Anisotropic cosmological model created by quantum polarization of vacuum

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A homogeneous anisotropic nonsingular space-time metric with a six-parameter symmetry group is found which can be created by the polarization of the vacuum of the quantum fields of matter by a self-consistent gravitational field in the absence of classical matter. The mean values of the energy-momentum tensors for massless conformally covariant fields and a massive scalar field are computed in this metric. The Green function for a massive scalar field is constructed.

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1. Recently cosmological models in which the contributions of the quantum effects of particle production and the polarization of vacuum by a strong gravitational field are taken into account in the effective energy-momentum tensor of matter have attracted much attention. As is well known (see, for example, the discussion of this question in Refs. 1 and 2), in the single-loop approximation, to take these contributions into account, it is sufficient to add to the right-hand sides of the Einstein equations the mean vacuum value $\langle T_i^k \rangle$ of the energy-momentum tensor for all the quantized fields. In this case if the number of elementary fields of matter $N \gg 1$, then, in the leading approximation in $1/N$, we can neglect the contribution to $\langle T_i^k \rangle$ from the gravitons, a contribution whose computation meets with certain difficulties because of the dependence of the result on the gauge (this contribution should, however, be taken into account in the next order in $1/N$).

As a result, the problem reduces to the solution of the system of equations

$$R_i^k - {}^1/_2 \delta_i^k R = 8\pi G (T_{i(0)}^k + \langle T_i^k \rangle), \tag{1}$$

where $T_{i(0)}^k$ is the energy-momentum tensor for classical matter and $\langle T_i^k \rangle$ is a complicated functional of the mean (self-consistent) space-time metric g_{ik}.[1)] If $T_{i(0)}^k = 0$, but $\langle T_i^k \rangle \neq 0$ and real, we shall say that the metric g_{ik} is created by the polarization of vacuum (and also by particle production if this process occurs).

Earlier, the equations (1) were considered either for isotropic cosmological models,[3–5] or in a class of homogeneous anisotropic metrics close to the classical Kasner metric with $R_i^k = 0$ (Refs. 6–8). In the present paper we shall find for the equations (1) with $T_{i(0)}^k = 0$, a new homogeneous anisotropic solution that does not reduce to the vacuum or isotropic solution.

2. Let us consider the "two-sphere metric"

$$ds^2 = a^2 (dt^2 - \mathrm{ch}^2\, t dx^2 - d\theta^2 - \sin^2\theta d\varphi^2), \tag{2}$$

where $a = \text{const}$, $-\infty < t < \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$, and the range of x will be specified below. The corresponding space-time is geodesically complete, symmetric ($R_{iklm;n} = 0$), and anisotropic:

$$C_{iklm} C^{iklm} = 16/3a^4,$$

where C_{iklm} is the Weyl tensor; the Ricci tensor is

$$R_i^k = -\frac{1}{a^2}\delta_i^k, \quad R = -\frac{4}{a^2}.$$

This space-time has the six-parameter symmetry group $O(2,1) \times O(3)$, and is a particular case of the spatially homogeneous T metrics[9,10] that for formal reasons do not fall within the Bianchi classification of three-dimensional homogeneous spaces. The metric (2) covers the entire space-time. The two-sphere metric as previously encountered as the solution to the Einstein equations with a cosmological constant.[11,12] In the present paper we assume the cosmological constant to be equal to zero (or very small compared to the Planck scales).

Let us now make the following identification of points (splicing) in the metric[2)] (2):

$$(t, x, \theta, \varphi) = (t, x + 2\pi, \theta, \varphi). \tag{3}$$

Then below we can assume that $-\pi \leqslant x < \pi$. The conformal diagram of the resulting space-time is shown in Fig. 1. In the general case $\langle T_i^k \rangle$ does not have to have the same structure as $R_i^k - \frac{1}{2}\delta_i^k R$, and then the equations (1) are inconsistent. Let us show that, for the metric (2) with the identification (3), the vacuum average $\langle T_i^k \rangle \propto \delta_i^k$. Here we shall not assume that the quantized matter fields making a contribution to $\langle T_i^k \rangle$ are free fields; in particular, $\langle T_i^k \rangle$ may also include internal graviton loops. Therefore, the assertion made above essentially falls outside the limits of the single-loop approximation.

Let us perform an Euclidean rotation: $r = \pi/2 + it$. Then (2) assumes the form

$$ds^2 = -a^2 (dr^2 + \sin^2 r dx^2 + d\theta^2 + \sin^2\theta d\varphi^2). \tag{4}$$

The condition (3) guarantees the absence of a conical singularity at $r = 0, \pi$. We shall assume that there exists a state

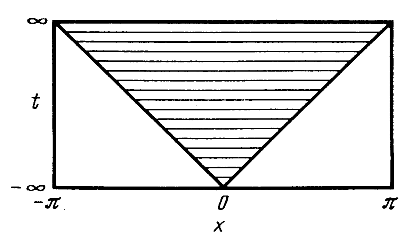

FIG. 1. Conformal diagram of the space-time continuum described by the metric (2) with the splicing (3). The lines $x = \pm\pi$ are identified. Each point of the diagram represents a two-dimensional sphere (θ,φ) with radius a^2. The region covered by the metric (I.1) is hatched.

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vector that is invariant under the operations of the full symmetry group of the metric (4). Then it follows from the invariance under $O(3)\times O(3)$ and the conservation condition $T^k_{i;k}=0$ that the only nonzero components of the average value $\langle T^k_i\rangle$ as computed over this state are

$$\langle T_r{}^r\rangle=\langle T_x{}^x\rangle=A,\quad \langle T_\theta{}^\theta\rangle=\langle T_\varphi{}^\varphi\rangle=B,\quad A,\ B=\text{const}.$$

Furthermore, since the radii of the two spheres are equal, there also exists a discrete symmetry connected with the interchange of the coordinates of the two spheres: $(r, x, \theta,\varphi)\to(\theta,\varphi-\pi, r, x+\pi)$. Hence we have $A=B$.

Thus,

$$\langle T_i{}^k\rangle=f(a^2)\delta_i{}^k, \tag{5}$$

where f is some function. This result remains valid when we return from r to t; in this case the range of x $(-\pi\leqslant x<\pi)$ remains the same as before. For the quantity a^2 we obtain from (1) the algebraic equation

$$1/a^2=8\pi Gf(a^2), \tag{6}$$

the positive roots of which determine self-consistent metrics of the form (2), which are created by vacuum polarization. The negative roots $(a^2<0)$ also have a physical meaning: they furnish self-consistent "two-pseudosphere metrics":

$$ds^2=|a^2|(\text{sh}^2\, r dt^2-dr^2-d\theta^2-\text{sh}^2\,\theta d\varphi^2), \tag{7}$$

where $-\infty<t<\infty$, $0\leqslant r$, $\theta<\infty$, $0\leqslant\varphi<2\pi$ (the metric (7) does not cover all space-time: there is a horizon at $r=0$). In the metric (7), in contrast to (2), the energy density $\langle T^0_0\rangle<0$. Therefore, the metric (7) is less interesting from the physical point of view: from it we cannot go over to the Friedmann solutions.

Perhaps the complex roots of (6) can be interpreted as describing the unstable solutions, the imaginary part of a^2 being connected with the decay probability. But this point needs to be investigated further.

Let us note that homogeneous metrics with splicings have been considered before (see, for example, Refs. 13 and 14), but the type and parameters of the splicing were always chosen arbitrarily, and imposed as initial conditions. Here we, apparently for the first time, encounter a situation in which the identification condition (3) follows unambigously from the equations (1) and the form of the metric (2). For other identifications, the metric (2) does not satisfy the equations (1) with $T^k_{i(0)}=0$.

3. Let us proceed to consider those cases in which we can compute f and find the roots of Eq. (6). Let us, to begin with, assume that all the quantized fields of matter are massless, noninteracting, and conformally covariant. Then in the single-loop approximation the trace $\langle T\rangle$ of the energy-momentum tensor is completely determined by the well-known conformal anomaly:

$$\langle T\rangle=-\frac{1}{2880\pi^2}\left[k_1C_{iklm}C^{iklm}+k_2\left(R_{ik}R^{ik}-\frac{1}{3}R^2\right)+k_3R_{;i}^{;i}\right]. \tag{8}$$

The constants k_1, k_2, and k_3 depend on the form of the quantum field; the formula (8) contains their sum over all the fields that have been taken into consideration. For the metric (2)

$$4f=\langle T\rangle=(k_2-4k_1)/2160\pi^2a^4. \tag{9}$$

If, following Starobinsky,[5] we introduce the quantity $H^2=360\pi/Gk_2$, the curvature on de Sitter's self-consistent quantum-mechanical solution, then we finally obtain

$$a^2=\frac{1}{3H^2}\left(1-4\frac{k_1}{k_2}\right). \tag{10}$$

The single-loop approximation is applicable if $|a^2|\gg G$, i.e., when the quantity $|k_2-4k_1|$ is sufficiently large. The root $a^2>0$ if $k_2>4k_1$, which is the case for, for example, photons ($k_1=-13$; $k_2=62$). Therefore, the polarization of the vacuum of a sufficiently large number of vector fields can sustain the metric (2). For scalar and spinor particles $k_2<4k_1$, and for them we arrive at a solution of the form (7). Let us note that the ratio of $\langle T^0_0\rangle$ in the metric (2) to $\langle T^0_0\rangle$ in de Sitter's quantum-mechanical solution is equal to $(1-4k_1/k_2)^{-1}$. For $k_1=0$ the vacuum energy densities in the two self-consistent solutions are equal.

4. Let us investigate the role played by a nonzero rest mass of quantized fields for the particular case of a massive free scalar field satisfying the equation

$$(\nabla_i\nabla^i+m^2-R/6)\Phi=0, \tag{11}$$

which, as is well known, is conformally covariant in the $m=0$ case. To determine f, we first find the causal Green function of this field in the metric (2) (such a problem has not been solved before). Let us go over to the Euclidean version of the metric (2), i.e., to the metric (4). We have

$$G(x_1{}^i, x_2{}^i)=\sum_\lambda \lambda^{-1}\Phi_\lambda(x_1{}^i)\Phi_\lambda{}^*(x_2{}^i), \tag{12}$$

where the Φ_λ are the regular solutions to the equation

$$(\nabla_i\nabla^i+m^2-R/6)\Phi_\lambda=\lambda\Phi_\lambda \tag{13}$$

with the normalization

$$\int d^4x\, g^{1/2}\Phi_\lambda\Phi_{\lambda'}{}^*=\delta_{\lambda\lambda'}.$$

In the metric (4)

$$\begin{aligned}&\Phi_\lambda(x^i)=a^{-2}Y_{lm}(\theta,\varphi)Y_{l'm'}(r,x),\\ &\lambda=a^{-2}[l(l+1)+l'(l'+1)+(ma)^2+{}^2/_3].\end{aligned} \tag{14}$$

Then

$$G(x_1{}^i, x_2{}^i)$$

$$=\frac{1}{(4\pi a)^2}\sum_l\sum_{l'}\frac{(2l+1)(2l'+1)P_l(\cos\alpha)P_{l'}(\cos\delta)}{l(l+1)+l'(l'+1)+(ma)^2+{}^2/_3}, \tag{15}$$

$$\cos\alpha=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2),$$

$$\cos\delta=\cos r_1\cos r_2+\sin r_1\sin r_2\cos(x_1-x_2).$$

The invariance of G under the interchange $(\alpha\rightleftarrows\delta)$ of the two spheres can be seen from (15). Let us simplify (15), using Dougall's expansion[15]:

$$\begin{aligned}&P_{i\beta_l-1/2}(-\cos\delta)=\frac{\text{ch}\,\pi\beta_l}{\pi}\sum_{l'=0}^{\infty}\frac{(2l'+1)P_{l'}(\cos\delta)}{(l'+{}^1/_2)^2+\beta_l{}^2},\\ &\beta_l=[l(l+1)+(ma)^2+{}^5/_{12}]^{1/2}.\end{aligned} \tag{16}$$

We obtain

$$G(x_1^i, x_2^i) = \frac{1}{16\pi a^2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\alpha)\frac{P_{i\beta_l-1/2}(-\cos\delta)}{\operatorname{ch}\pi\beta_l}. \tag{17}$$

Now we can return to the correct signature by setting $r = \pi/2 + it$. In the metric (2)

$$\cos\delta = -\operatorname{sh} t_1 \operatorname{sh} t_2 + \operatorname{ch} t_1 \operatorname{ch} t_2 \cos(x_1 - x_2) \tag{18}$$

and (17) gives the causal Green function (apart from the factor i). In Appendix I the formula (17) is derived in other coordinates.

Let us note that, outside the light cone (i.e., for $\cos\delta < 1$), $\operatorname{Im} G = 0$, as in flat space-time, which indicates that scalar-particle production does not occur in the metric (2). The Green function inside the light cone (i.e., for $\cos\delta > 1$ and imaginary δ) can be obtained through the analytic continuation of (17) as a function of $\cos\delta$; in this case the shift $\cos\delta \to \cos\delta - i\varepsilon$, $\varepsilon > 0$ must be made.

As the points approach each other, the function $G(x_1^i, x_2^i)$ diverges. Let us compute $\langle\Phi^2\rangle_{\rm reg}$. The regularization of $\langle\Phi^2\rangle$ requires two subtractions. We shall use the generally covariant regularization method consisting in the subtraction from G of the first two terms of the de Witt–Christensen expansion[1,16]:

$$\langle\Phi^2\rangle_{reg} = \lim_{x_1^i\to x_2^i}\left\{G(x_1^i, x_2^i) - \frac{1}{8\pi^2}\left[\frac{1}{\sigma} - \frac{1}{6}R_{ik}\frac{\sigma_{,i}\sigma_{,k}}{\sigma_{,l}\sigma^{,l}} + m^2\left(\frac{1}{2}\ln\left|\frac{m^2\sigma}{2}\right| + \gamma - \frac{1}{2}\right)\right]\right\}, \tag{19}$$

where $\sigma(x_1^i, x_2^i)$ is one half the square of the geodesic interval between the points x_1^i and x_2^i and γ is the Euler constant. In (19) the interval between the points is assumed to be spacelike, i.e., that $\sigma > 0$.

It follows from the de Witt–Christensen expansion that, as $m^2 \to \infty$,

$$\langle\Phi^2\rangle_{reg} = \frac{a_2}{16\pi^2 m^2} + O(m^{-4}),$$
$$a_2 = \frac{1}{180}(R_{iklm}R^{iklm} - R_{ik}R^{ik} + R_{;i}^{;i}) = \frac{1}{45a^4}. \tag{20}$$

Since the Green function obtained in (17) is in the form of a series, it is convenient to use the following computational procedure: We represent the subtrahends in (19) also in the form of power series in l. Assuming that the points x_1^i and x_2^i are separated only on one of the spheres (i.e., that $\alpha = 0$, $|\delta| \ll 1$, $\sigma = a^2\delta^2/2$), and using the Heine formula,[15] we have

$$\frac{1}{\sigma} = \frac{1}{a^2(1-\cos\delta)} - \frac{1}{6a^2} + O(\delta^2)$$
$$= \frac{1}{a^2}\sum_{l=0}^{\infty}(2l+1)Q_l(2-\cos\delta) - \frac{1}{6a^2} + O(\delta^2);$$
$$\ln\left|\frac{m^2\sigma}{2}\right| = \ln\frac{(ma)^2}{2} + \ln(1-\cos\delta) + O(\delta^2)$$
$$= \ln\frac{(ma)^2}{2} + \ln 2 - 1 - \sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}P_l(\cos\delta) + O(\delta^2). \tag{21}$$

Let us substitute (17) and (21) into (19) and introduce into the sum over l the cutoff parameter $e^{-\varepsilon l}$, $\varepsilon > 0$. Then

$$\langle\Phi^2\rangle_{reg} = \frac{1}{(4\pi a)^2}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\left\{\frac{\pi}{\operatorname{ch}\pi\beta_0}P_{i\beta_0-1/2}(-\cos\delta) - 2Q_0(2-\cos\delta) - 2\mu^2(\ln\mu + \gamma - 1) + \sum_{l=1}^{\infty}(2l+1)e^{-\varepsilon l}\left[\frac{\pi}{\operatorname{ch}\pi\beta_l}P_{i\beta_l-1/2}(-\cos\delta) - 2Q_l(2-\cos\delta) + \frac{\mu^2}{l(l+1)}P_l(\cos\delta)\right]\right\}; \quad \mu = ma. \tag{22}$$

For $\varepsilon \neq 0$ each term in the sum can be uniformly expanded in powers of δ about the point $\delta = 0$. In particular, for $\delta \to 0$

$$P_{i\beta_l-1/2}(-\cos\delta) \equiv F\left(\frac{1}{2} + i\beta_l, \frac{1}{2} - i\beta_l, 1; \frac{1+\cos\delta}{2}\right)$$
$$= -\frac{2\operatorname{ch}\pi\beta_l}{\pi}\left[\ln\frac{\delta}{2} + \gamma + \operatorname{Re}\psi\left(\frac{1}{2} + i\beta_l\right)\right] + O(\delta^2\ln\delta), \tag{23}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function and F is the hypergeometric function. Then let us go over in (22) to the limit $\delta \to 0$ and, lastly, to the limit $\varepsilon \to 0$. Finally, we obtain

$$\langle\Phi^2\rangle_{reg} = \frac{1}{(4\pi a)^2}\left\{-2\mu^2(\ln\mu + \gamma - 1) - 2\gamma - 2\operatorname{Re}\psi\left(\frac{1}{2} + i\beta_0\right) + \sum_{l=1}^{\infty}(2l+1)\left[2\psi(l+1) - 2\operatorname{Re}\psi\left(\frac{1}{2} + i\beta_l\right) + \frac{\mu^2}{l(l+1)}\right]\right\}, \tag{24}$$

where the β_l are defined in (16). For $l \gg 1$ the terms of the series in l in (24) behave like

$$\frac{\mu^4 + {}^2/_3\,\mu^2}{(l+{}^1/_2)^3} + \frac{-{}^2/_3\,\mu^6 - {}^1/_6\,\mu^4 + {}^1/_{15}\,\mu^2 + {}^{64}/_{2835}}{(l+{}^1/_2)^5} + \dots, \tag{25}$$

therefore, the series converges, and the quantity $\langle\Phi^2\rangle_{\rm reg}$ is finite.

Since the passage to the $\delta \to 0$ limit (the shifting of the points) preceded the passage to the $\varepsilon \to 0$ limit (the lifting of the momentum cutoff), the regularization procedure used by us is closer in spirit to the adiabatic method or the Pauli-Villars method than to the point splitting method proper. The equivalence of the method employed to the Pauli-Villars method or the method of adiabatic regularization follows from the fact that, first, as can be seen from (19), we subtracted from $\langle\Phi^2\rangle$ only those terms that we have the right to subtract (i.e., those terms which either do not depend on the space-time curvature, or are proportional to R), and, second, the expression (24) obtained above for the quantity $\langle\Phi^2\rangle_{\rm reg}$ has the correct asymptotic form (20) as $m^2 \to \infty$. The latter assertion is proved in the Appendix II.

For $m = \mu = 0$ a numerical calculation with the use of the formula (20) yields

$$(4\pi a)^2\langle\Phi^2\rangle_{reg}\approx-0.0265. \tag{26}$$

Let us note that the result $\langle\Phi^2\rangle_{\rm reg}=0$ for $m=0$ was obtained for the space-time under consideration by Page[17] in the Gaussian approximation. For $\mu=ma\ll 1$

$$\begin{aligned}(4\pi a)^2\langle\Phi^2\rangle_{reg}&\approx-0{,}0265+\mu^2(-2\ln\mu+C),\\ C&=2(1-\gamma)+\left(\frac{12}{5}\right)^{1/2}\operatorname{Im}\psi'\left(\frac{1}{2}+i\left(\frac{5}{12}\right)^{1/2}\right)\\ &+\sum_{l=1}^{\infty}(2l+1)\left[\left(l(l+1)+\frac{5}{12}\right)^{-1/2}\operatorname{Im}\psi'\left(\frac{1}{2}+i\left(l(l+1)+\frac{5}{12}\right)^{1/2}\right)+\frac{1}{l(l+1)}\right]\approx-1.756,\end{aligned} \tag{27}$$

where $\psi'(z)=d\psi(z)/dz$. Figure 2 shows the plot, obtained in a numerical calculation with the use of (24), of the quantity $(4\pi a)^2\langle\Phi^2\rangle_{\rm reg}$. The quantity $\langle\Phi^2\rangle_{\rm reg}$ vanishes at $\mu\approx0.094$, and attains its maximum value $\approx0.070(4\pi a)^{-2}$ at $\mu\approx0.43$.

The computation of $\langle T_i^k\rangle$ requires three subtractions, the last of which leads to the appearance of a conformal anomaly in the trace. Since for a classical massive quasiconformal scalar field, $T=m^2\Phi^2$, for a quantized field in the metric (2), (3)

$$\langle T_i^k\rangle={}^1/{}_4\delta_i^k\langle T\rangle,\qquad \langle T\rangle=m^2\langle\Phi^2\rangle_{reg}-\frac{1}{720\pi^2a^4}=4f, \tag{28}$$

where $\langle\Phi^2\rangle_{\rm reg}$ is given by the formula (24). For $\mu=0$ we come back to the formula (9) with $k_1=k_2=1$; for $\mu\gg1$ we have $\langle T\rangle\propto m^{-2}a^{-6}$. Figure 2 shows a plot of the quantity $(4\pi a^2)^2\langle T\rangle$. The quantity $\langle T\rangle$ changes sign at $\mu\approx0.60$. Therefore, massive scalar fields with $ma>0.6$ help sustain the metric (2), although the contribution from them is numerically small. For $\mu=1$ the quantity $(4\pi a^2)^2\langle T\rangle\approx6.9\times10^{-3}$.

5. Thus, we have shown that the polarization of the vacuum of the quantized fields of matter can create and sustain the metric (2) with the splicing (3). Particle production does not occur in the single-loop approximation. It is to be expected, however, that the solution (2), (3) will be unstable when the higher-order loops are taken into consideration, just as de Sitter's quantum-mechanical solution is unstable against the creation of the scalar mode.[5] In our case, besides the scalaron instability, there can also occur the usual gravitational instability whose principal mode is a homogeneous anisotropic perturbation that transfers (2) into the general class of T metrics:

$$ds^2=dt^2-a^2(t)\,dx^2-b^2(t)\,(d\theta^2+\sin^2\theta d\varphi^2). \tag{29}$$

Let us note that the class of metrics (29) contains as a particular case a metric that covers part of the de Sitter space-time:

$$a(t)=a_0\operatorname{sh}Ht,\quad b(t)=H^{-1}\operatorname{ch}Ht,\quad a_0=\text{const.}$$

We can, by analogy with the isotropic case,[5] surmise that the Eqs. (1) with $T_{i(0)}^k=0$ possess solutions that describe the process of decay and transition of the cases when this transition occurs via the intermediate de Sitter phase and when it does not. Thus, the metric (2) may turn out to be important in the investigation of the question how the quantum de Sitter state arose in the early universe and for the construction of alternative models of a universe without the de Sitter phase. There is also no doubt that the Eqs. (1) possess more complicated anisotropic solutions with $R_i^k\neq0$ produced through the quantum polarization of vacuum (the structure of the solutions with $R_i^k=0$ is furnished by the general Belinskiĭ-Lifshitz-Khalatnikov oscillation regime[18]).

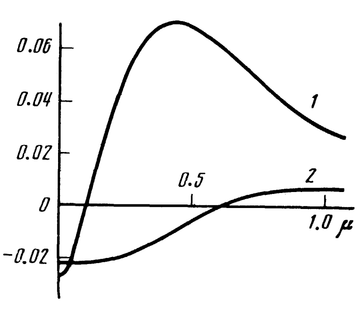

FIG. 2. Plots of the quantities $(4\pi a)^2\langle\Phi^2\rangle_{\rm reg}$ (curve 1) and $(4\pi a^2)^2\langle T\rangle$ (curve 2) as functions of $\mu=ma$ for a massive scalar field.

APPENDIX I

It is also convenient to use for the description of the two-sphere space-time continuum the metric

$$ds^2=a^2(d\tau^2-e^{2\tau}dy^2-d\theta^2-\sin^2\theta d\varphi^2), \tag{I.1}$$

$(-\infty<\tau,y<\infty)$, which covers one half of the manifold under consideration (see Fig. 1). The formulas for the transformation from (2) to (I.1) have the form

$$\tau=\ln(\operatorname{ch}t\cos x+\operatorname{sh}t),\quad \operatorname{ch}t\cos x+\operatorname{sh}t>0,\qquad y=\frac{\operatorname{ch}t\sin x}{\operatorname{ch}t\cos x+\operatorname{sh}t}. \tag{I.2}$$

Let us compute the quantum two-point function

$$\tilde G(x_1^i,x_2^i)=\langle\Phi(x_1^i)\Phi(x_2^i)\rangle$$

for the massive scalar field (11). The normalized positive-frequency basis for the solutions to (11) in the metric (I.1) is

$$\Phi_{plm}=\frac{\sqrt{\pi}}{2a}\exp\left[-\frac{\pi\beta_l}{2}\right]\eta^{1/2}H^{(1)}_{i\beta_l}(|p|\eta)\frac{e^{-ipy}}{\sqrt{2\pi}}Y_{lm}(\theta,\varphi),\qquad \eta=e^{-\tau}. \tag{I.3}$$

Then

$$\begin{aligned}\tilde G(x_1^i,x_2^i)&=\frac{1}{8a^2}\sum_{lm}\int_{-\infty}^{\infty}dp\,(\eta_1\eta_2)^{1/2}H^{(1)}_{i\beta_l}(|p|\eta_1)\\ &\times H^{*(1)}_{i\beta_l}(|p|\eta_2)\exp[-\pi\beta_l]e^{-ip(y_1-y_2)}Y_{lm}(\theta_1,\varphi_1)Y^*_{lm}(\theta_2,\varphi_2)\\ &=\frac{1}{8a^2\pi^3}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\alpha)\\ &\times\int_{-\infty}^{\infty}dp\,(\eta_1\eta_2)^{1/2}K_{i\beta_l}(-i|p|\eta_1)K_{i\beta_l}(i|p|\eta_2)e^{-ip(y_1-y_2)}\end{aligned} \tag{I.4}$$

where cos α is defined in (15), and we have used the formula

$$H_{i\beta_l}(|p|\eta) = -(2i/\pi)\exp[\pi\beta_l/2]K_{i\beta_l}(-i|p|\eta).$$

The p integral in (I.4) is computed in Ref. 19; it is equal to

$$\tfrac{1}{2}\pi|\Gamma(\tfrac{1}{2}+i\beta_l)|^2 F(\tfrac{1}{2}+i\beta_l, \tfrac{1}{2}-i\beta_l, 1; 1-\sigma/2);$$
$$\sigma=[(y_1-y_2)^2-(\eta_1-\eta_2)^2]/2\eta_1\eta_2 \quad \text{(I.5)}$$

for $\sigma > 0$. Taking into account the fact that outside the light cone (i.e., for $\sigma > 0$) the function $\widetilde{G}$ coincides with the causal Green function G (since they differ by the advanced Green function), we obtain

$$G(x_1^i, x_2^i) = \frac{1}{16\pi a^2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\alpha)\frac{P_{i\beta_l-1/2}(-1+\sigma)}{\operatorname{ch}\pi\beta_l}. \quad \text{(I.6)}$$

Finally, from (I.2) it follows that

$$1-\sigma = \operatorname{ch} t_1 \operatorname{ch} t_2 \cos(x_1-x_2) - \operatorname{sh} t_1 \operatorname{sh} t_2 = \cos\delta$$

and (I.6) reduces to (17). We should, in analytically continuing G into the light cone, make the shift $\sigma \to \sigma + i\varepsilon$, $\varepsilon > 0$.

APPENDIX II

Let us show that the quantity $S = (4\pi a)^2\langle\Phi^2\rangle_{\text{reg}}$, where $\langle\Phi^2\rangle_{\text{reg}}$ is determined from (24), has the following asymptotic form:

$$S = 1/45\mu^2 + O(\mu^{-4}), \quad \mu\to\infty. \quad \text{(II.1)}$$

From (24) it follows that

$$S = -2\mu^2(\ln\mu+\gamma-1) - 2\gamma - 2\operatorname{Re}\psi\left(\frac{1}{2}+i\left(\frac{5}{12}+\mu^2\right)^{1/2}\right)$$
$$+\sum_{l=1}^{\infty}(2l+1)\left[2\psi(l+1) - 2\operatorname{Re}\psi\left(\frac{1}{2}+i\left(l(l+1)+\frac{5}{12}+\mu^2\right)^{1/2}\right)+\frac{\mu^2}{l(l+1)}\right], \quad \text{(II.2)}$$

where γ is the Euler constant. For $\mu \gg 1$, let us expand in asymptotic series those ψ functions whose argument contains μ; then we obtain

$$S = -2\mu^2(\ln\mu+\gamma-1) - 2\gamma - 2\ln\mu - \frac{1}{3\mu^2}$$
$$+\sum_{l=1}^{\infty}(2l+1)\left\{2\psi(l+1) - \ln\left[\left(l+\frac{1}{2}\right)^2+\mu^2\right]\right.$$
$$-\frac{1}{12}\left[\left(l+\frac{1}{2}\right)^2+\mu^2\right]^{-1}$$
$$\left.+\frac{7}{480}\left[\left(l+\frac{1}{2}\right)^2+\mu^2\right]^{-2}+\frac{\mu^2}{l(l+1)}\right\}+O(\mu^{-4}). \quad \text{(II.3)}$$

Let us consider the auxiliary sums:

1) $$S_1(\mu^2) = \sum_{l=1}^{\infty}(2l+1)\left[\frac{\mu^2}{l(l+1)} - \ln\left(1+\frac{\mu^2}{(l+1/2)^2}\right)\right],$$
$$S_1(0) = 0, \quad \text{(II.4)}$$

$$\frac{dS_1(\mu^2)}{d\mu^2} = \sum_{l=1}^{\infty}(2l+1)\left[\frac{1}{l(l+1)} - \frac{1}{(l+1/2)^2+\mu^2}\right]$$
$$= \sum_{l=1}^{\infty}\left(\frac{1}{l}+\frac{1}{l+1}-\frac{1}{l+1/2+i\mu}-\frac{1}{l+1/2-i\mu}\right)$$
$$= 2\operatorname{Re}\psi\left(\frac{3}{2}+i\mu\right)+2\gamma-1,$$
$$S_1 = (2\gamma-1)\mu^2 + 4\int_0^{\mu} z\operatorname{Re}\psi\left(\frac{3}{2}+iz\right)dz$$
$$= (2\gamma-1)\mu^2 + 2\int_0^{\mu} z\left[\left(z^2+\frac{1}{4}\right)^{-1} + 2\operatorname{Re}\psi\left(\frac{1}{2}+iz\right)\right]dz. \quad \text{(II.5)}$$

Let us use the following formula, which can be derived from Binet's formulas[15]:

$$\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^2} - \int_0^{\infty}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{1}{2}-\frac{t}{12}\right)e^{-tz}dt.$$

Then

$$2\operatorname{Re}\psi\left(\frac{1}{2}+iz\right) = \ln\left(z^2+\frac{1}{4}\right) - \frac{1}{2(z^2+1/4)} + \frac{z^2-1/4}{6(z^2+1/4)^2}$$
$$-2\int_0^{\infty}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{1}{2}-\frac{t}{12}\right)e^{-t/2}\cos tz\,dt.$$

Substituting this into (II.5), we find that for $\mu \gg 1$

$$S_1 = 2\mu^2(\ln\mu+\gamma-1) + \frac{11}{6}\ln 2\mu - \frac{1}{12} + \frac{9}{32\mu^2}$$
$$-4\int_0^{\infty}\left(\frac{1}{1-e^{-t}}-\frac{1}{t}\right.$$
$$\left.-\frac{1}{2}-\frac{t}{12}\right)e^{-t/2}\left(\frac{\mu\sin\mu t}{t}+\frac{\cos\mu t}{t^2}-\frac{1}{t^2}\right)dt + O(\mu^{-4}).$$

For $t \to 0$ we have

$$\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{1}{2}-\frac{t}{12} = -\frac{t^3}{720}+O(t^5);$$

$$\lim_{\varepsilon\to 0}\int_0^{\infty}dt\, t^2 e^{-\varepsilon t}\sin t = -2, \quad \lim_{\varepsilon\to 0}\int_0^{\infty}dt\, t e^{-\varepsilon t}\cos t = -1.$$

Therefore, finally

$$S_1=2\mu^2(\ln\mu+\gamma-1)+\frac{11}{6}\ln 2\mu-\frac{1}{12}+4I+\frac{127}{480\mu^2}+O(\mu^{-4}), \tag{II.6}$$

where the integral

$$I=\int_0^\infty\left(\frac{1}{1-e^{-t}}-\frac{1}{t}-\frac{1}{2}-\frac{t}{12}\right)e^{-t/2}t^{-2}\,dt \tag{II.7}$$

cannot be expressed in terms of elementary functions.

$$2)\ S_2=\sum_{l=0}^\infty(2l+1)\left[2\psi(l+1)-2\ln(l+{}^1/_2)-{}^1/_{12}(l+{}^1/_2)^{-2}\right]. \tag{II.8}$$

Using the formula[15]

$$\ln n=\int_0^\infty(e^{-t}-e^{-nt})t^{-1}\,dt,$$

the Binet formula for $\psi(z)-\ln z$, and the expression (II.7), we have

$$S_2=4\sum_{l=0}^\infty\left(l+\frac{1}{2}\right)\left[\psi(l+1)-\ln(l+1)+\ln\frac{l+1}{l+{}^1/_2}-\frac{1}{24}\left(l+\frac{1}{2}\right)^{-2}\right]$$

$$=-4\sum_{l=0}^\infty\left(l+\frac{1}{2}\right)\int_0^\infty\left(\frac{e^{-t/2}}{1-e^{-t}}-\frac{1}{t}+\frac{t}{24}\right)e^{-t(l+{}^1/_2)}\,dt$$

$$=4\int_0^\infty\left(\frac{e^{-t/2}}{1-e^{-t}}-\frac{1}{t}+\frac{t}{24}\right)\frac{d}{dt}\left(\sum_{l=0}^\infty e^{-t(l+{}^1/_2)}\right)dt$$

$$=4\int_0^\infty\left(\frac{e^{-t/2}}{1-e^{-t}}-\frac{1}{t}+\frac{t}{24}\right)\frac{d}{dt}\left(\frac{e^{-t/2}}{1-e^{-t}}\right)dt$$

$$=4\lim_{\varepsilon\to0}\left[\frac{1}{2}\frac{e^{-t}}{(1-e^{-t})^2}\bigg|_\varepsilon^\infty+\left(-\frac{1}{t}+\frac{t}{24}\right)\frac{e^{-t/2}}{1-e^{-t}}\bigg|_\varepsilon^\infty-\int_\varepsilon^\infty\left(\frac{1}{t^2}+\frac{1}{24}\right)\frac{e^{-t/2}}{1-e^{-t}}\,dt\right]$$

$$=-4I+4\lim_{\varepsilon\to0}\left[\frac{1}{2\varepsilon^2}-\frac{1}{24}-\frac{1}{24}\int_\varepsilon^\infty\frac{e^{-t/2}}{1-e^{-t}}\,dt-\int_\varepsilon^\infty\left(\frac{1}{t}+\frac{1}{2}+\frac{t}{12}\right)e^{-t/2}t^{-2}\,dt\right]$$

$$=-4I-\frac{1}{6}(\gamma+\ln 2)+\frac{1}{12}. \tag{II.9}$$

$$3)\ S_3=\sum_{l=1}^\infty(2l+1)\left[\frac{1}{(l+{}^1/_2)^2}-\frac{1}{(l+{}^1/_2)^2+\mu^2}\right]$$

$$=2\,\mathrm{Re}\,\psi\left(\frac{3}{2}+i\mu\right)-2\psi\left(\frac{3}{2}\right)=2\ln\mu+2\gamma+4\ln 2-4+\frac{11}{12\mu^2}+O(\mu^{-4}) \tag{II.10}$$

for $\mu\gg1$.

$$4)\ S_4=\sum_{l=1}^\infty\frac{2l+1}{[(l+{}^1/_2)^2+\mu^2]^2}=2\int_0^\infty\frac{z\,dz}{(z^2+\mu^2)^2}+O(\mu^{-4})$$

$$=\mu^{-2}+O(\mu^{-4}) \tag{II.11}$$

for $\mu\gg1$.

Collecting the formulas (II.4), (6), and (8)–(11), we obtain

$$S=-2\mu^2(\ln\mu+\gamma-1)-2\ln\mu-2\ln 2+{}^1/_3-1/3\mu^2+S_1+S_2+{}^1/_{12}S_3+{}^7/_{480}S_4$$

$$=1/45\mu^2+O(\mu^{-4}),$$

which was to be proved.

[1)]Here and below we set $c=\hbar=1$.

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