,[5] but requires additional and more direct verification before this result can be used for practical calculations. An important point in this paper was the introduction of internal degrees of freedom on the string that effectively models the planar Feynman diagrams.

The two-dimensional gauge theory has been found to be useful from the heuristic point of view. Some of the results obtained for this model[6] indicate the existence of a string in the limit of infinite N.

From our point of view, the principal impediment along this promising path is the absence of simple and reliable results. In the present paper, we shall try to formulate this type of result for the strong-coupling expansion in $U(N)$ lattice gauge theory in a space of arbitrary number of dimensions.

We shall consider the strong-coupling expansion for the free energy, defined as

$$\exp\{N^2F(\beta)\} = \int\prod_{r\,\mu} d(U_{r\mu})\exp\left\{\beta N\sum_{r,\mu,\nu}\operatorname{tr}(U_{r,\mu}U_{r+\mu,\nu}U^{+}_{r+\nu,\mu}U^{+}_{r,\nu}+\text{h. c.})\right\} \quad (1)$$

where $U_{r,\mu}$ is the $U(N)$ group matrix corresponding to a link in d-dimensional lattice that begins at the site r and has the direction μ, and $d(U_{r,\mu})$ is the $U(N)$ invariant Haar measure.

We shall show (despite the doubts expressed in Ref. 7) that the strong-coupling expansion for $F(\beta)$ can be written in the form

$$F(\beta)=\sum_{\varkappa=0}^{\infty}\frac{1}{N^{2\varkappa}}\sum_{\sigma_\varkappa}\beta^{\text{Area}(\sigma_\varkappa)}\prod_{\xi\in\sigma_\varkappa}f_\xi, \quad (2)$$

where $\sigma_\varkappa$ is an oriented surface with $\varkappa$ handles, constructed from lattice plaquettes in accordance with the rules described in the next section, Area $(\sigma_\varkappa)$ is the number of plaquettes on the surface $\sigma_\varkappa$, and f_ξ are local integer-valued factors, each of which refers to a particular point ξ on the given surface, and depends only on the internal properties of the surface at this point.

An analogous expression is also available for the Wilson average:

$$W(C)=\sum_{\varkappa=0}\frac{1}{N^{2\varkappa}}\sum_{\sigma_\varkappa}\beta^{\text{Area}(\sigma_\varkappa)}\prod_{\xi\in\sigma_\varkappa}f_\xi, \quad \partial\sigma_\varkappa=C. \quad (3)$$

In the limit as $N\to\infty$, only the planar surfaces survive in (2) and (3), and we are left with the string picture of multicolored QCD.

The procedure used to derive these formulas is, in many ways, analogous to that used in the Weingarten lattice model for the free string,[8] but the rules for constructing the surfaces ("differential lattice geometry") are somewhat different.

The most important feature of our lattice string representation is that it can be generalized to the weak-coupling phase of $U(\infty)$ theory, which is separated from the strong-coupling phase by the Gross-Witten phase transition. This generalization is achieved by isolating the N gauge-invariant variables responsible for the transition.

In Sec. 2 we shall obtain the rules for the summation over lattice surfaces and will describe the internal structure of the surfaces.

In Sec. 3 we shall formulate the lattice string represen-

 0038-5646/83/121096-07$04.00

tation for the free energy. We shall put forward arguments suggesting that the local limit of our chromodynamic lattice string in the strong-coupling phase is described by the Nambu-Goto-Polyakov string.[9]

Sec. 4 is devoted to the generalization of our string construction to the weak-coupling phase of $U(\infty)$ gauge theory.

2. STRONG-COUPLING EXPANSION. $U(\infty)$ INTEGRALS AND CONSTRUCTION OF SURFACES FROM PLAQUETTES

In QCD, the representation in the form of a sum over free surfaces can be obtained from the strong-coupling expansion, just as in the case of the Weingarten lattice string.[8] The basic idea of this construction is the correspondence between the integral over the group variable on a link and the joining together of surfaces consisting of plaquettes along this link into pieces of the surface. In this section, we shall examine in detail this correspondence, and this will lead us to the rules for constructing the chromodynamic lattice surface.

The strong-coupling expansion is the expansion in terms of β of the right-hand side of Eq. (1). An arbitrary term in the expansion has the form

$$\beta^{\Sigma}\int\prod_{r,\alpha}d(U_{r,\alpha})\prod_{r,\mu,\nu}(\operatorname{tr}U_{r,\mu}U_{r+\mu,\nu}U^{+}_{r+\nu,\mu}U^{+}_{r,\nu})^{n_{r,\mu,\nu}},\qquad \Sigma=\sum_{r,\mu,\nu}n_{r,\mu,\nu},\quad n_{r,\mu,\nu}=0,1,2,\ldots \tag{4}$$

We must now integrate in (4) with respect to all the variables on the links $U_{r,\alpha}$. Each such integral is evaluated for several U-matrices contained in the plaquettes, including the given link. It is clear that the number of U-matrices in each integral must be equal to the number of U^{+}-matrices in order to ensure that the result of integration is nonzero.

The plaquettes combine into different clusters in the course of integration. This picture is well removed from the representation in the form of a sum over free surfaces, because the resulting surfaces become branched, i.e., the "observed" self-crossings.

In the case of the Weingarten lattice string, it can be shown that any plaquette configuration near a given link can be represented by freely crossing pieces of surface, each of which consists of two plaquettes joined along a common link. This is possible because of the Gaussian nature of the measure for the U-matrices in the Weingarten model.

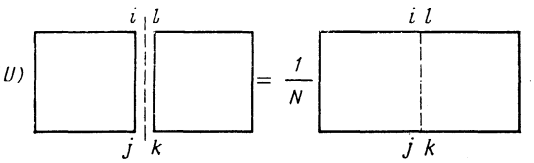
In our case, the measure is not Gaussian, but we shall show that an analogous procedure can be used to join together surfaces consisting of plaquettes, but with a certain complication of the differential lattice geometry of these surfaces. In other words, the recipe for joining together the plaquettes will be somewhat different from that in the Weingarten model.

We begin with some simple examples and then describe the general picture.

The simplest configuration consists of two plaquettes with a common link. The corresponding integral is

$$d(U)U_j^{\ i}U_i^{+k}=\frac{1}{N}\delta_i^{\ i}\delta_j^{\ k}, \tag{5}$$

in complete analogy with the Weingarten model. The joining together of the plaquettes can be represented schematically by the diagram



(6)

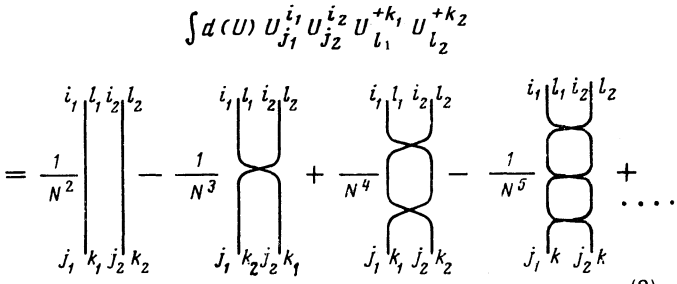
The plaquettes are joined together in accordance with the structure of the tensor $\delta_i^{\ i}\delta_j^{\ k}$.

Another, less trivial, example involves four plaquettes with a common link (see the left-hand side of the equation in Fig. 1). The corresponding $U(N)$ integral is

$$\begin{aligned}\int d(U)U_{j_1}^{\ i_1}U_{j_2}^{\ i_2}U_{l_1}^{+k_1}U_{l_2}^{+k_2}&=\frac{1}{N^2-1}\delta_{l_1}^{\ i_1}\delta_{j_1}^{\ k_1}\delta_{l_2}^{\ i_2}\delta_{j_2}^{\ k_2}\\&-\frac{1}{N(N^2-1)}\delta_{l_1}^{\ i_1}\delta_{j_2}^{\ k_1}\delta_{l_2}^{\ i_2}\delta_{j_1}^{\ k_2}+(i_1,j_1\leftrightarrow i_2,j_2)\\&\approx\frac{1}{N^2}\delta_{l_1}^{\ i_1}\delta_{j_1}^{\ k_1}\delta_{l_2}^{\ i_2}\delta_{j_2}^{\ k_2}\\&-\frac{1}{N^3}\delta_{l_1}^{\ i_1}\delta_{j_2}^{\ k_1}\delta_{l_2}^{\ i_2}\delta_{j_1}^{\ k_2}+\frac{1}{N^4}\delta_{l_1}^{\ i_1}\delta_{j_1}^{\ k_1}\delta_{l_2}^{\ i_2}\delta_{j_2}^{\ k_2}\\&-\frac{1}{N^5}\delta_{l_1}^{\ i_1}\delta_{j_2}^{\ k_1}\delta_{l_2}^{\ i_2}\delta_{j_1}^{\ k_2}+\ldots+(i_1,j_1\leftrightarrow i_2j_2)\end{aligned} \tag{7}$$

where we have expanded the coefficients in front of the two different invariant tensors in terms of $1/N$. We note that the structure of the first tensor is the same as in the Weingarten model and corresponds to Wick's pairing for pairs of U-matrices. The second tensor is quite new.

It is useful to represent (7) with the aid of the following diagrams



(8)

The crossing lines in the graphs show, as we shall see below, the "history" of the permutations of the indices $(k_1\leftrightarrow k_2)$.

We note that the tensor structure of the first and second terms on the right-hand side of (8) is equivalent to the structure of the third and fourth terms, respectively. However, the interpretation in terms of the joining of plaquettes will be different, and is given in Fig. 1.

The joining of plaquettes in the first terms on the right-hand side of Fig. 1 is analogous to the situation in the Weingarten model.

The second term represents a different tensor structure, typical of $U(N)$ gauge theory. The most natural surface that embodies this structure is obtained by joining plaquettes into a saddle-type configuration.

The third term has the same tensor structure as the first,

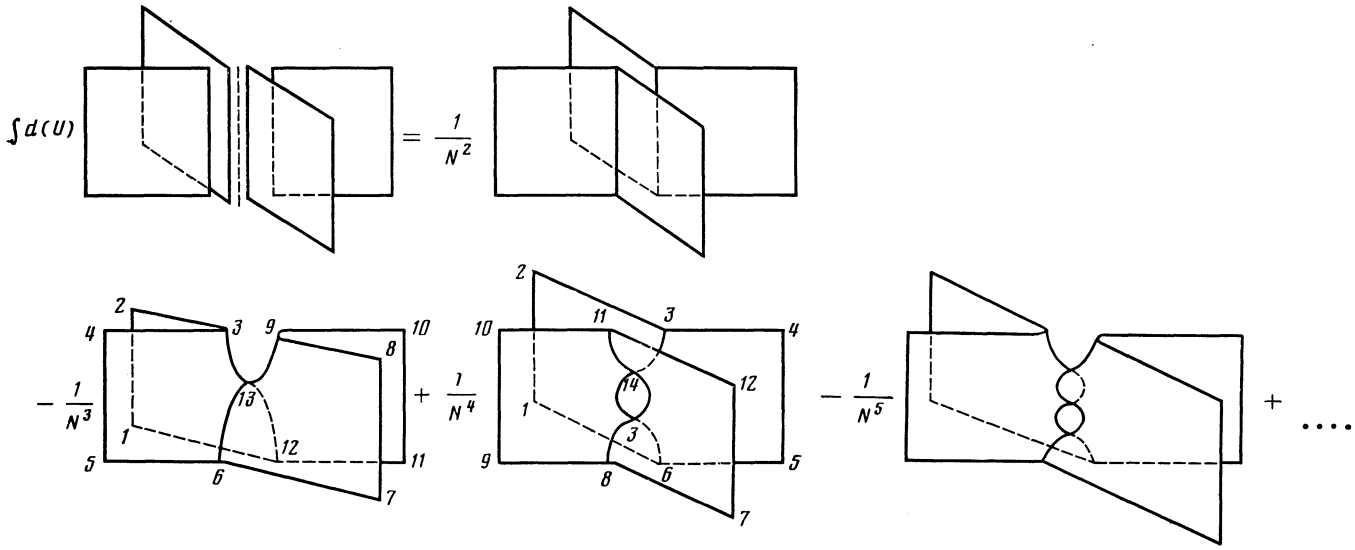


FIG. 1.

but a different order in $1/N$. We shall represent it by two pieces of the surface, each of which is made by joining two plaquettes into a single surface with the aid of a tube.

The fourth term on the right-hand side of Fig. 1 is a nonplanar surface.

It is, of course, immaterial where, on a given link, the saddle points are located. To be specific, we shall suppose that all these new vertices on the surface (we shall call them vertices on links, in contrast to vertices on surfaces, which are situated at the sites of the d-dimensional lattice) are situated in an infinitesimal neighborhood of the midpoint of the link. Of course, in our figures, all the resulting saddles and tubes have finite dimensions.

We must now describe the general case of an arbitrary number of plaquettes near a given link. The geometry of the corresponding surfaces will arise in close analogy with the four-plaquette case examined above. We shall also be able to evaluate the coefficients in (8) and its generalizations, and the numbers f_ξ in (2) and (3). As in the case of the Weingarten model, the Euler theorem will enable us to represent the $1/N$ expansion in the form of a topological expansion in terms of our surfaces, and the leading term in $1/N$ will then correspond to the sum over planar surfaces.

Let us therefore consider the $U(N)$ group integral in the following form:

$$J_{\{jl\}}^{\{ik\}} = \int dU^+ dU \delta(U^+U-I)\, U_{j_1}^{\ i_1} U_{l_1}^{\ +k_1} \cdot \ldots \cdot U_{j_n}^{\ i_n} U_{l\ n}^{\ +k_n}$$
$$= \int d\alpha e^{N \operatorname{tr} \alpha} \int dU^+ dU e^{-N \operatorname{tr} \alpha U^+ U} U_{j_1}^{\ i_1} U_{l_1}^{\ +k_1} \cdot \ldots \cdot U_{j_n}^{\ i_n} U_{l\ n}^{\ +k_n} \tag{9}$$

where the $U(N)$ group measure was determined by using a δ-function link and then exponentiating with the aid of the anti-Hermitian matrix field α.

We can now perform the Gaussian U-integration. In accordance with Wick's theorem, we have

$$J_{\{jl\}}^{\{ik\}} = -\frac{1}{N^n} \int d\alpha e^{N \operatorname{tr}(\alpha - \ln \alpha)} \Big\{ [\delta_{l_1}^{\ i_1} (\alpha^{-1})_{j_1}^{\ k_1}] \cdot \ldots \cdot [\delta_{l_n}^{\ i_n} (\alpha^{-1})_{j_n}^{\ k_n}] + \text{other Wick pairings} \Big\}. \tag{10}$$

The integral with respect to α can readily be evaluated by the Schrödinger-Dyson method which, in this case, yields

$$\int d\alpha \frac{d}{d\alpha_{ij}} (e^{N \operatorname{tr}(\alpha - \ln \alpha)} F(\alpha)) = \int d\alpha \Big\{ \Big[N\delta_{ij} - N(\alpha^{-1})_{ij} + \frac{d}{d\alpha_{ij}} \Big] F(\alpha) \Big\} \cdot e^{N \operatorname{tr}(\alpha - \ln \alpha)} = 0, \tag{11}$$

where $F(\alpha)$ is an arbitrary function of the matrix α (with suitable properties at infinity).

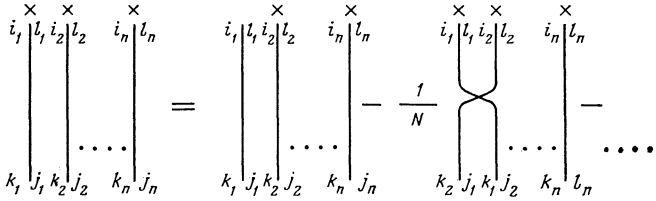
Let us apply (11) to the first term in (10) (all the other terms are similar). If we take

$$F(\alpha) = [\delta_{l_1}^{\ i_1} \delta_{j_1}^{\ k_1}]\, [\delta_{l_2}^{\ i_2} (\alpha^{-1})_{j_2}^{\ k_2}] \cdot \ldots \cdot [\delta_{l_n}^{\ i_n} (\alpha^{-1})_{j_n}^{\ k_n}] \tag{12}$$

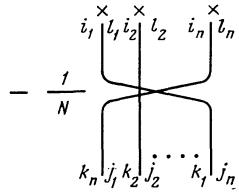
we obtain

$$\langle [\delta_{l_1}^{\ i_1} (\alpha^{-1})_{j_1}^{\ k_1}) \cdot \ldots \cdot [\delta_{l_n}^{\ i_n} (\alpha^{-1})_{j_n}^{\ k_n}) \rangle_\alpha$$
$$= \langle [\delta_{l_1}^{\ i_1} \delta_{j_1}^{\ k_1}]\, [\delta_{l_2}^{\ i_2} (\alpha^{-1})_{j_2}^{\ k_2}] \cdot \ldots \cdot [\delta_{l_n}^{\ i_n} (\alpha^{-1})_{j_n}^{\ k_n}] \rangle_\alpha \tag{13}$$
$$- \frac{1}{N} \langle [\delta_{l_1}^{\ i_1} (\alpha^{-1})_{j_1}^{\ k_2}]\, [\delta_{l_2}^{\ i_2} (\alpha^{-1})_{j_2}^{\ k_1}]\, [\delta_{l_3}^{\ i_3} (\alpha^{-1})_{j_3}^{\ k_3}] \cdot \ldots \rangle_\alpha - \ldots$$

This equation can be represented graphically by



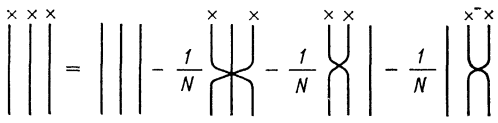
(14)



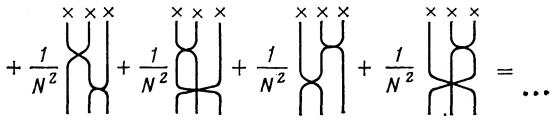
The crosses in this expression represent the α^{-1}-matrices.

The identity given by (14) tells us that we can remove one cross from any diagram (replacing α^{-1} by δ) and sub-

tract from the resulting diagram all the diagrams that are obtained by interchanging this cross with any other (with the factor $1/N$), which signifies the interchange of the corresponding indices. Repeated application of this procedure gives the $1/N$-expansion of the coefficients in the group integrals, such as (8). For example,



(15)



We note that some of these terms may have an equivalent tensor structure: for example, in the last row in (15),the first term is equal to the fourth, and the second to the third.

The type of plaquette junction, and the resulting pieces of the surface, follow from the structure of the corresponding terms in (14) or (15): as in the case of (8) and Fig. 1, such diagrams give the "skeleton" (lines of plaquette junction) for the corresponding representation of the surfaces in the figures. Each term is a set of handles separated by tubes.

We shall now interpret the appearance of the factor $1/N$ in (8), (14), and (15) with a view to the further application of Euler's theorem to the development of the topological expansion. We note that the factor $1/N$ appears after each interchange of two line ends. However, each interchange leads to the formation on the resulting surface of two additional edges and one new vertex. We can therefore interpret this factor in the same way as in the Weingarten model, in which each surface is accompanied by the factor

$$N^{F+V-L}, \tag{16}$$

where F is the number of faces (plaquettes) on the surface, V is the number of vertices, and L is the number of edges.

In our case, the result of substitution in (14) is

$$N^{\Delta V-\Delta L}=N\cdot N^{-2}=1/N. \tag{17}$$

Thus, the expression in (16) is also valid in our case.

We now proceed to our last remark before we finally formulate the string representation for the lattice QCD.

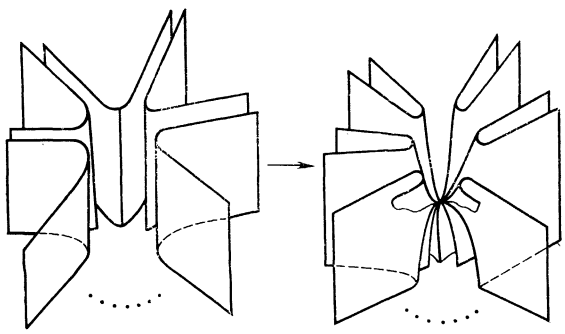
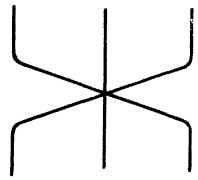


FIG. 2.

Let us compare the second and third terms in the last row of (15). As already noted, they have the same tensor structure, but the corresponding surfaces are somewhat different: they contain two simple saddles (with two "valleys") that follow in different order along the given lattice link. However, we have already agreed to place all saddles at the center of the link, so that there will be no difference between the corresponding surfaces. The two terms can then be represented by a single term and the same skeleton graph:



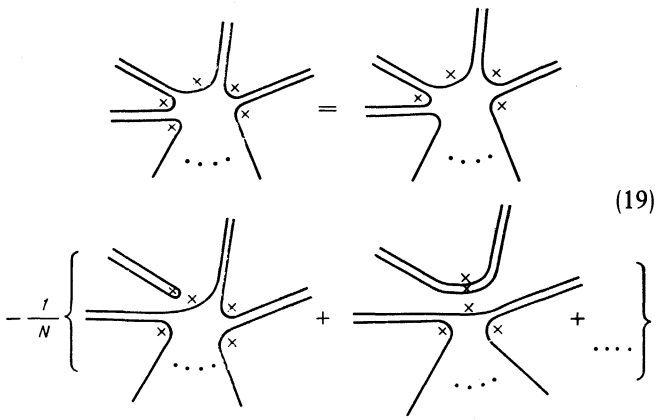
(18)

The corresponding surfaces will then take the form of a three-valley saddle (sometimes referred to as the "monkey saddle"). The arbitrary case of an n-valley saddle is shown in Fig. 2. Thus, three-valley saddles appear in the sum over the surfaces with a factor of 2.

Any diagram consisting of any number of plaquette pairs containing a given link can be reduced in an analogous manner. The corresponding surface can be looked upon as consisting of different n-valley saddles separated by tubes. Each n-valley saddle is accompanied by an integer factor f_n that depends only on n.

We shall now evaluate the f_n. It is clear from the last row in (15) that different terms with equivalent tensor structure differ only by the order of permutation of the crosses. Thus, we see that $|f_n|$ is equal to the number of all the possible ways of forming the n-valley saddle by a successive interchanging of the crosses. Such permutations must not form closed cycles of the form $(1\leftrightarrow 2)$, $(2\leftrightarrow 1)$ or $(1\leftrightarrow 2)$, $(2\leftrightarrow 3)$, $(3\leftrightarrow 1)$ because such cycles result in the formation of tubes separating saddles. For example, the third term on the right-hand side of (8) contains two two-valley saddles separated by a tube (see Fig. 1), so that they cannot join to form a three-valley saddle.

To evaluate f_n, consider the formation of an n-valley saddle from a configuration of $2n$ pairs of plaquettes, shown in Fig. 2. A graphical representation similar to (14) can then be represented by Fig. 2 as viewed from above:



(19)

If we continue such operations with the crosses, we shall obtain, among other things, all the permutation var-

iants that lead to the saddle of Fig. 2. It is clear that the first term on the right-hand side of (19) cannot produce a saddle of this kind, and that the remaining terms lead to the following recurrence relation for f_n:

$$f_n=-\sum_{k=1}^{n-1} f_{n-k}f_k \tag{20}$$

with the normalizing condition

$$f_0=f_1=1. \tag{21}$$

The factors $1/N$ were not taken into account in these expressions because we now know how to take them into account for each surface.

Consider the generating function

$$f(t)=\sum_{n=0}^{\infty} f_n t^n, \tag{22}$$

and rewrite (20) in the form

$$f(t)^2-f(t)-t=0. \tag{23}$$

As a result, we obtain

$$f(t)=\frac{1+(1+4t)^{1/2}}{2}, \tag{24}$$

$$f_n=\frac{(-1)^n}{-2(2n-1)}\frac{(2n)!}{(n!)^2}, \quad n=1,2,3,\ldots \tag{25}$$

the numbers

$$f_1=1, \quad f_2=-1, \quad f_3=2, \quad f_4=-5, \quad f_5=14 \tag{26}$$

have already appeared in Ref. 9; here they play the role of internal local factors on a surface. They are local because they appear independently at each vertex on a link on a given surface, and they are internal characteristics of the surface because n (the number of valleys in the saddle) can be determined by an internal observer on the surface. The number n is related to the internal angle φ of the surface at a given vertex on a link (or to the internal "curvature") by

$$\varphi=2\pi n. \tag{27}$$

We are now in a position to formulate our representation for the free energy in QCD in terms of the sum over noninteracting surfaces with internal local structure. This will be done in the next section.

3. FORMULATION OF THE REPRESENTATION OF THE SUM OVER FREE SURFACES AND THE POSSIBLE LOCAL LIMIT

In the last section we formulated the rules for constructing surfaces from the plaquettes of a d-dimensional Euclidean lattice. The plaquettes adjacent to a given link are joined together in accordance with the skeleton diagrams, such as (14) and (15), which form the $1/N$-expansion for the single-plaquette integrals. They produce singular saddles and tubes (cf. Figs. 1 and 2), and also simple junctions, indicated in (6). All these saddles and tubes lie at the midpoint of a link and form new vertices of the surface, to which we have referred as vertices on links (in addition to the site vertices that appear in the Weingarten model). Each group of n simple saddles (two-valley saddles) merges into a single n-valley saddle if this merging does not modify the topology of the surface (does not annihilate the tubes).

We shall now formulate the sum over the free surfaces for the $U(N)$ gauge theory in the case of free energy:

$$F(\beta)=\sum_{\varkappa=0} N^{-2\varkappa}\sum_{\sigma_\varkappa}\beta^{\mathrm{Area}(\sigma_\varkappa)}\prod_{\xi\in\sigma_\varkappa} f_{n_\xi}, \tag{28}$$

where $\sigma_\varkappa$ is the closed connected surface with $\varkappa$ handles (Euler number equal to $2-2\varkappa$) and ξ are the "coordinates" of the vertex on a link for the given surface (in the local limit, we obtain the pair of coordinates ξ_1, ξ_2 that parametrize the surface).

The sum over the connected surfaces for the free energy appears by close analogy with the Weingarten model: the strong-coupling expansion for the statistical sum is given in terms of all (and not merely connected) surfaces, and, by taking the logarithm of the statistical sum, we retain only connected surfaces in the sum.

In the limit of large N, only the planar surfaces $\sigma(\varkappa=0)$ survive in (28), and we obtain

$$F(\beta)\underset{N\to\infty}{=}\sum_{\sigma}\beta^{\mathrm{Area}(\sigma)}\prod_{\xi\in\sigma} f_{n_\xi}. \tag{29}$$

Analogous formulas can be written down for the Wilson average [see (3)]. Care must be taken when the factors f_n are determined near the boundary C, where the local angles $\varphi=2\pi n$ are not well defined.

We are now in a position to put forward some ideas on the possible local limit for the chromodynamic lattice string (29).

It would appear that the existence of a local internal structure on the surface, i.e., a structure that can be determined by a two-dimensional observer by measurements involving a few neighboring surface plaquettes, may lead to a local limit in the form of the Nambu-Goto-Polyakov string:[10]

$$F\sim\sum_{\sigma}\exp\left\{-\mathrm{const}\int_{\sigma} d^2\xi\sqrt{g}\right\}, \tag{30}$$

where g_{ab} is the metric tensor on the surface.

The principal argument for the proposition that (30) is the local limit for (29) is the existence of a unique local internal invariant of the lowest dimensionality—the surface area.

One could object to this suggestion by noticing that the individual terms in the sum given by (29) are not even positive-definite. However, a fixed point for (29) at which a local limit is possible may also correspond to complex values of the coupling constant $1/\beta$, and so this type of objection is not serious.

4. GENERALIZATION TO THE CASE OF WEAK COUPLING

It is well known that the strong-coupling expansion in the $U(\infty)$ lattice gauge theory cannot be continued into the physically interesting region of weak coupling because of the Gross-Witten phase transition. Our lattice string is therefore required in the modification that we shall examine in this section.

Let us begin with the simple example of a single-pla-

quette theory. Introducing the Lagrangian multiplier α and integrating with respect to U^+ and U, we obtain the following expression that is analogous to (10):

$$Z=\int dU^+dU\delta(I-U^+U)\exp\{\beta N\,\mathrm{tr}\,(U^++U)\}$$

$$=\int d\alpha\exp\{N\,\mathrm{tr}\,(\alpha-\ln\alpha)+\beta^2N\,\mathrm{tr}\,\alpha^{-1}\}. \qquad (31)$$

In the limit of infinite N, direct expansion in terms of β yields

$$Z=\int d\alpha e^{N\,\mathrm{tr}(\alpha-\ln\alpha)}\sum_{k=0}^{\infty}\frac{(\beta^2N^2)^k}{k!}\left(\frac{\mathrm{tr}\,\alpha^{-1}}{N}\right)^k$$

$$=\sum_{k=0}^{\infty}\frac{(\beta^2N^2)^k}{k!}=e^{N^2\beta^2}, \qquad (32)$$

where we have used the factorization property for large N and Eq. (21). We see that direct integration with respect to α of each term in the expansion in terms of β gives the correct result only for the phase of the strong coupling.

We now consider the integral with respect to α in (32) in terms of integration with respect to the angle variables and the eigenvalues:

$$\alpha_{ij}=\Omega_{ik}{}^+\lambda_k\Omega_{kj},\quad d\alpha_{ij}=d(\Omega)\left(\prod_i d\lambda_i\right)\left(\prod_{i\neq j}(\lambda_i-\lambda_j)\right). \qquad (33)$$

Integrating over the unitary matrix Ω in the first row in (32), after using the factorization property, and summing over k, we obtain

$$Z=\int\prod_i d\lambda_i\exp\left\{N\sum_i(\lambda_i-\ln\lambda_i+\beta^2\lambda_i^{-1})+2\sum_{i\neq j}\ln|\lambda_i-\lambda_j|\right\}. \qquad (34)$$

We are thus left with only N (and not N^2) variables of integration λ_i, and we can use the method of steepest descents to evaluate the integral. The corresponding expression for the saddle point is

$$1-\frac{1}{\lambda}+2\int\frac{d\lambda'\rho(\lambda')}{\lambda-\lambda'}=\beta^2\frac{1}{\lambda^2}. \qquad (35)$$

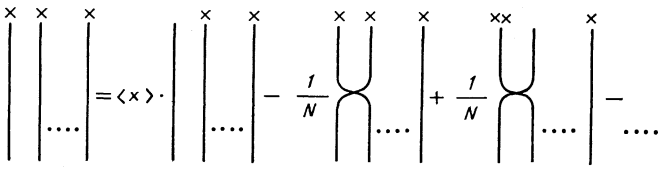
The spectral function $\rho(\lambda)$ describing the distribution of eigenvalues of the α-matrix can be found by standard methods (by investigating its analytic properties). The solution given by (35) exhibits a third-order transition for $\beta_c=1/2$ (the Gross-Witten phase transition). This enables us to calculate colorless quantities for any of the two phases.

We thus see that the operations of integration over eigenvalues and exponentiation of the β-expansion are noncommutative, and this is responsible for the phase transition.

We shall now apply this approach to the realistic case of the α-dimensional $U(\infty)$ lattice gauge theory. We therefore substitute each $\alpha_{r,\mu}$ in (10) in the form given by (33), and integrate with respect to the angle variable $\Omega_{r,\mu}$ for fixed $\lambda_{r,\mu}$. This can be done with the aid of the Schwinger-Dyson equation that follows from invariance under the left shift on the group

$$\delta\Omega_{ij}=\varepsilon_a\tau_{ik}{}^a\Omega^{kj}. \qquad (36)$$

Applying (36) to the first term on the right of (10), we obtain the following recurrence relation [cf. (14)]:



(37)

A cross in this expression represents the matrix $\Omega_{ik}{}^+\lambda_k{}^{-1}\Omega_{kj}$, two crosses represent the matrix $\Omega_{ik}{}^+\lambda_k{}^{-2}\Omega_{kj}$; and the symbol $\langle\times\rangle$ denotes $N^{-1}\Sigma_k\lambda_k{}^{-1}$. All the other designations are analogous to (14).

We see that (37) is very close to (14). The only exception is the last term written out in (37), and the fact that now $\langle\times\rangle\neq1$, and is a known function of the N variables λ_i (analogously, $\langle\times\times\rangle=N^{-1}\Sigma_k\lambda_k{}^{-2}$, and so on). Consequently, the structure of the free surfaces regains its principal features, but the factors on the surface are now definite functions of $\lambda_i^{r,\mu}$. The string will therefore experience the effects of the external field, which depends on the lattice coupling.

Two simplifications arise in the limit of infinite N. Firstly, only the planar surfaces survive in the corresponding sum over the surfaces. Secondly, we are left with only N variables of integration on each link and, consequently, there is a saddle point $\lambda_i^{r,\mu}=\lambda_i^*$ that does not depend on the lattice coordinate because of the translational and rotational lattice invariance.

To evaluate λ_i^*, we write the partition function in the form

$$Z=\int\prod_i d\lambda_i\exp\left\{NV\left[\sum_i(\lambda_i-\ln\lambda_i)+\frac{2}{N}\sum_{i\neq j}\ln|\lambda_i-\lambda_j|+NF(\beta,\lambda)\right]\right\}, \qquad (38)$$

where V is the number of links in the lattice and $F(\beta,\lambda)$ is the sum over all the closed planar surfaces with the structure determined from (37). The equation for the saddle-point value λ_i^* is

$$1-\frac{1}{\lambda}+2\int\frac{\rho(\lambda')d\lambda'}{\lambda-\lambda'}=-\frac{\partial F(\beta,\lambda)}{\partial\lambda}. \qquad (39)$$

The problem thus reduces to the evaluation of $F(\beta,\lambda)$, i.e., the sum over the planar surfaces for arbitrary λ_i. We note that the variables $\lambda_i^{r,\mu}$ are gauge-invariant.

Of course, the principal question in our approach is that of the existence of the local limit for the chromodynamic string. Our formulas can be used as the starting point for this type of construction.

The other, more pragmatic, utilization of (37)–(39) is the strong-coupling expansion that is valid for arbitrary β (modified strong coupling expansion).

The authors are indebted to A. A. Migdal, A. M. Polyakov, E. Witten, S. P. Novikov, I. K. Kostov, S.B. Khokhlachev, R. L. Mkrtchyan, and A. B. Zamolodchikov for useful discussions.

[1]E. Witten, Nucl. Phys. B **160**, 57 (1979); Yu. M. Makeenko and A. A. Migdal, Nucl. Phys. B **188**, 269 (1981).

[2]K. Wilson, Phys. Rev. D **10**, 2445 (1974).
[3]G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).
[4]J. L. Gervais and A. Neveu, Phys. Lett. B **80**, 255 (1979); Y. Nambu, Phys. Lett. B **82**, 372 (1979); A. M. Polyakov, Phys. Lett. B **82**, 247 (1979).
[5]A. A. Migdal, Phys. Lett. B **96**, 333 (1980); A. A. Migdal, Nucl. Phys. B **189**, 253 (1980).
[6]V. A. Kazakov and I. K. Kostov, Nucl. Phys. B **179**, 199 (1980); V. A. Kazakov, Nucl. Phys. B **179**, 283 (1980); V. A. Kazakov and I. K. Kostov, Phys. Lett. B **105**, 453 (1981).
[7]D. Weingarten, Phys. Lett. B **90**, 285 (1980).
[8]D. Weingarten, Phys. Lett. B **90**, 280 (1980).
[9]J. Samuel, J. Math. Phys. **21**, 2695 (1980).
[10]A. M. Polyakov, Phys. Lett. B **103**, 207 (1981).

Translated by S. Chomet