

# Low-temperature phase of an Ising spin glass

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A three-dimensional model of an Ising spin glass with an oscillatory long-range interaction is analyzed. There is a marginally stable low-temperature phase, which has nothing in common with the Edwards-Anderson phase. In particular, the magnetic susceptibility corresponding to complete equilibrium is  $\chi = c/T$  at all  $T$ .

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1. One of the basic questions in the theory of spin glasses is whether there exists a thermodynamically equilibrium state different from a paramagnetic state. The Sherrington-Kirkpatrick model,<sup>1</sup> which has recently been studied quite thoroughly,<sup>2-5</sup> cannot answer this question since several of its properties differ from those of the real system; specifically, its behavior is nonergodic,<sup>4,5</sup> which means that as the size of the system becomes infinite the phase space of the system breaks up into regions separated by infinitely high energy barriers. A thermodynamic average in this model is equivalent to an average over a small part of the phase space. In a real system with a finite interaction range all the energy barriers are of finite height, so that a thermodynamic average calculated in the Sherrington-Kirkpatrick model has absolutely nothing in common with the equilibrium properties of the real physical system. In the present paper we suggest an alternative three-dimensional model for a spin glass with a large but finite interaction range. A corresponding one-dimensional model was analyzed in Ref. 6.

In classical spin glasses (dilute solutions such as  $\text{Cu}_{1-x}\text{Mn}_x$  and  $\text{Au}_{1-x}\text{Fe}_x$ ) the atoms of magnetic impurities interact with each other by the RKKY law:

$$H = \frac{1}{2} \sum_{i,j} s_i s_j V(r_i - r_j), \quad (1)$$

$$V(r) = V_0 r^{-3} e^{-r/l} \cos 2p_0 r;$$

where  $r$  is the distance between magnetic impurities,  $p_0$  is the Fermi momentum of the electrons in the metal matrix, and  $l$  is the electron mean free path.

We seek the average field exerted on a spin by its neighbors:

$$\langle h^2 \rangle_j \sim \sum_i V^2(r_i - r_j).$$

For interaction (1), this sum is dominated by the nearest neighbors, of which there are  $N \sim 10$  in a real three-dimensional system. We alter the form of the interaction in such a way that  $\langle h^2 \rangle$  is determined primarily by the region  $r \sim \chi^{-1} \gg c^{-1/3}$  ( $c$  is the spin density):

$$V = V_0 (\chi p_0 / 2\pi r) e^{-\chi r} \sin p_0 r. \quad (2)$$

It can be hoped that this interaction will be a valid approximation of (1) of order  $1/N$ .

Furthermore, there are real substances in which magnetic impurities do interact in accordance with (2). These are

primarily rare earth elements in which a helicoidal magnetic structure<sup>7</sup> can form with a direction which is not fixed with respect to the lattice axes.<sup>8</sup> Above the point of the helicoidal magnetic transition,  $T_h$ , impurity atoms having a localized magnetic moment (for example, the atoms of a transition metal or of another rare earth metal) interact with each other through the exchange of virtual magnetic excitations. In the range of applicability of the Landau theory the field of the virtual excitations has the free energy<sup>7</sup>

$$F = \Theta \{ \alpha^{-1} [ (\nabla^2 + p_0^2) \Phi ]^2 + \tau \Phi^2 + \Phi^4 \}, \quad (3)$$

$$\Theta \sim T_h, \quad \alpha \sim (T_h / \epsilon_F)^{1/2}, \quad \tau = (T - T_h) / T_h.$$

In the presence of impurities, the phase transition occurs at a temperature above  $T_h$ :  $\tau_c \sim c a^3$  ( $c$  is the impurity density, and  $a$  is the lattice constant). At  $\tau_c \gg (p_0 a)^2 (\Theta / \epsilon_F)^{1/6}$ , the fluctuational renormalization of the spectrum of the field  $\Phi$  can be ignored, so that the interaction between impurity atoms takes the form in (2).

The physical situation is slightly different in substances such as Y and Sc (Refs. 9–12). In their pure state these substances do not form a helicoidal structure, but at low temperatures the spectrum of their magnetic excitations has a significant gap at a finite momentum. It has been asserted<sup>10</sup> that this gap is nearly symmetric with respect to momentum direction. At low densities of magnetic impurities, helicoidal structures can form in these substances.<sup>9</sup> It has been shown experimentally<sup>11,12</sup> that at low densities of Er, Gd, and Tb a phase with properties reminiscent of those of a classical spin glass can form; in the Er case, the spin glass is an Ising spin glass.<sup>12</sup>

In this paper we examine the low-temperature phase of a system of Ising spins which interact with each other in accordance with (2). The basic idea of our approach is to single out the slow dynamic variables of the amplitude and phase type which describe the spin configuration. We wish to emphasize that we are singling out these variables at the level of the spins  $s_i$ , not at the level of their averages  $\overline{\langle s_i \rangle^2}$ , as in the Edwards-Anderson approach.<sup>13,1-5</sup>

2. To calculate the partition function over the configurations of the Ising spins it is convenient to introduce an auxiliary field variable: the complex field  $\Psi(x)$ , whose real part is the molecular field acting on the spin at the point  $x$ ,

$$Z = \sum_{\{s\}} \exp(-H(s)/T)$$

$$= \int \sum_{(s)} \exp \left\{ -\frac{V_0}{T} \left[ \frac{1}{4\kappa^2 p_0^2} |(\nabla^2 + p_0^2) \Psi|^2 + |\Psi|^2 \right] + S_i [\mathcal{H} + \text{Re} \Psi(x_i)] / T \right\} D\Psi, \quad (4)$$

where  $\mathcal{H}$  is the external magnetic field. Summing over  $s$ , we find

$$Z = \int \exp[-H(\Psi)] D\Psi, \quad (5)$$

$$H(\Psi) = \frac{V_0}{T} \left\{ \frac{1}{4\kappa^2 p_0^2} |(\nabla^2 + p_0^2) \Psi|^2 + |\Psi|^2 - \sum_i \delta(x-x_i) \times \ln \text{ch} [V_0 (\text{Re} \Psi(x_i) + \tilde{\mathcal{H}}) / T] \right\}, \quad \tilde{\mathcal{H}} = \mathcal{H} / V_0.$$

The summation in (5) is over the random positions ( $x_i$ ) of the spins. In the average-field approximation the state  $\langle \Psi \rangle = 0$  becomes unstable at  $T_c = cV_0$ . We restrict the discussion below to temperatures  $T \sim T_c$ ; in this temperature range the logarithm of the hyperbolic cosine in (5) can be replaced by the first two terms of its expansion:

$$H(\Psi) = \frac{1}{c} \left\{ \frac{1}{4\kappa^2 p_0^2} |(\nabla^2 + p_0^2) \Psi|^2 + \tau |\Psi|^2 + \frac{1}{2c^2} |\Psi|^4 + \sum_i \delta(x-x_i) \text{Re} \Psi^2 \right\}, \quad (6)$$

where  $\tau = T/T_c - 1$  satisfies  $\tau \lesssim 1$ . Two distinct low-temperature phases can form in this system: a  $Q$  phase (an Edwards-Anderson phase) and a  $\Psi$  phase. This question has been studied in detail elsewhere.<sup>14</sup> Let us determine the structure of each of these phases. The  $\Psi$  phase corresponds to the formation of a condensate in the form of a wave  $\Psi_0 = \rho \exp(i\mathbf{p}_0 \mathbf{r} + i\varphi)$ , where  $\varphi$  is a slow variable. The mean-field theory gives us  $\varphi = \text{const}$ . To determine the structure of the  $Q$  phase we take an average of the free energy over the positions of the impurities by the replica method. We find the effective Hamiltonian

$$H(\Psi) = \frac{1}{c} \sum_a \left\{ \frac{1}{4\kappa^2 p_0^2} |(\nabla^2 + p_0^2) \Psi_a|^2 + \tau |\Psi_a|^2 + \frac{1}{2c^2} |\Psi_a|^4 \right\} - \sum_{a,b} \frac{1}{2c^2} \text{Re} \Psi_a^2 \text{Re} \Psi_b^2. \quad (7)$$

To continue we need to assume that  $\gamma = p_0^2 \kappa / 4\pi c$  is small, in the approach of Ref. 14. Under this condition we can discard all types of perturbation-theory diagrams at  $\tau \gg \gamma^{2/3}$ , while at  $\tau \lesssim \gamma^{2/3}$  we need retain only the ladder diagrams. The latter lead to an instability of the paramagnetic phase at  $\tau = \gamma^{2/3}$ . As in the theory of superconductivity, this instability gives rise to an order parameter  $Q_{ab}$ , which is the adjoint of  $\text{Re} \Psi_a \text{Re} \Psi_b$ , and  $u_a$ , which is the adjoint of  $(\text{Re} \Psi_a)^2$ .

The  $Q$  phase which arises here is analogous<sup>14</sup> to the Edwards-Anderson spin glass. As the temperature is lowered into the region  $\tau < 0$ ,  $|\tau| \gg \gamma^{2/3}$ , however, the  $\Psi$  phase becomes preferred from the energy standpoint (in the approximation of an average field we would have  $F_\Psi = -c^2 \tau^2 / 2 < F_Q = -c^2 \tau^2 / 4$ ); at  $-\tau \sim \gamma^{2/3} \ll 1$ , there is accordingly a first-order transition from the  $Q$  phase to the  $\Psi$  phase. The

condition  $\gamma \ll 1$  is very important for the occurrence of this first-order transition; this condition is apparently never satisfied in a real spin glass. Furthermore, as the high-temperature series show,<sup>15</sup> a theory with an order parameter  $Q_{ab}$  has a lower critical dimensionality greater than three, so that the  $Q$  phase is not an actual thermodynamic phase in a three-dimensional system. On the other hand, the only condition required for the existence of the low-temperature  $\Psi$  phase (as shown below) is the far weaker condition  $T p_0 \kappa^2 / c \ll 1$ .

At  $\gamma \gtrsim 1$  the mean-field theory, which leads to the conclusion that there is a  $Q$  phase, has absolutely no range of applicability. It can thus be assumed that the transition from the paramagnetic phase to the  $\Psi$  phase occurs in two steps, as does the analogous transition<sup>16</sup> from an isotropic liquid to a nematic liquid crystal to a smectic  $A$  liquid crystal, which is described by Hamiltonian (7) without the average fields. In terms of  $\Psi$ , the smectic  $A$  state means that we have  $\Psi = \rho \exp(i\mathbf{p}_0 \mathbf{x} + \varphi)$ , where  $\varphi$  is a slowly changing variable. At high temperatures, dislocations of the field  $\varphi$  arise. (The field  $\varphi$  is defined accurate to  $2\pi$ .) In this state (the nematic state) we have  $\langle \cos(\varphi(x) - \varphi(0)) \rangle \sim \exp(-x)$ , but the quantity  $\Psi + \nabla \Psi$  is still a good order parameter. At even higher temperatures there is a transition to an isotropic liquid, and all the  $\Psi$  correlation functions begin to fall off exponentially with distance. The smectic  $A$  state corresponds to the  $\Psi$  state in the system under consideration in the present paper.

At low temperatures the system contains only closed dislocation loops of finite length  $l$ . At distances greater than  $l$  the system can be described in terms of the phase  $\varphi$ :  $\Psi = \rho \exp i\varphi$ . It is natural to suggest that the same qualitative picture is correct when the random term (that which is not diagonal in the replica indices) is retained in Hamiltonian (7). The following phase Hamiltonian is defined over distances greater than the dimension of the dislocation loops,  $l$ :

$$H(\varphi) = \sum_a \frac{1}{2t} \left\{ \frac{A}{4} [(\nabla \varphi)^2 - r]^2 + B(\nabla^2 \varphi)^2 \right\} + \sum_{a,b} v \cos \varphi_a \cos \varphi_b, \quad (8)$$

$$r = A = B = 1.$$

(Here we have changed variables:  $x \rightarrow x/2p_0$ ,  $\varphi \rightarrow \varphi/2$ .) Under the conditions  $\gamma \ll 1$ ,  $\tau \gg \gamma^{2/3}$  the fluctuations of the modulus of  $\Psi$  can be ignored, and Hamiltonian (8) follows from Hamiltonian (7), with

$$t = p_0 \kappa^2 / 2c\tau, \quad v = c\tau^2 / 16p_0^3. \quad (9)$$

The minimum characteristic dimension  $l$  over which (8) holds is the dimension over which we may ignore the fluctuations of  $\rho$ . The Green's function of these fluctuations is

$$G_\rho^{-1} = (p^2 - p_0^2)^2 / 4p_0^2 \kappa^2 + \tau,$$

so that  $l = p_0 / \kappa \tau^{1/2}$ . For  $\gamma \gtrsim 1$  the quantities  $t$ ,  $v$ , and  $l$  can be expressed in terms of the unrenormalized parameters by means of scaling indices:

$$t = p_0 \kappa^2 / c\tau^\mu, \quad v^{1/2} = c\tau^\sigma, \quad l = p_0 \kappa^{-1} \tau^{-\nu}. \quad (10)$$

Unfortunately, the scaling indices (9) differ from those for the transition described by the part ( $H_0$ ) of Hamiltonian

(7) which is diagonal in the replica indices. To show this, we assume the opposite, i.e., that the random part of Hamiltonian (7) does not influence the indices. We use Toner's assertion<sup>16</sup> that  $H_0$  belongs to the same universality class as the  $XY$  model. In this case we have  $\mu = \nu = 2/3$  and  $\sigma = 0.8$  (the index  $\sigma$  can be expressed<sup>17</sup> in terms of the scaling dimensionality of the irreducible tensor  $S_\alpha S_\beta - \delta_{\alpha\beta} S^2$ , where  $S_\alpha$  is the order parameter of the  $XY$  model; see Ref. 18). The dimensionless quantity which characterizes the random potential is  $\nu l^4 \sim \tau^{-1.1} \sim l^{1.5}$ . We see that the random potential increases with increasing characteristic dimension and is therefore an important variable, whose incorporation alters the transition indices. The behavior of the system at large distances is determined in one way or another by Hamiltonian (8). Let us examine this behavior for  $\nu l^4 \ll 1$ . Although the condition  $\nu l^4 \sim 1$  apparently always holds in a real Ising glass, we may hope that a universal behavior established for  $\nu l^4 \ll 1$  will also hold at  $\nu l^4 \sim 1$ . Furthermore, Hamiltonian (8) with  $\nu$  as an adjustable parameter arises in several other problems, in particular, in a description of a Heisenberg spin glass with spin-orbit interactions. The approach of the present paper can also be used for a Heisenberg spin glass (and also a planar spin glass), but the results turn out to be very different for all three types of spins.

3. The part of Hamiltonian (8) which is diagonal in the replica indices has been studied by Grinstein and Pelcovits.<sup>19</sup> It is the same as the Hamiltonian which describes a smectic  $A$  liquid crystal. Although the coefficients  $A$  and  $B$  have the dimensionality of length, it can be shown that the parameters of the perturbation theory are the dimensionless quantities  $Ar^{1/2}$  and  $Br^{1/2}$ . Grinstein and Pelcovits<sup>19</sup> showed that after a renormalization and a switch to large characteristic dimensions the Hamiltonian (8) remains the same in form, while the coefficients  $A$  and  $B$  are logarithmically renormalized:

$$A = (1 + 5t\xi/64\pi)^{-\nu}, \quad B = (1 + 5t\xi/64\pi)^{\nu}, \quad \xi = \ln x. \quad (11)$$

Let us examine the behavior of the random  $\nu$  term in the course of the renormalization as it was carried out in Ref. 20, where a study was made of a two-dimensional sine-Gordon model. We break  $\varphi$  up into two parts:  $\varphi = \varphi_0 + \tilde{\varphi}$ , where  $\varphi_0$  is the "slow" field, i.e., that which has only momenta smaller than  $l^{-1}\xi^{-1}$  in the momentum representation, and  $\tilde{\varphi}$  is the "fast" field. We integrate the partition function over  $\tilde{\varphi}$ , using as the zeroth approximation the Hamiltonian  $H_{\tilde{\varphi}}$ :

$$H_{\tilde{\varphi}} = \frac{1}{2t} \{A (\nabla\varphi_0 \nabla\tilde{\varphi})^2 + B (\nabla^2\tilde{\varphi})^2\}. \quad (12)$$

In first order in  $\nu$  we find, after taking an average over  $\tilde{\varphi}$ ,

$$\langle \exp [ \nu \cos \varphi_a \cos \varphi_b ] \rangle_{\tilde{\varphi}} = \exp [ \nu \cos \varphi_a^0 \cos \varphi_b^0 (1 - \langle \varphi_a^2 \rangle / 2 - \langle \varphi_b^2 \rangle / 2) ] = \exp \{ \nu [ 1 - \xi t / 4\pi (AB)^{1/2} ] \cos \varphi_a^0 \cos \varphi_b^0 \}. \quad (13)$$

As we will see below, the parameter of the perturbation theory is not  $\nu$  but  $\tilde{\nu} = \nu (Ar)^{1/4} B^{-3/4}$ : a quantity with a normal dimensionality of four. We thus find a renormalization-group equation for  $\tilde{\nu}$ :

$$d\tilde{\nu}/d\xi = 4\tilde{\nu} [ 1 - \tilde{\nu} / (ABr^2)^{1/2} - \tilde{\nu}^2 / 8 \tilde{\nu} A^{1/2} B^{-3/2} r^{-1/2} ], \quad \tilde{\nu} = t / 16\pi. \quad (14)$$

The last term in square brackets falls off  $\propto \xi^{-1}$ , so that it is unimportant to the discussion below. The expression  $\tilde{\nu} (ABr)^{1/2}$  increases  $\propto \xi^{1/5}$ , so that  $\nu$  falls off at sufficiently large characteristic dimensions; a random potential of this type therefore decays.

We will now calculate the renormalization of the effective Hamiltonian in second order in  $\nu$ , and we will see that this renormalization generates some new and "dangerous" random terms. Expanding the partition function we find that we must take an average of the following expression over  $\tilde{\varphi}$ :

$$\frac{\nu^2}{2} \sum_{a,b,c,d} \int [\langle \cos \varphi_a(\mathbf{x}) \cos \varphi_b(\mathbf{x}) \cos \varphi_c(\mathbf{x}') \cos \varphi_d(\mathbf{x}') \rangle_{\tilde{\varphi}} - \langle \cos \varphi_a(\mathbf{x}) \cos \varphi_b(\mathbf{x}) \rangle_{\tilde{\varphi}} \langle \cos \varphi_c(\mathbf{x}') \cos \varphi_d(\mathbf{x}') \rangle_{\tilde{\varphi}}] d\mathbf{x} d\mathbf{x}'. \quad (15)$$

After the average is taken over  $\tilde{\varphi}$ , the expression summed in (15) must be of the form  $(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})I_1 + (\delta_{ac} + \delta_{ad} + \delta_{bc} + \delta_{bd})I_2 + I_3$ , where the  $I_{1,2,3}$  do not contain replica indices. The summation leaves us with only the term  $I_1$ . We evaluate the integral in (15) for various relations among  $a, b, c$ , and  $d$ , and we write it in the form of a sum of this type. The averaging over  $\tilde{\varphi}$  must be carried out with Hamiltonian (12). The Green's function of the field  $\tilde{\varphi}_a$  is

$$G_{ab} = t [k^2 Q_a^2 + q^4]^{-1} \delta_{ab}, \quad Q_a = \nabla \varphi_a^0. \quad (16)$$

Here  $k$  and  $q$  are the components of the momentum  $\mathbf{p}$  which are respectively longitudinal and transverse with respect to the vector  $\mathbf{Q}_a$ . We can now carry out the averaging in (15). We add to the effective Hamiltonian an increment  $H_1$ , given by

$$H_1 = \frac{\nu^2}{4} \sum_{a,b} \int \cos [Q_a(\mathbf{x} - \mathbf{x}')] \cos [Q_b(\mathbf{x} - \mathbf{x}')] d^2\mathbf{x} d^2\mathbf{x}' \times \{ \exp [G_{aa}(\mathbf{x} - \mathbf{x}')] - 1 \}^2 \exp [ -G_{aa}(0) - G_{bb}(0) ]. \quad (17)$$

A contribution of this type to the Hamiltonian substantially changes the Green's functions. To determine the new Green's functions, we vary  $H_1$  twice with respect to  $\mathbf{Q}_a$  and  $\mathbf{Q}_b$ . Retaining only those terms which are not diagonal in the replica indices, we find

$$\frac{\delta^2 H_1}{\delta Q_a^i \delta Q_b^j} = S_1(\mathbf{Q}) \delta_{ij} + S_2(\mathbf{Q}) Q_a^i Q_b^j + S_3(\mathbf{Q}) Q_a^i Q_b^i.$$

Below we will need the value of  $\delta^2 H / \delta Q_a \delta Q_b$  only at  $\mathbf{Q}_a = \mathbf{Q}_b$ . In this case we have

$$S_1 = r\hbar^2 = \frac{\nu^2}{8} \int [x^2 - (\mathbf{x}\mathbf{Q})^2] \{ \exp [G(\mathbf{x})] - 1 \}^2 \exp [ -2G(0) ] d^2\mathbf{x},$$

$$S_2 + S_3 = \frac{\nu^2}{4} \int (\mathbf{x}\mathbf{Q})^2 \{ \exp [G(\mathbf{x})] - 1 \}^2 \exp [ -2G(0) ] d^2\mathbf{x}.$$

The integral determining  $S_2 + S_3$  diverges logarithmically at  $\tilde{\nu} = 1$  (i.e., at the point with  $dv/d\xi = 0$ ) and is  $\nu^2 l^8 (Ar/B)^{3/2} \xi$ . Terms of the type  $S_2$  (or  $S_3$ ) lead to a correction

$$G_2 = t [k^2 Q_a^2 + q^4]^{-2} k^2 S_2$$

to the Green's function. This correction has in the limit  $k \propto q^2 \rightarrow 0$  a singularity of the same type as that in  $G$ , so that we can determine the relative magnitude of the correction  $H_1$ . Comparing  $G_2$  and  $G$ , we find that the effective charge is  $\tilde{v} = v l^4 (Ar)^{1/4} B^{-3/4}$ , as was assumed in the derivation of Eq. (14).

For  $\tilde{t} \approx 1$ , the integral in the expression for  $h^2$  is determined by short distances; it yields

$$h^2 \approx v^2 l^2 (Ar/B)^{1/2}.$$

The expressions giving  $S_i$  in terms of  $v$  are valid only for  $\tilde{t} \approx 1$ ,  $v^{-1} dv/d\xi \ll 1$ , but their qualitative consequences should hold over the entire region  $\tilde{t} \lesssim 1$ .

The terms of the type  $S_1$  make a contribution to the Green's function which is considerably more singular than that from  $S_{2,3}$ :

$$G_i = t [rk^2 + q^4]^{-2} q^2 h^2 r.$$

This singular increment leads to highly divergent diagrams, i.e., to a significant change in the properties of the system at long range, while terms of the type  $S_2$  and  $S_3$  do not alter the logarithmic nature of the theory. In general,  $H_1$  consists of the sum of Legendre polynomials of index  $2n$  of  $\mathbf{Q}_a \mathbf{Q}_b$ ; each polynomial corresponds separately to a random-anisotropy field of order  $2n$ . The detailed structure of this series is unimportant for our purposes, since the Green's function is determined exclusively by  $(\delta^2 H_1 / \delta Q_a \delta Q_b)_{Q_a = Q_b}$ . It is the dependence of  $H_1$  on the angle between  $\mathbf{Q}_a$  and  $\mathbf{Q}_b$  which gives rise to the diagrams which are highly divergent at small momenta. We can thus set  $\mathbf{Q}_a^2 = \mathbf{Q}_b^2 = 1$  in  $H_1$ ; furthermore, as we will see below, it is sufficient to retain only the first term in the expansion of  $H_1$  in  $(\mathbf{Q}_a - \mathbf{Q}_b)^2$ , so we will examine the Hamiltonian

$$H = \frac{1}{2t} \sum_a \left\{ \frac{A}{4} ((\nabla \varphi_a)^2 - r)^2 + B (\nabla^2 \varphi_a)^2 \right\} + \frac{r h^2}{2} \sum_{ab} (\nabla \varphi_a - \nabla \varphi_b)^2, \quad (18)$$

where  $h^2 = v^2 l^6 (Ar/B)^{1/2}$ .

4. The theory described by Hamiltonian (18) is logarithmic in five-dimensional space and exhibits a zero-charge behavior in it. We are thus forced to resort to the  $\varepsilon$ -decomposition method. We carry out a renormalization in the five-dimensional space. We break up the field into a fast component  $\tilde{\varphi}$  and a slow component  $\varphi_0$ :  $\varphi = \varphi_0 + \tilde{\varphi}$ ,  $\nabla \varphi_0 = \mathbf{Q}$ . The terms linear in  $\tilde{\varphi}$  can be discarded since  $\varphi$  carries a large momentum, and these terms contribute nothing to the integration over  $\tilde{\varphi}$ . In the single-loop approximation it is sufficient to retain terms quadratic in  $\tilde{\varphi}$ . (The higher orders in  $\tilde{\varphi}$  necessarily lead to more loops.)

Collecting the necessary terms, we have

$$H_{\tilde{\varphi}} = \sum_a \frac{1}{2t} \left\{ A (\nabla \tilde{\varphi}_a \mathbf{Q}_a)^2 + \frac{1}{2} A (\nabla \tilde{\varphi}_a)^2 (\mathbf{Q}_a^2 - 1) + B (\nabla^2 \tilde{\varphi}_a)^2 \right\} - \sum_{a,b} (\nabla \tilde{\varphi}_a \Delta \tilde{\varphi}_b) h^2 r. \quad (19)$$

To check the renormalization procedure we must evaluate

$$H_2(\mathbf{Q}) = \ln \int D\tilde{\varphi} \exp(-H_{\tilde{\varphi}})$$

in second order in  $\tilde{\varphi}$ . For the vectors  $\mathbf{Q}_a$  which remain constant over space the functional integral over  $\tilde{\varphi}$  can be evaluated immediately; it makes the following contribution to  $H_2$ :

$$H_2^{(1)} = - \sum_a \frac{A^2}{16} (\mathbf{Q}_a^2 - r)^2 \int G^3(\mathbf{p}) q^6 \frac{d^3 \mathbf{p}}{(2\pi)^3} = - \sum_a \frac{A}{8t} (\mathbf{Q}_a^2 - r)^2 6w \xi, \quad (20)$$

$$w = t^2 h^2 (Ar)^{1/2} B^{-3/2} / 128 \pi^2.$$

To determine the correction to the  $B (\nabla^2 \varphi)^2$  term, we must allow slow variations of  $\mathbf{Q}$ . We therefore evaluate  $\delta^2 H_2 / \delta Q_a(\mathbf{x}) \delta Q_b(\mathbf{x}')$ . This expression is given by a diagram (see Fig. 2 below) which diverges quadratically at large momenta. For our purposes, however, we are interested in its second derivation with respect to the momentum ( $q$ ) which passes through it, so that the increment in  $H_2$  is of the form  $(\nabla \mathbf{Q}_a)^2$ . This quantity diverges logarithmically. Evaluating it, we find the final expression for  $H_2$ :

$$H_2 = \sum_a \left\{ - \frac{A}{8t} (\mathbf{Q}_a^2 - r)^2 6w + \frac{1}{2t} B (\nabla \mathbf{Q}_a)^2 w \right\} \xi. \quad (21)$$

In the single-loop approximation, we might point out, there are no diagrams which renormalize  $S_1$ . Furthermore, no such diagrams at all are seen in the lower orders of perturbation theory, so we will assume  $dh^2/d\xi = 0$ . In a space of dimensionality  $d < 5$  this result means that  $h^2$  does not have an anomalous dimensionality. The equations of the renormalization group for  $A$  and  $B$  follow from a comparison of  $H_2$  in (21) and  $H$  in (18). We write them immediately in a space of dimensionality  $5 - \varepsilon$ :

$$dA/d\xi = -6wA, \quad dB/d\xi = wB, \quad (22)$$

$$dw/d\xi = \varepsilon w - 11/2 w^2.$$

In the derivation of the latter equation we made use of the circumstance that the normal dimensionality of  $w$  is  $\varepsilon$ , as follows from the scale transformation

$$(A, B) \rightarrow (A, B) A^{4-d}, \quad r \rightarrow r A^{-1}.$$

Equations (22) have a fixed point at  $w = 2\varepsilon/11$ . At this fixed point we have  $A \sim x^{-12\varepsilon/11}$ ,  $B \sim x^{2\varepsilon/11}$ , so that the renormalized Green's function is

$$G_R^{-1} = \bar{A} k^\alpha + \bar{B} q^\beta, \quad (23)$$

where  $\alpha = 2 + 6\varepsilon/11$ ,  $\beta = 4 - 2\varepsilon/11$ . We estimate  $\bar{A}$  and  $\bar{B}$  for  $d = 3$ . The strongly divergent diagrams in Figs. 1 and 2

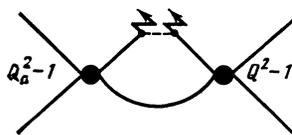


FIG. 1.

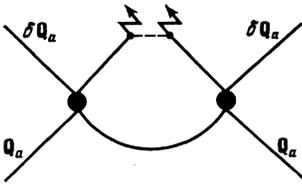


FIG. 2.

introduce in  $G_0^{-1} = Ak^2 + Bq^4$  a correction that becomes of the order of  $G_0^{-1}$  at

$$|k| \sim q^2 \sim q_1^2 = {}^3/_{64}\pi^{-1}t^2h^2.$$

At smaller values of  $|k|$  and  $q^2$ , expression (23) becomes valid. Joining these two expressions, we find

$$\bar{A} \approx q_1^{4-2\alpha}, \quad \bar{B} \approx q_1^{4-\beta}. \quad (24)$$

At  $\varepsilon = \varepsilon^* = 11/4$  fluctuation spectrum becomes isotropic. The fact that a spectrum becomes isotropic in the sense  $\alpha = \beta$  does not imply a complete isotropy, since  $\bar{A} \gg \bar{B}$ , but this "numerical anisotropy" is unimportant for the scaling relations below. The value of  $\varepsilon^*$  is calculated in first order of the  $\varepsilon$  decomposition, so we do not have to take it seriously. On the other hand, there can be no doubt that there does in fact exist a point  $\varepsilon$  at which the spectrum becomes isotropic. We now present some arguments which imply  $\varepsilon^* \leq 2$ , i.e., that the spectrum is isotropic in three-dimensional space.

We first derive some exact scaling relations, without using the  $\varepsilon$ -decomposition method. For this purpose we assume that the vertices  $A$  and  $B$  are power functions of the momentum:  $A \sim k^{\alpha-2}$ ,  $B \sim q^{\beta-4}$ . The Green's function is then

$$G^{-1} = \bar{A}k^\alpha + \bar{B}q^\beta.$$

As before, the diagrams in Figs. 1 and 2 are the most singular contributions to  $A$  and  $B$ . Evaluating their divergence order and equating it to the order of the singularity in the quantity ( $A$  or  $B$ ) to which they contribute, we find two scaling relations. These relations are dependent and equivalent to

$$5+d=2\alpha+\beta/\alpha. \quad (25)$$

We were able to derive this relation because the nonlinear vertex and one of the coefficients in the Green's function are determined by the same quantity,  $A \approx \bar{A}k^{\alpha-2}$ , by virtue of the symmetry of the problem. We are thus left with only two independent indices:  $\alpha$  and  $\beta$ .

To now calculate the renormalization of the wave vector of the structure (the quantity  $r$ ), we use the renormalized spectrum. This renormalization is  $\langle(\nabla\bar{\varphi})^2\rangle$ , so it is finite at  $d > d^*$ , where  $d^*$  is determined by

$$3+d^*=2\beta-\beta/\alpha. \quad (26)$$

In a space of dimensionality  $d^*$  the integral over the momentum which determines  $\langle(\nabla\bar{\varphi})^2\rangle$  diverges logarithmically at small momenta. Substituting (25) into (26), we find  $\alpha = \beta = 2 + d^*/2$  at the point  $d^*$ ; i.e., the spectrum becomes isotropic at the same dimensionality at which the renormalization of the period of the structure diverges.

We note now that the quantity  $\langle(\nabla\bar{\varphi})^2\rangle$ , evaluated with the unrenormalized Green's function, diverges logarithmically in three-dimensional space. We would not expect this divergence to be removed by a renormalization of the spectrum which renders the fluctuation spectrum softer. We checked this assumption in first-order perturbation theory in  $h^2$  directly in the three-dimensional space and found that the correction is of the same sign as the main term. This means that we have  $\varepsilon^* \leq 2$  and that the spectrum is isotropic in three-dimensional space. To derive scaling relations in a space of dimensionality below the critical dimensionality, we assume that the Green's function is of the form  $G_R \propto p^{-\beta}$ , while the wave vector of the structure is  $r \propto p^\gamma$ . Evaluating the divergence of the diagrams which renormalize the vertex (Fig. 1) and the Green's function (Fig. 2), we find the two scaling relations

$$\alpha + \gamma - 2\beta + 4 + d = 0, \quad 2\alpha + 2\gamma - 4\beta + 4 + d = 0, \quad (27)$$

from which we in turn find  $\beta = \alpha + \gamma = 2 + d/2$ , i.e., in three-dimensional space,

$$G(r) = [\bar{A}k^{2/3} + \bar{B}q^{2/3}]^{-1}, \quad \bar{A} \approx q_1^{-3}, \quad \bar{B} \approx q_1^{1/3}. \quad (28)$$

We can write equations of the renormalization group for  $r$ , assuming that its change is slow and making use of the isotropic Green's function in (28):

$$dr/d\xi = -t^2 h^2 r / \bar{A}^{2/3} \bar{B}^{1/3} \eta_1, \quad \eta_1 \approx 25, \quad (29)$$

It follows that  $r$  and, equivalently, the correlation function  $\langle\nabla\varphi(x)\nabla\varphi(0)\rangle = C(x)$  have a power-law dependence on the distance:

$$C(x) \sim x^{-\Delta}; \quad \Delta \approx 2.5q_1^2. \quad (30)$$

These expressions are valid only for  $d = d^*$  and  $q_1^2 \ll 1$ , for which the  $x$  dependence of  $C$  is weak. At  $d < d^*$ , another solution of the equations of the renormalization group can arise with an index  $\gamma$  which depends on only the dimensionality of the space—not on  $h^2$ . At the very largest distances, where  $C(x) \rightarrow 0$ , Green's function (28) should become completely isotropic.

It follows from the power-law behavior of  $r$  that the low-temperature phase is a genuine thermodynamic phase which is different from the paramagnetic phase, since in this phase we have the correlation function  $\langle\mathbf{Q}(0)\mathbf{Q}(x)\rangle$ , which falls off slowly with distance, in a power-law manner, while in the paramagnetic phase all the correlation functions fall off exponentially rapidly. We might also note that the integral determining  $\langle(\nabla\bar{\varphi})^2\rangle_T$  is itself determined primarily by the region of large momenta and is finite, so that thermal fluctuations do not annihilate  $\langle\mathbf{Q}\rangle_T$ , in contrast with the situation in the paramagnetic phase, where we of course have  $\langle\mathbf{Q}\rangle_T = 0$ . The  $\Phi$  phase is furthermore not similar to the Edwards-Anderson phase; in it, for example, the Edwards-Anderson order parameter is  $\langle s \rangle_T^2 = \langle \Phi \rangle_T^2 = 0$ , but, in contrast with the paramagnetic phase, we have

$$\langle \cos[\bar{\varphi}(0) - \bar{\varphi}(x)] \rangle_T = \exp \left[ - \int G(\mathbf{p}) (1 - \cos \mathbf{p}\mathbf{x}) d^3\mathbf{p} / (2\pi)^3 \right] \sim \exp[-x^{1/3} \bar{A}^{-2/3} \bar{B}^{-1/3} \eta_2] \sim \exp[-3\bar{\xi}(q_1 x)^{1/3}], \quad (31)$$

$$\bar{A} \approx q_1^{-3}, \quad \bar{B} \approx q_1^{1/2}, \quad \eta_2^{-1} \approx 16.$$

This result means, in particular, that the average molecular field acting on a spin vanishes, so that the complete-equilibrium susceptibility obeys the Curie law,  $\chi_{eq} = c/T$ . The non-linear susceptibility  $\chi_{eq} = -\partial^2 \chi_{eq} / \partial \mathcal{H}^2$ , another important characteristic of a spin glass, is proportional to the integral

$$\int \langle \cos[\bar{\varphi}(0) - \bar{\varphi}(\mathbf{x})] \rangle_T d^3 \mathbf{x}$$

and is finite at all  $T$ , as follows from (31).

5. We have shown that Hamiltonian (8) with  $vl^4 \ll 1$  does in fact describe a genuine thermodynamic phase of a spin glass which is different from the paramagnetic phase. The phase transition between these two phases apparently stems from the nucleation of infinitely long dislocations of the field  $\varphi$ . Both the energy and the anisotropy of a linear dislocation are proportional to its length  $L$  in a one-dimensionally periodic system<sup>21-23</sup>:  $F(L) = (a - Tb)L$ , where  $a$  and  $b$  are determined by the short-range order. Consequently, the renormalizations of the spectra at small momenta caused by the last term in (8) should not result in the nucleation of dislocations at  $T < a/b$ .

There is the possibility that there is some combination of the parameters  $\kappa$ ,  $p_0$ ,  $p_0 \ll \kappa^4 c^{-4}$ , for which we would have  $vl^4 \ll 1$  and  $\gamma \ll 1$ . In this case the fluctuations of the modulus of  $\rho$  are always small, and we can determine  $v$  from Eq. (9). To describe the long-range behavior of the system we do not have to make any further assumptions. This relation between  $\kappa$  and  $p_0$  corresponds to a nearly ferromagnetic long-range interaction between the impurity atoms, which leads to the formation of a helicoidal structure of very large period. The transition to a state of this sort must be of first order (unfortunately, we cannot cite any corresponding physical systems as examples).

In a real Ising spin glass the case  $vl^4 \sim 1$  apparently holds at all times, so that the existence of a low-temperature phase cannot be regarded as rigorously proved. If, however, there exists a temperature region in which dislocations of infinite length are not created then the large-scale behavior is still described by Hamiltonian (18), so that we have

$$G(\mathbf{p}) \sim p^{-7/2}, \quad \langle (\nabla \varphi)_{\mathbf{x}} \rangle^2 \neq 0.$$

At temperatures above the point of the dislocation transition, the phase variable  $\varphi$  is not determined. Near the transition, the long-range behavior of the system can be described by Hamiltonian (18) with  $\nabla \varphi \rightarrow \mathbf{Q}$ , where  $\mathbf{Q}$  in this case is an arbitrary three-dimensional vector. As for the Heisenberg model with disorder, analysis by the renormalization-group method shows that in a four-dimensional space the spin rigidity  $\rho_s = B$  falls off with increasing characteristic dimension in accordance with

$$\rho_s = \rho_{s0} [1 - g_0 \ln(1/q)],$$

while the effective charge increases in proportion to  $\rho_s^{-1}$  ("asymptotic freedom"). This result appears to mean that in four-dimensional space and, especially, in three-dimensional space this phase is actually not different from the paramagnetic phase.

Let us examine the applicability of this model, with interaction (2), to real substances. The most important point for our purposes was the formation of a one-dimensionally periodic structure with an arbitrarily directed wave vector  $\mathbf{Q}$ . For this purpose the interaction apparently does not have to be exactly spherically isotropic, as in (2); it is instead sufficient that the anisotropy be small in comparison with the ratio  $\kappa/p_0$ . In the case of highly anisotropic interactions such as indirect exchange in indirect-gap semiconductors,<sup>24</sup>

$$V(\mathbf{r}) = V_0 r^{-s/2} \left[ \sum_i \exp(i\mathbf{k}_i \mathbf{r}) \exp(-\alpha r) \right] \quad (32)$$

the wave vector  $\mathbf{Q}$  is tied to the crystallographic axes, and the thermodynamic fluctuations of the phase  $\varphi$  have the customary quadratic spectrum. Consequently,  $\langle \varphi^2 \rangle_T$  is finite, and the average spins are not zero,  $\langle s_i \rangle \neq 0$ ; the equilibrium magnetic susceptibility changes slope at  $T = T_c$ . Accordingly, the similarity which interactions (2) and (32) seem to have at first glance leads to spin glasses which behave very differently. We might also note here that the description of a spin glass offered by Edwards and Anderson,<sup>13</sup> in terms of quantities which are bilinear in the spins,  $\langle S \rangle_T^2$ , is overly crude in our opinion, since it completely ignores the internal structure of this state [in particular, interactions (2) and (32) are indistinguishable in this approach]. We know that a successful description of a highly fluctuating system requires finding slow dynamic variables which describe the deviation of the state of the system from equilibrium. In the case of spin glasses we usually do not know the structure of this state, but this ignorance by no means implies that the structure is unimportant for the system itself. The approach outlined in this paper constitutes an attempt to identify this internal structure for spin glasses of one type (a recent experiment<sup>25</sup> indicates the existence of a helical magnetic structure in the spin glass  $\text{Cu}_{1-x}\text{Mn}_x$ ).

The state which we have studied does not have an order parameter in the Edwards-Anderson sense,  $\langle s_i \rangle^2 = 0$ , but it is a genuine low-temperature thermodynamic phase. Unfortunately, the equilibrium correlation functions which distinguish this phase are related to measurable quantities in an extremely complicated way. At the same time it is clear that the spin dynamics in this phase is qualitatively different from a paramagnetic dynamics. When we thus bring up the old question of whether a spin glass exists as a thermodynamic phase or is simply a dynamic phenomenon we come up with a paradoxical answer (for our case of Ising spins): This phase exists, but it is manifested only in the dynamic quantities.

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