Propagation of a plane front in an inhomogeneous medium

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The correction to the velocity of a plane front (phase boundary, domain wall, or soliton wall) due to inhomogeneity of the medium is calculated. It is shown that at low velocities, the dynamics of the motion of the front are significantly nonlinear at all temperatures (in contrast to the motion of a periodic structure).

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We consider the motion of a plane d-dimensional surface in a $(d+1)$-dimensional medium ($d=1,2$) with inhomogeneities. The different physical situations that lead to such a problem can be divided into two groups. To the first belong weakly inhomogeneous processes associated with the motion of various planar (linear) boundaries that arise in equilibrium thermodynamic systems (this can be a domain wall in uniaxial ferro- and antiferromagnets (FM, AFM), the interface of two phases that are close to equilibrium, a soliton wall in an almost commensurate system,[1] such as a charge density wave (CDW) or adsorbed film). The second group consists of those processes of excitation progagation in active chemical and biological systems,[2] for which the plane front of the excitation is (in a uniform medium) a stable and a well-defined object (this applies, in particular, to the process of excitation momentum transfer in heart tissue in the case of sufficiently high speed of propagation[3]). It is very important that the processes of the second group can frequently be described in complete analogy with those of the first in spite of the essential nonequilibrium nature of the active medium.

Let the surface be given by the equation $z=\zeta(\mathbf{x},t)$ where $(z,\mathbf{x})=\mathbf{R}$ is a vector in $(d+1)$-dimensional space, while, in the absence of motion and defects, $\zeta(\mathbf{x},t)=\text{const}$. Then the equation of motion (the medium is assumed to be strongly dissipative) has the form

$$\frac{1}{\Gamma_0}\frac{\partial\zeta}{\partial t}=D\nabla^2\zeta+E+h(\zeta,\mathbf{x})-\frac{\partial U(\zeta,\mathbf{x})}{\partial\zeta}+f(\mathbf{x},t) \qquad (1)$$

(we assume hereafter that the kinetic coefficient $\Gamma_0=1$). The random functions $h(\mathbf{R})$ and $U(\mathbf{R})$ are characterized by the mean values

$$\overline{h(\mathbf{R})}=\overline{U(\mathbf{R})}=0,\ \overline{h(\mathbf{R})h(\mathbf{R}')}=\gamma_h K_h(\mathbf{R}-\mathbf{R}'),$$

$$\overline{U(\mathbf{R})U(\mathbf{R}')}=\gamma_U K_U(\mathbf{R}-\mathbf{R}')$$

(the bar signifies averaging over the inhomogeneities of the medium), where the constants γ_h, γ_U are so defined that in the momentum $\mathbf{p}$ representation, $K_h(\mathbf{p})$ and $K_U(\mathbf{p})\to 1$ as $\mathbf{p}\to 0$. The first term on the right side of (1) describes the resistance of the surface to bending (it is assumed that the stability of the surface is not destroyed during the motion). The second term, which corresponds to a "force" that sets the surface in motion, has the meaning of an external magnetic field for FM, of the difference between the thermodynamic potentials of two phases for the problem of the motion of a phase interface, or of the external electric field for the almost commensurate CDW. The third term in (1) arises in the presence of defects in the medium, distinguishing its states on the opposite sides of the front (magnetic impurities in an FM, any frozen-in impurities in two-phase systems, inhomogeneities in the refractory period in a biological medium). These defects, which sense only the presence of the wall itself (non-magnetic impurities in the FM, various defects in the CDW, inhomogeneities of the diffusion coefficient in the active medium) make a contribution only to the fourth term [$U(\zeta,\mathbf{x})$ is the change in the generalized free energy of the wall, brought about by the defects]. Finally, the last term in Eq. (1) describes thermodynamic random forces defined by the correlator.

$$\langle f(\mathbf{x},t)f(\mathbf{x}',t')\rangle=2T\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$$

(such forces are also present in active media, but the parameter T is not identical with the temperature).

Our problem will consist in the calculation of the front-velocity correction necessitated by the defects. We shall use the method developed by Larkin and Ovchinnikov for the description of the motion of a vortex lattice in superconductors[4] (a similar calculation for strongly incommensurate CDW can be found in Ref. 5). The difference of our problem from those mentioned above is that the random fields $h(\zeta,\mathbf{x})$ and $U(\zeta,\mathbf{x})$ contain the entire spectrum of harmonics in the variable ζ:

$$h(\zeta,\mathbf{x})=\int h_q(\mathbf{x})e^{iq\zeta}\frac{dq}{2\pi},\quad U(\zeta,\mathbf{x})=\int U_q(\mathbf{x})e^{iq\zeta}\frac{dq}{2\pi}. \qquad (2)$$

This leads to interesting temperature effects.

We begin with the case $T=0$. Substituting

$$\zeta(\mathbf{x},t)=vt+\zeta_1(\mathbf{x},t),\ (\overline{\zeta_1(\mathbf{x},t)}=0)$$

in (1) and (2) we obtain (similar to Refs. 4 and 5) the first correction to the velocity in the form

$$v-E=\int dt\int\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{dq}{2\pi}[\gamma_h K_h(q,\mathbf{k})+q^2\gamma_U K_U(q,\mathbf{k})]e^{iqvt}iqG_0(\mathbf{k}t), \qquad (3)$$

where the response function is

$$G_0(\mathbf{k},t)=\theta(t)\exp(-D\mathbf{k}^2t).$$

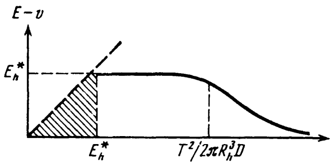

FIG. 1.

Let the correlators $K_h(\mathbf{p})$ and $K_U(\mathbf{p})$ $[\mathbf{p}=(\mathbf{q},\mathbf{k})]$ differ significantly from 1 as $p\sim R_h^{-1}$, R_U^{-1}, respectively (in the case of defects that are small in scale in comparison with the front thickness, ξ, we have $R_h, R_u\sim\xi$; in the opposite case, R_h, and R_U are determined by the scale of the corresponding defects). Then, under the condition $v\ll DR_{h,U}^{-1}$, which is nonrestrictive for weak disorder, we can neglect the dependence of $K_{h,U}$ on $\mathbf{k}$, and the integrals over $\mathbf{k}$ and t in (3) can easily be calculated, while the principal contribution to the integral over q, is made by the region $q\sim R_h^{-1}, R_U^{-1}$, since, by virtue of the analyticity of $K_{h,U}(\mathbf{R})$ in $\mathbf{R}$ at $R\ll R_{h,U}$, we have

$$K_{h,U}(\mathbf{p})\sim 0(\exp(-pR_{h,U})),\quad pR_{h,U}\gg 1.$$

As a result we obtain

$$v=E-D^{-d/2}v^{(d-2)/2}[a_h\gamma_hR_h^{-(d+2)/2}+a_U\gamma_UR_U^{-(d+6)/2}];\qquad a_{h,U}\sim 1. \tag{4}$$

As is seen from (4), the contributions of the defects of the h- and U-type have essentially different dependences on their scales. It appears that the U defects can be significant only when the existence of h defects is forbidden by the symmetry. Equation (4) is valid at $(E-v)/v\ll 1$, i.e., at $E\gg E$, where the critical field is

$$E_c\sim D^{-d/(4-d)}(\gamma_hR_h^{-(d+2)/2}+\gamma_UR_U^{-(d+6)/2})^{2/(4-d)}.$$

At $E<E_0$, motion become impossible, and a pinning of the front on the defects takes place (similar to the cases in Refs. 4 and 5).

We now consider the temperature effects and limit ourselves to the case of a linear front in a plane medium ($d=1$), when these effects are most important. The expression for the correction to the velocity at $T\neq 0$ differs from (3) only (under the condition $v\ll DR_{h,U}^{-1}$) by the appearance in the integrand of the factor

$$S_q(t)=\langle\exp iq(\zeta_2(t)-\zeta_2(0))\rangle,$$

where $\zeta_2(x,t)$ are the thermal fluctuations of the front. Calculation of $S_q(t)$ is carried out in trivial fashion and gives

$$S_q(t)=\exp\left[-Tq^2\left(\frac{2}{\pi D}\right)^{1/2}t^{1/2}\right].$$

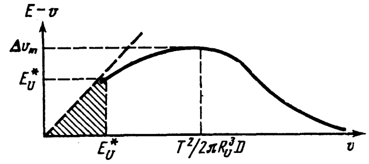

FIG. 2.

Substituting $S_q(t)$ under the integral (3) and integrating over t and k ($d=1$), we obtain

$$v-E=-\int_0^\infty\frac{dq}{2\pi}(q/v)^{1/2}\left(\frac{\pi}{2}\right)^{1/2}\times[\gamma_hK_h(q)+q^2\gamma_UK_U(q)]F\left[\frac{Tq^{3/2}}{(2\pi vD)^{1/2}}\right], \tag{5}$$

where

$$F(z)=\cos z^2+\sin z^2-2(\cos z^2C(z)+\sin z^2S(z)),$$

while $C(z)$ and $S(z)$ are the Fresnel integrals. At $T^2R_{h,U}^{-3}\ll 2\pi vD$ the difference of $F(z)$ from 1 is significant and we return to Eq. (4). In the opposite limiting case $T^2R_{h,U}^{-3}\gg 2\pi vD$ we can replace $K_{h,U}(q)$ by 1; as a result, we obtain

$$v=E-\left[\frac{\pi^{1/2}}{6\sqrt{2}}\gamma_h\frac{D^{1/2}}{T}+\alpha v^{2/3}\gamma_U\frac{D^{7/6}}{T^{1/3}}\right],$$

$$\alpha=\frac{\pi^{1/6}\Gamma^2(1/3)}{2^{7/6}3^{2/3}}\approx 0{,}75. \tag{6}$$

Thus, the impurity correction to the velocity decreases significantly at high temperature; however, it falls off as $v\to 0$, more slowly than v, as before. This means that at $E\lesssim E_{h,U}^*$ where

$$E_h^*\approx 0{,}2\gamma_hD^{1/2}/T;\quad E_U^*\approx 0.4\gamma_U^3D^{7/2}/T^7, \tag{7}$$

a transition takes place to another, significantly nonlinear dynamical regime. Thus, the result (6) is applicable at $E_{h,U}^*\ll v\ll T^2R_{h,U}^{-3}/2\pi D$, i.e., the region of its applicability exists at $T\gg T_{h,U}^*$, where $T_h^*=\gamma_h^{1/3}D^{1/2}R_h$, $T_U^*=\gamma_U^{1/3}D^{1/2}\times R_u^{1/3}$. The dependence of $E-v$ on v in this range of temperatures for the cases of h and U defects are shown respectively in Figs. 1 and 2 ($\Delta v_m=(E_U^*)^{1/3}(T^2/2\pi R_U^3D)^{2/3}$).

We emphasize that the region of nonlinear dynamics in weak fields (the shaded regions on the diagrams) exists at any T; this is connected with the presence of such arbitrarily small vectors q in the spectrum of random forces $h(\zeta,x)$ and $U(\zeta,x)$. For comparison, we show that in the case of a strongly incomensurate CDW $(K(q,k)=1/2\delta(q+q_0)+1/2\delta(q-q_0))$ at $T\gg D^{1/3}\gamma^{1/3}q_0^{-1}$ the region of significantly nonlinear behavior disappears and the impurity corrections at $E\to 0$ remain small. In conclusion, we note [see (6), (7)] that the h defects affect the dynamics of the front at the high temperatures much more strongly than the U defects.

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