Distortions of the distribution function of collisionless particles by high-frequency gravitational waves

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Equations for the correlation functions of fluctuations in the spectra of relativistic collisionless particles are obtained from the combined system of Einstein's equations and the Vlasov equation. It is shown that the interaction of high-frequency gravitational waves with collisionless particles leads to diffusion of their spectrum in the momentum space. The distortions in the spectrum of the microwave background radiation in a cosmological model with high-frequency gravitational waves are discussed. Bounds are obtained on the spectral characteristics of background gravitational waves.

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The evolution of gravitational waves in an expanding cosmological model was considered for the first time by E. M. Lifshitz in a pioneering paper.[1] Many years later[2–5] there followed investigations of linear effects of interaction of gravitational waves with collisionless particles—neutrinos—after decoupling from electron-positron pairs and with photons of the microwave background after hydrogen recombination.

In the present paper we consider, in the collisionless matter components in an expanding cosmological model, nonlinear effects due to the interaction of ultrarelativistic (or massless) particles with tensor excitations of the space-time metric. The proposed approach is analogous to the methods of the quasilinear relaxation theory for the particle distribution function in an electron plasma developed in Refs. 6–8. The interaction of electrons with collective plasma excitations (plasmons) has much in common with the process of particle scattering by tensor perturbations of the metric (gravitons). The behavior of collisionless particles in a cosmological model with a background of high-frequency gravitational waves must be described by means of the relativistic analog of Vlasov's kinetic equation with self-consistent field, the system of Einstein's equations playing the part of Poisson's equation.[3,4,9] In the framework of the quasilinear approximation, which takes into account only the interaction between the collective excitations and the particles, this system of equations reduces to a diffusion equation for the particles in the momentum space and to an equation for the rate of growth or damping of the energy density of the gravitational waves. It is found that the diffusion equation for the particles in the momentum space is analogous to Kompaneets' equation,[10] which was considered in Refs. 11 and 12 in connection with the problem of spectral distortions of the background radiation resulting from its interaction with hot electrons of the cosmological plasma. The analogy arises because of the similarity of the mechanisms of energy transfer of the particles to the high-frequency region: the quadratic Doppler effect ($\Delta\nu/\nu\sim v^2/c^2$) in the models of Refs. 11 and 12 and $\Delta\nu/\nu\sim\varphi^2/c^4$ in the considered situation (here, $\varphi^2\sim h_\alpha^\beta h_\beta^\alpha$ is the analog of the square of the perturbation of the gravitational potential, where h_α^β are the tensor perturbations of the space-time metric).

In Sec. 1 of the present paper we obtain the basic equations for the correlation functions of the perturbations of the particle distribution function and the tensor fluctuations of the space-time metric. In Sec. 2 we discuss the conditions of applicability of the quasilinear approximation, and we analyze the Fokker-Planck equation for the particle distribution function. Section 3 is devoted to calculation of the diffusion coefficient of the particles in the momentum space in the approximation of a stochastic diffusion process. On the basis of a solution of the self-consistent linear problem of the dynamics of gravitational waves and perturbations of the particle distribution function, we consider in Sec. 4 the nonlinear distortions of the spectrum of the collisionless subsystem of the cosmological substrate under the influence of the energy density of the high-frequency gravitational waves. Section 5 is devoted to a discussion of the astrophysical consequences of the considered effect.

1. QUASILINEAR EQUATIONS FOR THE METRIC AND THE PARTICLE DISTRIBUTION FUNCTION

We shall describe the behavior of collisionless gravitating particles in an expanding cosmological model by the combined system of Einstein's equations and the Vlasov equation[2–5]:

$$\left(p^i\frac{\partial}{\partial x^i}-\Gamma_{ik}{}^{\alpha}p^ip^k\frac{\partial}{\partial p^\alpha}\right)F(x^i,p^\alpha)=0,\qquad(1)$$

$$R_{ik}-{}^1/_2g_{ik}R=\varkappa T_{ik},\qquad(2)$$

where $F(x,p)$ is the particle distribution function in the phase space; x^i ($i=0,1,2,3$) are the coordinates; p^α ($\alpha=1,2,3$) are the momenta; g_{ik} is the metric tensor; Γ_{ik}^α are Christoffel symbols of the second kind; R_{ik} is the Ricci tensor; T_{ki} is the energy-momentum tensor of the relativistic collisionless particles[9]:

$$T^{ik}=c\int\frac{\sqrt{-g}}{p^0}p^ip^kF(x,p)\,d^3p,\qquad(3)$$

$\varkappa$ is the gravitational constant; and $g\equiv\det\|g_{ik}\|$.

In accordance with the methods of Refs. 6–8, we represent the particle distribution function $F(x, p)$ and the geometrical characteristics of space-time (g_{ik}, R_{ik}) as sums of regular and fluctuating terms:

$$g_{ik}=\bar{g}_{ik}+h_{ik}, \quad \Gamma_{ik}{}^{l}=\bar{\Gamma}_{ik}{}^{l}+S_{ik}{}^{l}, \quad R_{ik}=\bar{R}_{ik}+P_{ik}, \tag{4}$$

$$F(x, p)=\bar{F}(x, p)+\Phi(x, p),$$

$$\bar{R}_{ik}=\bar{\Gamma}_{ik,l}{}^{l}+\bar{\Gamma}_{ik}{}^{l}\bar{\Gamma}_{lm}{}^{m}-\bar{\Gamma}_{il,k}{}^{l}-\bar{\Gamma}_{il}{}^{m}\bar{\Gamma}_{mk}{}^{l}, \tag{5}$$

$$S_{ik}{}^{l}=\frac{1}{2}G^{lm}(h_{mi;k}+h_{mk;i}-h_{ik;m}), \qquad G_{i}{}^{p}(\delta_{k}{}^{i}+h_{k}{}^{i})=\delta_{k}{}^{p},$$

where the semicolon denotes the covariant derivative in the space with the metric $\bar{g}_{ik}$; the comma denotes the partial derivative; $S_{ik}^{\,l}$ and P_{ik} are the linear and nonlinear perturbations of the Christoffel symbols and the Ricci tensor; and the bar over a symbol denotes averaging over the random phases of the perturbations or over a four-dimensional relativistically invariant volume:

$$\langle f\rangle\equiv\bar{f}=\frac{1}{\sqrt{-g}V^{(4)}}\int f(x,p)\sqrt{-g}\,dV^{(4)}. \tag{6}$$

In what follows, we assume that the fluctuating terms in (4) and (5) satisfy the condition

$$\langle h_{ik}\rangle=\langle\Phi(x, p)\rangle=0. \tag{7}$$

Using the representation (4), (5), we separate in the Christoffel symbols and the Ricci tensor the terms linear $(S_{ik}^{\,l(1)}, P_{ik}^{(1)})$ and quadratic $(S_{ik}^{\,l(2)}, P_{ik}^{(2)})$ in the amplitude of the metric perturbations:

$$S_{ik}{}^{l}=S_{ik}^{l(1)}+S_{ik}^{l(2)}, \qquad P_{ik}=P_{ik}^{(1)}+P_{ik}^{(2)}, \tag{8}$$

$$\langle P_{ik}^{(1)}\rangle=\langle S_{ik}^{l(1)}\rangle=0, \qquad \langle P_{ik}^{(2)}\rangle\neq0, \qquad \langle S_{ik}^{l(2)}\rangle\neq0. \tag{9}$$

Then, substituting (4), (5), and (8) in the system of equations (1) and (2) and restricting ourselves to terms of order h and h^2, we obtain quasilinear equations for the particle distribution function and the space-time metric:

$$p^{i}\frac{\partial\bar{F}}{\partial x^{i}}-(\bar{\Gamma}_{ik}{}^{\alpha}+\langle S_{ik}^{\alpha(2)}\rangle)p^{i}p^{k}\frac{\partial\bar{F}}{\partial p^{\alpha}}=\left\langle S_{ik}^{\alpha(1)}p^{i}p^{k}\frac{\partial\Phi}{\partial p^{\alpha}}\right\rangle, \tag{10}$$

$$\bar{R}_{i}{}^{p}\langle G_{p}{}^{k}\rangle-\frac{1}{2}\delta_{i}{}^{k}\bar{R}_{l}{}^{m}\langle G_{m}{}^{l}\rangle=\varkappa[\bar{T}_{i}{}^{p}\langle G_{p}{}^{k}\rangle+\langle G_{p}{}^{k}\tau_{i}{}^{p}\rangle]$$
$$-\langle P_{i}{}^{l}G_{l}{}^{k}\rangle+\frac{1}{2}\delta_{i}{}^{k}\langle P_{l}{}^{m}G_{m}{}^{l}\rangle, \tag{11}$$

where τ_i^p is the perturbed energy-momentum tensor of the particles.

Next, subtracting (10) and (11) from Eqs. (1) and (2), we arrive at the following system of linear equations giving a self-consistent description of the change in the particle distribution function and the fluctuations of the metric:

$$p^{i}\frac{\partial\Phi}{\partial x^{i}}-S_{ik}^{\alpha(1)}p^{i}p^{k}\frac{\partial\bar{F}}{\partial p^{\alpha}}-\bar{\Gamma}_{ik}{}^{\alpha}p^{i}p^{k}\frac{\partial\Phi}{\partial p^{\alpha}}=0, \tag{12}$$

$$P_{i}{}^{l}G_{l}{}^{k}-\langle P_{i}{}^{l}G_{l}{}^{k}\rangle+\bar{R}_{i}{}^{l}(G_{l}{}^{k}-\langle G_{l}{}^{k}\rangle)-\frac{1}{2}\delta_{i}{}^{k}\{P_{l}{}^{m}G_{m}{}^{l}-\langle P_{l}{}^{m}G_{m}{}^{l}\rangle$$
$$+\bar{R}_{l}{}^{m}(G_{m}{}^{l}-\langle G_{m}{}^{l}\rangle)\}=\varkappa\{G_{l}{}^{k}\tau_{i}{}^{l}-\langle G_{l}{}^{k}\tau_{i}{}^{l}\rangle+\bar{T}_{i}{}^{l}(G_{l}{}^{k}-\langle G_{l}{}^{k}\rangle]\}. \tag{13}$$

The properties of the system (11)–(13) were investigated in Refs. 13–15 in the framework of the hydrodynamic approximation. In this case, there is no interaction of the gravitational waves with the matter, and in the linear approximation the solution of (13) agrees with the results of Ref. 1. In the framework of the kinetic description of the interaction of collisionless particles with tensor excitations of the metric a fundamental difference from the conclusions of Refs. 1 and 13–15 arises. As can be seen from (12), the perturbation of the particle distribution function is proportional to the gradient of the background distribution function in the momentum space, and an influence of the particles on the gravitational waves is absent only when $\partial F/\partial p^\alpha=0$. In the general case, $\Phi(x,p)\propto\partial F/\partial p^\alpha$, and Eq. (10) actually reduces to a Fokker-Planck equation in the momentum space. Below, we shall consider this question in more detail for a model of the Universe that is homogeneous and isotropic on the average.

2. PARAMETERS OF THE QUASILINEAR THEORY. THE FOKKER-PLANCK EQUATION FOR THE PARTICLE DISTRIBUTION FUNCTION

It follows from (10) and (11) that the response of the particle distribution function to the collective excitations of the space-time metric is determined by the correlation functions of the gravitational-wave amplitudes. In these equations, the operation of averaging is to be understood as calculation of integrals of the type

$$\langle f\rangle=\frac{1}{L^{3}\sqrt{-\bar{g}}}\int d^{3}x\frac{1}{T}\int f(x^{\alpha},t)\sqrt{-\bar{g}}\,dt. \tag{14}$$

This operation is meaningful only when the characteristic scale of the fluctuations is appreciably less than the radius of curvature of the Universe. However, in the theory of gravitational instability[1] the scale ct of the cosmological horizon automatically appears, separating at any time t the regular and oscillator regimes. In the case $\zeta=\lambda/ct\ll1$, the tensor perturbations of the metric oscillate in time with the characteristic frequency $\omega^2\gg c^2\bar{R}$ and, therefore, satisfy the condition of stochasticity. At the same time, if we take into account in (10) and (11) the averaged squares of the fluctuation amplitudes we find only asymptotic solutions for the particle distribution function in the phase space. Making the assumption that $\bar{F}(x,p)$ depends only on the modulus of the momentum, we consider its transformation in time due to the interaction of the particles with the high-frequency gravitational waves.

We assume that the interval between events of the four-dimensional space-time is conformal to the interval of Minkowski space:

$$ds^2=a^2(\eta)(d\eta^2-dx^2-dy^2-dz^2).$$

Then for the perturbations $h_\alpha^{\,\beta}(x^\mu,\eta)$ and $\Phi(x^\alpha, p^\beta, \eta)$ we can use the Fourier representation

$$h_{\alpha}{}^{\beta}(x^{\mu},\eta)=\int h_{\alpha(\mathbf{k})}^{\beta}e^{i\mathbf{kx}}\,d^{3}k, \tag{15}$$

$$\Phi(p^{\alpha},x^{\beta},\eta)=\int\Phi_{\mathbf{k}}(p^{\alpha},\eta)e^{i\mathbf{kx}}\,d^{3}k. \tag{16}$$

Substituting (15) and (16) in (12) and introducing a spherical coordinate system in the space of wave vectors, we transform (12) to the integral equation

$$\Phi_{\mathbf{k}}(q^2,\theta,\chi,\eta)=e^{-ik\eta\cos\theta}\int d\eta' e^{ik\eta'\cos\theta}S_{ik}^{\alpha(1)}\frac{p^i p^k}{p^0}\frac{\partial \bar{F}}{\partial p^\alpha}, \quad (17)$$

where $q^2 = \gamma_{\alpha\beta}\, p^\alpha p^\beta$, and $\gamma_{\alpha\beta}$ is the metric of three-dimensional space; θ and χ are the polar and azimuthal angles of the spherical coordinate system.

Then, using (17), we reduce Eq. (10) to the form

$$\frac{\partial \bar{F}(q^2,\eta)}{\partial\eta}=\frac{\partial}{\partial p^\alpha}D^{\alpha\varepsilon}\frac{\partial \bar{F}}{\partial p^\varepsilon}, \quad (18)$$

where

$$D^{\alpha\varepsilon}=-(2\pi)^3\int d^3k e^{-ik\eta\cos\theta}\left[2S^{\alpha}_{0\beta(\mathbf{k})}p^\beta+S^{\alpha}_{\delta\gamma(\mathbf{k})}\frac{p^\gamma p^\delta}{p^0}\right]^* \times\int e^{ik\eta'\cos\theta}\left(2S^{\varepsilon}_{0\mu(\mathbf{k})}p^\mu+S^{\varepsilon}_{\mu\nu(\mathbf{k})}\frac{p^\mu p^\nu}{p^0}\right)d\eta' \quad (19)$$

is the diffusion tensor, and $S^{\delta}_{\alpha\beta(\mathbf{k})}$ is the Fourier transform of the perturbations of the Christoffel symbols.

It is obvious that the diffusion tensor $D^{\alpha\varepsilon}$ must be isotropic in the momentum space of the particles. Indeed, the appearance of anisotropy in $D^{\alpha\varepsilon}$ would signify the existence of preferred directions in the momentum space, which is incompatible with the assumed symmetry of the particle distribution function $\bar{F}(q^2, \eta)$ and the metric $\bar{g}_{ik}(\eta)$.

By virtue of the isotropy of $\bar{F}(q^2, \eta)$ the derivative $\partial\bar{F}/\partial p^\alpha$ can be represented in the form

$$\partial\bar{F}/\partial p^\alpha=s_\alpha\partial\bar{F}/\partial q,$$

with the unit vector $s_\alpha = \gamma_{\alpha\beta}\, p^\beta/q$. Then from (18)

$$\frac{\partial\bar{F}}{\partial\eta}=\frac{1}{q^2}\frac{\partial}{\partial q}q^2 D(\eta,q)\frac{\partial\bar{F}}{\partial q}, \quad D(\eta,q)=s_\alpha s_\varepsilon D^{\alpha\varepsilon}, \quad (20)$$

where the contraction of the diffusion tensor is calculated by solving the system of linear equations (12) and (13).

Thus, the interaction of the collisionless gravitating particles with the high-frequency gravitational waves is accompanied by a transformation of the distribution function in the phase space. We recall that we have obtained this result in the quasilinear approximation for the combined system of Einstein's equations and the Vlasov equation, this approximation taking into account the self-consistent variation of $\bar{F}(\eta, q^2)$ and the energy density of the gravitational waves in the expanding cosmological model. In the following section, we shall discuss this effect in the framework of random diffusion processes.[16]

3. APPROXIMATION OF RANDOM DIFFUSION PROCESSES FOR THE INTERACTION OF THE COLLISIONLESS PARTICLES WITH THE HIGH-FREQUENCY GRAVITATIONAL WAVES

Following the methods of Ref. 16, we consider the motion of the collisionless particles in the space with metric $g_{ik} = \bar{g}_{ik} + h_{ik}$ by means of the geodesic equations

$$dp^\alpha/ds-\Gamma_{ik}{}^{\alpha}p^i p^k=0, \quad (21)$$

where the perturbations h_{ik} of the metric are a random Gaussian field defined on the background $\bar{g}_{ik}(\eta)$. Taking into account the representation (8), we write (21) in the form

$$p^0 dp^\alpha/d\eta-\Gamma_{ik}{}^{\alpha}p^i p^k=S_{ik}{}^{\alpha}p^i p^k. \quad (22)$$

Introducing the particle distribution function $F(p^\alpha, \eta)$, we go over from (22) to the equation for the probability density of the distribution of the random fields p^α (Ref. 16):

$$\frac{\partial F(p^\alpha,\eta)}{\partial\eta}+\frac{\partial}{\partial p^\alpha}\{(v^\alpha+A^\alpha)F(p^\alpha,\eta)\} -\frac{\partial^2}{\partial p^\alpha\,\partial p^\beta}\{\Psi^{\alpha\beta}F(p^\alpha,\eta)\}=0, \quad (23)$$

where

$$\Psi^{\alpha\beta}=\left\langle\left(2S_{0\gamma}{}^{\alpha}p^\gamma+S_{\delta\gamma}{}^{\alpha}\frac{p^\delta p^\gamma}{p^0}\right)^*\int\left(2S_{0\delta}{}^{\beta}p^\delta+S_{\delta\varepsilon}{}^{\beta}\frac{p^\delta p^\varepsilon}{p^0}\right)d\eta'\right\rangle,$$

$$A^\alpha=\frac{\partial}{\partial p^\beta}\Psi^{\alpha\beta}, \quad v^\alpha=2\frac{a'}{a}p^\alpha.$$

It can already be seen from (23) that the function $\Psi^{\alpha\beta}$ is in fact the diffusion tensor of the particles in the momentum space and identical to the tensor $D^{\alpha\beta}$. Thus, in the framework of this approximation we arrive at a Fokker-Planck equation for $\bar{F}(p^\alpha, \eta)$ identical with (18). However, there is a fundamental difference between the quasilinear description of the interaction of the particles with the collective excitations of the space-time metric and the approximation of random diffusion processes. The point is that in the framework of the former the correlation characteristics of the random field h_{ik} are determined from the self-consistent system of equations for the gravitational waves and the perturbations of the particle distribution function. In the latter, these characteristics are specified as initial conditions. Therefore, when one is considering the interaction of collisionless particles with gravitational waves the quasilinear equation is more informative than the approximation of random diffusion processes.

4. LINEAR AND QUASILINEAR TRANSFORMATION OF THE PARTICLE DISTRIBUTION FUNCTION

To calculate the diffusion coefficient of the particles in the momentum space, we shall need to solve the system of linear equations for the perturbations of the space-time metric and the particle distribution function $\Phi(x, p^\alpha)$. In this stage, we arrive at the well-known problem of the growth of perturbations in an expanding cosmological model in the framework of the kinetic description.[4] According to Ref. 4, the amplitudes of the tensor perturbations in the high-frequency approximation $k\eta \gg 1$ have the representation

$$h_{\alpha(\mathbf{k})}^{\beta}(\eta)=C_\alpha{}^{\beta}\nu_{\mathbf{k}}(\eta)=C_\alpha{}^{\beta}\left\{\nu_{\alpha(\mathbf{k})}(\eta)-\frac{1}{k\eta^2}\frac{\partial}{\partial\eta}\eta\nu_{\alpha(\mathbf{k})}(\eta)\right\},$$
$$\nu_{\alpha(\mathbf{k})}(\eta)=\frac{1}{\eta}(c_k e^{-ik\eta}+c_k^* e^{ik\eta}), \quad (24)$$

where the coefficients C_α^β characterize the polarization of the gravitational waves and satisfy the conditions[1]

$$k^\alpha C_\alpha{}^{\beta}=k_\beta C_\alpha{}^{\beta}=0, \quad C_\alpha{}^{\alpha}=0.$$

Note that the appearance in (24) of superadiabatic damping of the amplitude of the tensor perturbations is due to the appearance of the fluctuations of the distribution function of the collisionless gravitating particles. Using (24), we find

$\Phi(p^\alpha, \eta)$ from Eq. (17). For this, we choose in the space of wave vectors a spherical coordinate system, in which the z axis is directed along the vector k, and we take into account the representation (5). As a result, we obtain

$$\Phi_k(\eta, q, \theta, \chi) = e^{-ik\eta\cos\theta} \int \left\{ C_\beta{}^\alpha \frac{p_\alpha p^\beta}{q} \nu_k'(\eta') + \frac{i\nu_k(\eta')}{2} (k_\gamma C_\beta{}^\alpha + k_\beta C_\gamma{}^\alpha - k^\alpha C_{\gamma\beta}) \frac{p_\alpha p^\beta p^\gamma}{q^2} \right\} \frac{\partial \bar{F}}{\partial q} e^{ik\eta'\cos\theta} d\eta'. \qquad (25)$$

Thus, (24) and (25) settle the linear problem of the growth of perturbations and enable us to turn to the calculation of the diffusion coefficient of the collisionless particles in the momentum space. It is worthy of note that the behavior of $\Phi(\eta, q)$ is determined by the nature of the unperturbed energy distribution of the particles. If $\partial \bar{F}/\partial q = 0$, the perturbations of the particle distribution function do not depend on h_{ik} and are determined solely by the initial conditions. We assume in what follows that for $\eta = \eta_0 \partial F/\partial q \neq 0$ it is possible to find from (20) and (24) an asymptotic ($\eta \gg \pi_0$) expression for $D^{\alpha\varepsilon}(\eta, q, \theta, \chi)$. Because this is cumbersome, we restrict ourselves to a discussion of the final expression for $D(\eta, q)$ obtained from (19) after integration over the angular variables:

$$D(\eta, q) = \frac{16}{5\eta^3} (2\pi)^4 q^2 \int_0^\infty c_k^* c_k k^2 \, dk. \qquad (26)$$

Thus, the kinetic equation for the distribution function of the collisionless particles after the change of variables

$$dy = \frac{16}{5} (2\pi)^4 \int c_k^* c_k k^2 \, dk \frac{d\eta}{\eta^3} \qquad (27)$$

reduces to Kompaneets' equation,[10] which describes the diffusion of the quanta of the background radiation in the frequency space:

$$\frac{\partial \bar{F}}{\partial y} = \frac{1}{q^2} \frac{\partial}{\partial q} q^4 \frac{\partial \bar{F}}{\partial q}. \qquad (28)$$

If for $y = 0$ the particle distribution function satisfies the equilibrium Planck or Fermi distribution

$$F(q, y=0) = \text{const} \left\{ \exp\left[\frac{c}{T} \frac{a(\eta)}{a(\eta_0)} (\gamma_{\alpha\beta} p^\alpha p^\beta)^{1/2} \right] \mp 1 \right\}^{-1},$$

the solution of Eq. (28) can be represented in the form[12]

$$F(q, y) = \frac{1}{(4\pi y)^{1/2}} \int_0^\infty \frac{dq'}{q'} F(q', y=0) \exp\left[-\frac{1}{4y} \left(\ln \frac{q}{q'} + 3y \right)^2 \right]. \qquad (29)$$

As follows from (29), the interaction of the quanta of the background radiation with high-frequency gravitational waves in a hot cosmological model shifts the spectrum of the radiation to the region $E > kT$, conserving the particle number. Such a transformation of the spectrum is analogous to the distortions[10–12] that arise as a result of the interaction of the background radiation with hot plasma electrons at red shifts $z \lesssim 10^4$–10^5. The reasons for the similarity are obvious. In the case of Compton scattering by electrons, the distortions of the radiation spectrum arise as a result of the quadratic Doppler effect: $\Delta\nu/\nu \sim kT/mc^2$. In the considered situation, the analog of the parameter kT/mc^2 is φ^2/c^4. Thus, scattering of the quanta by the fluctuations of the gravitational field leads to the extraction of energy from the high-frequency gravitational waves. Below, we shall consider some cosmological consequences of this effect.

5. BOUNDS ON THE SPECTRAL CHARACTERISTICS OF BACKGROUND GRAVITATIONAL WAVES IN THE UNIVERSE

As the collisionless subsystem of the cosmological substrate, we consider the microwave electromagnetic background after the epoch of hydrogen recombination. In the following estimates, we shall disregard the restrictions[17,18] on the amplitude of the gravitational waves on the scale $\lambda \sim ct_{\text{rec}}$ ($t_{\text{rec}} \sim$ the age of the Universe at $z = 10^3$) based on the data on the small-scale anisotropy of the microwave radiation, thereby permitting the possibility of secondary heating of the cosmological plasma. In addition, we shall assume that the contemporary matter density ρ_m does not exceed 1–3% of the critical matter density $\rho_{cr} = 4.7 \cdot 10^{-30}$ g/cm^3, taking the Hubble constant to be $H_0 = 50$ km $\cdot$ sec^{-1} $\cdot$ Mpc^{-1}. It is obvious that the level of the distortions of the microwave spectrum, which is determined in accordance with (29) by the parameter y, depends on the choice of the initial spectrum $g_0(k)$ of the gravitational waves. Omitting a discussion of the mechanisms of generation of tensor perturbations of the metric in the expanding Universe, we consider below two possible limiting cases of the behavior of $g_0(k)$: a power-law spectrum cut off at short wavelengths,

$$g_0(k) = \frac{b_0^2}{(2\pi)^4} k^m \Theta(k_{max} - k), \qquad (30)$$

and a delta-function spectrum

$$g_0(k) = \frac{c_0^2}{(2\pi)^4} \delta(k - k_0) \Theta(k_0 - k_{rec}), \qquad (31)$$

where k_{rec} is the dimensionless wave vector of a perturbation whose scale is equal to the horizon at the time of recombination; b_0^2 and c_0^2 are amplitudes; $k_{\max}$ is the cutoff parameter at short wavelengths; and $\Theta(x)$ is the Heaviside function.

Bearing in mind that at the recombination time η_{rec} there is a suppression of the short-wavelength part of the spectrum of the background gravitational waves by $(1 + k/k_{\text{rec}})^2$ times, we find from (30) and (31) the value of the parameter y at $\eta = \eta_{\text{rec}}$:

$$y = \frac{16}{5} \frac{b_0^2 k_{rec}}{|m+1|} \left\{ \left(\frac{k_{max}}{k_{rec}} \right)^{m+1} \Theta(m+1) + \Theta(-1-m) \right\}, \quad m \neq -1, \qquad (32)$$

$$y = \frac{16}{5} b_0^2 k_{rec}^4 \ln \frac{k_{max}}{k_{rec}}, \qquad m = -1$$

for the power-law spectra, and

$$y = \frac{16}{5} c_0^2 k_{rec}^4 \Theta(k_0 - k_{rec}) \Theta(k_{max} - k_0) \qquad (33)$$

for the delta-function spectra.

Following Refs. 13–15, 17, and 18, we introduce the energy density of the gravitational waves at the epoch of hydrogen recombination,

$$\varepsilon_g=\frac{c^2}{4\pi G a^2\eta^2}\int g_0(k)k^4\left(1+\frac{k}{k_{rec}}\right)^{-2}dk, \tag{34}$$

and we consider the behavior of

$$W_g=(4\pi G a^2\eta^2/c^2)\,\varepsilon_g$$

for different specifications of the initial spectrum. We obtain

$$W_g^{(1)}=\frac{b_0^2k_{rec}^{m+5}}{|m+3|}\left\{\left(\frac{k_{max}}{k_{rec}}\right)^{m+3}\Theta(m+3)+\Theta(-m-3)\right\},$$

$$m\neq-3, \tag{35}$$

$$W_g^{(1)}=b_0^2k_{rec}^2\ln\frac{k_{max}}{k_{rec}},\qquad m=-3$$

for the power-law spectra and

$$W_g^{(2)}=c_0^2k_{rec}^4\left(\frac{k_0}{k_{rec}}\right)^2\Theta(k_0-k_{rec})\Theta(k_{max}-k_{rec}) \tag{36}$$

for the delta-function spectra.

Substituting (32) in (35), we find the connection between W_g and the parameter y in the considered model. For the power-law spectra,

$$W_g(y)=y\cdot\frac{5}{32}\frac{x^2}{\ln x},\qquad m=-1,$$
$$W_g(y)=y\cdot\frac{5}{8}\ln x,\qquad m=-3, \tag{37}$$

$$W_g^{(1)}(y)=\frac{5}{16}y\left|\frac{m+1}{m+3}\right|\{\Theta(-m-3)+x^{m+3}\Theta(m+3)\Theta(-m-1)$$
$$+x^2\Theta(m+1)\}. \tag{38}$$

For the delta-function spectra

$$W_g^{(2)}(y)=\frac{5}{16}y\left(\frac{k_0}{k_{rec}}\right)^2\Theta(k_0-k_{rec})\Theta(k_{max}-k_0). \tag{39}$$

In these expressions, $x=k_{max}/k_{rec}$.

For the following estimates, we use the $\binom{0}{0}$ component of Eqs. (10) in the metric

$$\bar{g}_{ik}(\eta)=a^2(\eta)\,\mathrm{diag}(1,-1,-1,-1),$$

namely,

$$\frac{3}{a^2}\left(\frac{a'}{a}\right)^2=\frac{8\pi G}{c^2}\varepsilon_r+\frac{2}{a^2\eta^2}W_g, \tag{40}$$

where ε_r is the energy density of the electromagnetic radiation. As follows from (40), the expansion rate of the Universe is described by the well-known law $a(\eta)\propto\eta$ irrespective of the relationship between W_g and ε_r. Then, assuming that for an estimate of W_g we can take $\varepsilon_g\sim\varepsilon_r$, we find from (37) and (38) the connection (shown in Fig. 1) between the parameter y and the scale k_{max} of the cutoff of the spectrum. In Fig. 1, the broken curve shows the experimental bounds on the level of the spectral distortions of the background radiation in the Wien range taken from Ref. 19. As can be seen from Fig. 1, the parameter y in the case of a "flat" spectrum of the initial metric perturbations ($m=-3$) and $\varepsilon_g\sim\varepsilon_r$ decreases logarithmically with increasing k_{max} and agrees with the curve of the bounds only for $k_{max}=10^{14}k_{rec}$.

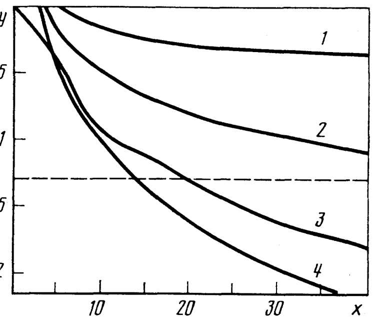

FIG. 1. Bounds on the parameter y of the distortion of the spectrum as a function of the cutoff scale $x=k_{max}/k_{rec}$ for different powers m: 1) $m=-3$, 2) $m=-2$, 3) $m=-1$, 4) $m=0$; the broken line corresponds to $y=0.05$.

For delta-function spectra of the background gravitational waves under the assumption $\varepsilon_g\sim\varepsilon_r$ the dependence $y(k_{max})$ differs little from the variant with $m=0$. We note that the assumption $\varepsilon_g\sim\varepsilon_r$ in the model with flat spectrum of the initial perturbations automatically leads to an amplitude of the gravitational waves of scale λ at the time they go below the horizon of order $h_g\sim\sqrt{y}$. For $\Omega=10^{-2}$, the value of h_g is close to the amplitude of adiabatic perturbations needed for the formation of the large-scale structural units of the matter in the Universe at such low densities of the nonrelativistic matter.

It should be emphasized that besides the bounds considered above on the spectral characteristics of the background gravitational waves we can obtain nontrivial information about the parameters of the initial state of the Universe described in the framework of the models of Refs. 20 and 21 by the Einstein-de Sitter solution

The analysis of the graviton production processes in an inflationary cosmological model made in Ref. 20 shows that the contemporary energy density of gravitons is determined by the following parameters of the maximal-symmetry solution:

$$\varepsilon_g/\varepsilon_r\approx(2/3\pi)\,\mu^2\ln x, \tag{41}$$

where $\mu=H_0l_g$, and H_0 is the reciprocal radius of curvature, which characterizes the rate of exponential expansion of the Universe, and l_g is the Planck length. The spectrum of gravitational waves obtained in the same paper corresponds to the variant we discussed earlier with $m=-3$. Taking $\varepsilon_g\approx\varepsilon_r$ in the range of wave vectors $r_{rec}<k<k_{max}$ and assuming $x\approx10^{13}$–10^{14} (see Fig. 1), we obtain bounds on the duration t_1 of the de Sitter stage:

$$H_0ct_1\geqslant3\div4.$$

In obtaining this estimate, we assumed that the spectrum of gravitational waves is cut off at $k<k_{rec}$. Otherwise the angular fluctuations in the temperature of the background radiation that develop in the approximation linear in the wave amplitude would undoubtedly exceed the observed limit.

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