Spin waves in noncollinear ferromagnetic structures at $T=0$

A. V. Chubukov

M. V. Lomonosov Moscow State University

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We use the Holstein-Primakoff transformation to evaluate the thermodynamical characteristics and the elementary excitation spectrum in uniaxial ferromagnets with a single ion anisotropy of the "easy axis" and "easy plane" type in transverse magnetic fields. We assume the magnitude of the single site spin to be arbitrary and the small parameter in the problem to be the ratio of the relativistic and exchange energies. The formulae obtained in the paper generalize the results of recent papers [E. Rastelli and P. A. Lindgaard, J. Phys. **C12**, 1899 (1979); E. Rastelli and A. Tassi, J. Appl. Phys. **53**, 3(II) (1982)].

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One of the main problems in the theory of magnetism is the calculation of the ground state energy and of the spectrum of the elementary excitations. This problem can be solved relatively simply (see Ref. 1) only in the case of collinear[1)] ferromagnetic structures (CFS), i.e., systems for which the stationary states of the spin Hamiltonian are characterized by definite values of the z-component of the total spin S^z and the ground state necessarily corresponds to the maximum z-component value of the total spin. In systems with a different symmetry (noncollinear ferromagnetic structures—NFS) the situation is appreciably more complicated and there is as yet no complete quantum-mechanical description of the spin wave spectrum. Most simply one can see the cause of this difficulty by changing from the spin operators to boson operators using any of the well known transformations,[2–6] writing the Hamiltonian $\hat{\mathscr{H}}$ in terms of particle creation and annihilation operators, and diagonalizing the quadratic terms in the operators. In CFS the second-quantization Hamiltonian commutes with the particle-number operator $\hat{N}$ so that the determination of the ground-state energy and of the single-particle excitation spectrum is trivial; On the other hand, in NFS this commutator is different from zero and the energy characteristics found by using only the harmonic terms will be substantially renormalized by the anharmonicities. It was assumed for this reason until recently in the study of NFS that the spin per site S is sufficiently large; the anharmonic terms are then small in the parameter $1/S$, and one can either neglect them completely (quasi-classical approximation[7,8]) or limit oneself to second-order perturbation theory.[9] However, the spin wave spectrum determined in this way shows only a poor agreement with experiment, since usually NFS have a spin $S \approx 1$.[10]

At the same time, since the noncollinearity of the structure[2)] is caused by the fact that the anisotropy axis direction and the equilibrium spin position are not the same, it is clear that we can use as the "noncollinearity" parameter the quantity $\delta \sim \mu^2/JV_0$ (μ is the Bohr magneton, J the exchange integral, V_0 the volume of the elementary cell, and we assume the dimensionless anisotropy constant to be of the order of unity). In crystals with not too low a Curie temperature we have then $\delta \ll 1$; this means that the renormalization of the spectrum due to anharmonicity in such crystals is small for any S, so that one can obtain rather exact results by restricting oneself to a finite number of terms in the perturbation-theory series.[3)]

In our opinion, amongst the different kinds of NFS of most interest are the following objects: I) Uniaxial ferromagnets with single-ion "easy axis" type magnetic anisotropy (EAFM) in a transverse magnetic field. II) Uniaxial ferromagnets with single-ion "easy plane" kind of anisotropy (EPFM) with a magnetic field along the anisotropy axis.

The first kind of system was studied in Refs. 7 and 11 to 15, where two approaches to the solution of the problem stated were explicitly developed: on the one hand, in the papers by a number of authors[13–15] the wave functions of the ground state and the low-lying excited states were constructed for the cases $S=\frac{1}{2}$ and $S=1$ in the self-consistent-field approximation and the magnetization in the direction of the magnetic field H (y axis) and the magnitude of the angle between the equilibrium direction of the magnetic moment and the y axis were determined using the self-consistency conditions. A somewhat different approach was developed in Refs. 7, 11, and 12. In Ref. 7 the ground state energy and the spin wave spectrum were constructed to first order in $1/2S$ without limitations on the field strength; in Refs. 11 and 12 the same quantities were evaluated using perturbation theory in the limits of small and strong fields for arbitrary spin magnitudes.

For type II NFS the problem of finding the spin wave spectrum when there is no external magnetic field was considered in a number of papers[4,5,17–21] accurate to terms $\sim \mathscr{O}(\delta^2)$. They used the Holstein-Primakoff (HP),[4,5,9] Dyson-Maleev (DM),[17–21] and Lindgaard-Danielson[4,5,18] transformations to change from spin to boson operators. However, the authors of these papers were not able to sum completely the perturbation-theory series (i.e., solve the problem for arbitrary spin). In this connection they used for the calculation of the coefficients in the terms in the spectrum which are quadratic in δ an expansion in $1/2S$. For instance, in a recent paper by Rastelli and Tassi,[17] who used the DM formalism, these coefficients were evaluated up to terms $\sim O(1/(2S)^2)$.

In the present report we determine the spin wave spectrum and the magnetic characteristics (magnetization and

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susceptibility) of type I and II NFS for any magnitude of the spinper site and in an arbitrary external field. We choose the HP transformation[4)] to change from spin to boson operators; for the given problem it seems to be more convenient than the DM transformation, as the latter leads to the appearance of a non-Hermitean Hamiltonian. The use of the normal quantum-mechanical perturbation theory[22] enables us to evaluate the main corrections to the magnetization and the susceptibility due to the zero-point oscillations, and also to determine the spin wave spectrum accurate to terms $\sim O(\delta^2)$.

The sequence of treatment in the remainder of the paper is as folows: in Sec. 1 we consider EAFM in a transverse field: we show how to construct the random phase approximation and we consider the effect of anharmonicity. In Sec. 2 we consider similarly EPFM in an external magnetic field along the anisotropy axis. In the Conclusion we consider the behavior of a system at finite albeit low temperatures.

To improve clarity we relegate the details of the calculations and the relatively complicated expressions to Appendices 1 to 5.

1. UNIAXIAL FERROMAGNET WITH EASY-AXIS SINGLE-ION ANISOTROPY

We choose the z axis along the easy magnetization axis and the y axis along the direction of the external magnetic field $\mathbf{H}$. Taking into account the exchange and dipole interaction the Hamiltonian of the system can then be written in the following form:[1]

$$\mathcal{H}=-\frac{J}{2}\sum_{1,\Delta}\mathbf{S}_1\mathbf{S}_{1+\Delta}-\beta\mu^2V_0^{-1}\sum_{1}(S_1^z)^2+2\mu H\sum_{1}S_1^y$$
$$+2\mu^2\sum_{1,1'}R_{11'}^{-5}\left[R_{11'}^2\mathbf{S}_1\mathbf{S}_{1'}-3(\mathbf{R}_{11'}\mathbf{S}_1)(\mathbf{R}_{11'}\mathbf{S}_{1'})\right],\qquad(1)$$

where $\mathbf{S}_l$ is the spin operator in the l-th site, J the exchange integral, Δ a vector connecting lattice sites for which J is nonzero, μ the Bohr magneton, β the dimensionless anisotropy constant, and V_0 the volume of the elementary cell; $\mathbf{R}_{ll'}$ is the radius vector connecting the sites l and l'.

When there is no external field the equilibrium position of the magnetic moment is along the z axis; when H increases the moment is deflected in the direction of the y axis and when the field reaches the critical value $H=H_{cr}$ it merges with the y axis.

Using standard methods[7,8,14] we change to a primed set of coordinates such that the z' axis is along the equilibrium position of the spin S (which, in turn, is determined self-consistently as a result of solving the problem) and the y axis remains in the $y'=z'$ plane, and we express the Hamiltonian (1) of the ferromagnet in terms of the boson operators $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$, using the HP transformation[2]

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{1s}+\mathcal{H}_{2s}+\mathcal{H}_{ss},\qquad(2)$$

where $\mathcal{H}_0$ does not contain the operators a_k^+ and a_k:

$$\mathcal{H}_0=J(0)S^2N\left[-{}^1/_2-\tilde{\beta}\delta+\tilde{\beta}\delta\cos^2\theta-2h\cos\theta\right.$$
$$\left.+2\delta V_0\sum_R(R^2-3(z')^2)/R^5\right];\qquad(3)$$

$\mathcal{H}_{1s}$ and $\mathcal{H}_{2s}$ are, respectively, the linear and quadratic forms in the operators $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$:

$$\mathcal{H}_{1s}=-2iJ(0)S(SN/2)^{1/2}(h-\tilde{\beta}\delta\cos\theta)\sin\theta a_0^++\text{H.c.}\qquad(4)$$

$$\mathcal{H}_{2s}=\sum_{\mathbf{k}}A_{\mathbf{k}}a_{\mathbf{k}}^+a_{\mathbf{k}}+{}^1/_2B_{\mathbf{k}}a_{\mathbf{k}}a_{-\mathbf{k}}+{}^1/_2B_{\mathbf{k}}^*a_{\mathbf{k}}^+a_{-\mathbf{k}}^+,\qquad(5)$$

where

$$A_{\mathbf{k}}=J(0)S[1-\nu(\mathbf{k})+2h\cos\theta$$
$$+2\tilde{\beta}\delta(1-{}^3/_2\cos^2\theta)+8\pi\delta\sin^2\theta_{\mathbf{k}}],\qquad(6)$$

$$B_{\mathbf{k}}=J(0)S\delta\tilde{\beta}[(1-1/2S)^{-1/2}\cos^2\theta+(8\pi/\tilde{\beta})\sin^2\theta_{\mathbf{k}}e^{-2i\varphi_{\mathbf{k}}}]\qquad(7)$$

(θ_k and φ_k are the polar and azimuthal angles of the wave vector $\mathbf{k}$).

We have used in Eqs. (3) to (7) the following notation:

$$J(\mathbf{k})=J\sum_{\Delta}e^{i\mathbf{k}\Delta},\ \nu(\mathbf{k})=\frac{J(\mathbf{k})}{J(0)},$$
$$h=\frac{\mu H}{J(0)S},\ \tilde{\beta}=\beta\left(1-\frac{1}{2S}\right),\ \delta=\frac{\mu^2}{J(0)V_0},\qquad(8)$$

θ' is the angle between the z' and y axes (the quantity δ is the small parameter of the problem). Furthermore

$$\mathcal{H}_{ss}=\sum_{n=3}^{\infty}\mathcal{H}_s^{(n)},\qquad(9)$$

where $\mathcal{H}_s^{(n)}$ is an n-th order form relative to the canonical form of the operators $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$.

We give here the explicit expression for the operator $\mathcal{H}_{s,ex}^{(4)}$ due to the exchange terms in the Hamiltonian:

$$\mathcal{H}_{s,ex}^{(4)}=\sum_{1234}\Phi(12,34)a_{\mathbf{k}_1}^+a_{\mathbf{k}_2}^+a_{\mathbf{k}_3}a_{\mathbf{k}_4}\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4),\qquad(10)$$

$$\Phi(12,34)=-\frac{J(0)}{8N}\{\nu(1-3)+\nu(1-4)+\nu(2-3)$$
$$+\nu(2-4)-4S[1-(1-1/2S)^{1/2}][\nu(1)+\nu(2)+\nu(3)+\nu(4)]\}.\qquad(11)$$

We give in Appendix 1 the other terms of $\mathcal{H}_s^{(n)}$ used in the exposition that follows.

We must also determine the form of the unitary transformation[1] which diagonalizes the quadratic form in the Hamiltonian $\mathcal{H}$. It is clear, however, that under such a transformation the coefficients $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are renormalized by the contributions from the anharmonic terms $\mathcal{H}_s^{(n)}$ ($A_{\mathbf{k}}\to\tilde{A}_{\mathbf{k}}, B_{\mathbf{k}}\to\tilde{B}_{\mathbf{k}}$), and the renormalized quantities themselves turn out to be functions of the parameters of the unitary transformation.

As a result, the diagonalization of the quadratic form in terms of the new boson operators $C_{\mathbf{k}}^+$ and $C_{\mathbf{k}}$ leads to the following set of integral equations for the coefficients[5)] $\tilde{A}_{\mathbf{k}}$ and $\tilde{B}_{\mathbf{k}}$:

$$\tilde{A}_{\mathbf{k}}=A_{\mathbf{k}}-\frac{J(0)}{N}\sum_{\mathbf{p}}\frac{\tilde{A}_{\mathbf{p}}-\varepsilon_{\mathbf{p}}}{2\varepsilon_{\mathbf{p}}}\left\{1+\nu(\mathbf{k}-\mathbf{p})\right.$$
$$-4S\left[1-\left(1-\frac{1}{2S}\right)^{1/2}\right]$$

$$[\nu(\mathbf{k})+\nu(\mathbf{p})]\Big\}-\frac{3}{2}J(0)S\beta\delta\left(1-\frac{1}{2S}\right)^{-1/2}\times\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)$$

$$\times\cos^2\theta\frac{1}{N}\sum_{\mathbf{p}}\frac{\tilde{B}_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}+\frac{3}{2}\frac{J(0)S}{N^2}\sum_{\mathbf{pq}}\frac{\tilde{B}_{\mathbf{p}}\tilde{B}_{\mathbf{q}}}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{q}}}\Big\{\Big[\frac{1}{2}\left(1-\left(1-\frac{1}{S}\right)^{1/2}\right)+\left(1-\frac{1}{2S}\right)^{1/2}-1\Big][\nu(\mathbf{k})+\nu(\mathbf{p})+\nu(\mathbf{q})]-\frac{1}{6}\Big[1-\left(1-\frac{1}{2S}\right)^{1/2}\Big]^2$$

$$\times[\nu(\mathbf{k})+2\nu(\mathbf{p})+2\nu(\mathbf{q})+\nu(\mathbf{k}+\mathbf{p}+\mathbf{q})+\nu(\mathbf{k}+\mathbf{p}-\mathbf{q})+\nu(\mathbf{k}-\mathbf{p}+\mathbf{q})+\nu(\mathbf{k}-\mathbf{p}-\mathbf{q})]\Big\}+\frac{4\pi}{\beta}\delta^2 f_1+O\left(\delta\sum_{\mathbf{p}}\frac{\tilde{A}_{\mathbf{p}}-\varepsilon_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\right), \tag{12}$$

$$\tilde{B}_{\mathbf{k}}=B_{\mathbf{k}}+\frac{J(0)}{4N}\sum_{\mathbf{p}}\frac{\tilde{B}_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\Big\{\Big[\nu(\mathbf{k}-\mathbf{p})+\nu(\mathbf{k}+\mathbf{p})-4S\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)\times(\nu(\mathbf{k})+\nu(\mathbf{p}))\Big]-2\beta\delta(3\cos^2\theta-2)\Big\}+\frac{4\pi}{\beta}\delta^2 f_2+O\left(\delta\sum_{\mathbf{p}}\frac{\tilde{A}_{\mathbf{p}}-\varepsilon_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\right), \tag{13}$$

$$\varepsilon_{\mathbf{k}}=(\tilde{A}_{\mathbf{k}}^2-|\tilde{B}_{\mathbf{k}}|^2)^{1/2}. \tag{14}$$

In what follows we show that

$$\sum_{\mathbf{p}}\frac{\tilde{B}_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\sim\delta,\quad\sum_{\mathbf{p}}\frac{\tilde{A}_{\mathbf{p}}-\varepsilon_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\sim\delta^{3/2},$$

so that the terms dropped from (12) and (13) are, indeed, small. To make the exposition more lucid and to avoid the appearance of too cumbersome expressions, we assume hereafter, when evaluating the spin wave spectrum (but not the magnetic characteristics), that $\beta\gg4\pi$. For this reason we do not give here the explicit expressions for the coefficients $f_{1,2}$ in (12) and (13).

The solution of the set of integral equations will be sought in the form of a power series in the small parameter δ. One notes then easily that in each step of the iteration process the system splits into two independent equations: an algebraic equation (12) and an integral equation (13) whose kernel factorizes, i.e., the substitution $\tilde{B}_{\mathbf{k}}=a+b(1-\nu(\mathbf{k}))$ leads to a set of two algebraic equations for the quantities a and b.

As a result we get after intermediate calculations

$$\tilde{A}_{\mathbf{k}}=A_{\mathbf{k}}^{(0)}+A_{\mathbf{k}}^{(1)}\delta+A_{\mathbf{k}}^{(2)}\delta^{3/2}+A_{\mathbf{k}}^{(3)}\delta^2+O(\delta^{5/2}), \tag{15}$$

$$\tilde{B}_{\mathbf{k}}=B_{\mathbf{k}}^{(1)}\delta+B_{\mathbf{k}}^{(2)}\delta^{3/2}+B_{\mathbf{k}}^{(3)}\delta^2+O(\delta^{5/2}), \tag{16}$$

where

$$A_{\mathbf{k}}^{(0)}=J(0)S(1-\nu(\mathbf{k})), \tag{17}$$

$$A_{\mathbf{k}}^{(1)}=J(0)S\Big[(2h/\delta)\cos\theta+2\beta\left(1-\frac{3}{2}\cos^2\theta\right)+8\pi\sin^2\theta_{\mathbf{k}}\Big], \tag{18}$$

$$B_{\mathbf{k}}^{(1)}=J(0)S\tilde{\beta}[(1+\gamma(S)(1-\nu(\mathbf{k})))\cos^2\theta+(8\pi/\tilde{\beta})\cdot\sin^2\theta_{\mathbf{k}}e^{-2i\varphi_{\mathbf{k}}}], \tag{19}$$

while

$$\gamma(S)=(2S)^{-1}\left(1-\frac{1}{2S}\right)^{-1/2}-\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)W,$$

$$W=\frac{1}{N}\sum_{\mathbf{p}}\frac{1}{1-\nu(\mathbf{p})}\approx1.51 \tag{20}$$

is Watson's integral. Expressions for the other coefficients of the expansion of the solutions of the set (12), (13) in powers of δ are given in Appendix 2.

As a result the Hamiltonian of the ferromagnetic is

$$\mathscr{H}=\tilde{\mathscr{H}}_0+\sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}C_{\mathbf{k}}^+C_{\mathbf{k}}+(i\mathscr{H}_1(\theta,h)C_0^++\mathscr{H}_D+\text{H.c.}). \tag{21}$$

The quantity $\mathscr{H}_0$ is here the unrenormalized energy of the ferromagnetic ground state with zero-point oscillations taken into account:

$$\tilde{\mathscr{H}}_0=\mathscr{H}_0+\frac{1}{4}\sum_{\mathbf{k}}(A_{\mathbf{k}}+\tilde{A}_{\mathbf{k}})\frac{\tilde{A}_{\mathbf{k}}-\varepsilon_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}}-\frac{1}{8}\sum_{\mathbf{k}}\Big[(B_{\mathbf{k}}+\tilde{B}_{\mathbf{k}})\frac{\tilde{B}_{\mathbf{k}}^*}{\varepsilon_{\mathbf{k}}}+\text{H.c.}\Big]. \tag{22}$$

Moreover,

$$\mathscr{H}_1(\theta,h)=2J(0)S\left(\frac{SN}{2}\right)^{1/2}\Big\{(\beta\delta\cos\theta-h)+\Big[h\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)-\beta\delta\cos\theta\left(1-\left(1-\frac{1}{2S}\right)^{1/2}+\frac{1}{S}\left(1-\frac{1}{2S}\right)^{-1/2}\right)\Big]\times\left(\frac{1}{2N}\sum_{\mathbf{k}}\frac{\tilde{B}_{\mathbf{k}}+2\tilde{A}_{\mathbf{k}}-2\varepsilon_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}}\right)+O(\delta^2)\Big\}\left(\frac{\tilde{B}_0}{|\tilde{B}_0|}\left(\frac{\tilde{A}_0-\varepsilon_0}{2\varepsilon_0}\right)^{1/2}+\left(\frac{\tilde{A}_0+\varepsilon_0}{2\varepsilon_0}\right)^{1/2}\right)\sin\theta, \tag{23}$$

and the term $\mathscr{H}_D$ determines the contribution from all anharmonic terms (reduced to canonical form); the explicit form of this operator is given in Appendix 3.

It is also clear that the equilibrium position of the spin (i.e., the angle $\theta=\theta_p(h)$) must be determined from the condition

$$\langle\psi_0|S_{y'}|\psi_0\rangle=0, \tag{24}$$

where ψ_0 is the wave function of the ground state of the Hamiltonian (1).

1a. Random phase approximation (RPA)

If one compares the transformation given above with the standard procedure of quasilinearization in the RPA (see Ref. 8) one verifies easily that this approximation corresponds to neglecting the quantity $\mathscr{H}_D$ in (21). This neglect is justified since the anharmonic terms are small in the noncollinearity parameter δ and one can expect their contribution to the renormalization of the spectrum to be small compared to (14). Indeed, the analysis of the dropped terms given in the next subsection shows that the RPA leads to correct results if one restricts oneself in Eqs. (12), (13), and (23) to merely quantities of first order in δ. Moreover, one notes easily that in the RPA framework the equilibrium position of the spin must be determined from the condition $\mathscr{H}_1(\theta_p(h),h) = 0$. As a result we get the following simple expressions for the magnetization and the susceptibility:

$$M_H = (\mu S/V_0) \begin{cases} h/h_0, & h \leqslant h_0 \\ 1, & h \geqslant h_0 \end{cases}, \tag{25a}$$

$$\chi_{HH} = \begin{cases} 1/\beta, & h < h_0 \\ 0, & h > h_0 \end{cases}, \tag{25b}$$

where $h_0 = \tilde{\beta}\delta$.

It is important that even in the RPA framework the critical field for the phase transition turns out be equal to $h_0 = \tilde{\beta}\delta \sim 1 - 1/2S$, a natural result, since there is no anisotropy for $S = \frac{1}{2}$. Moreover, it is clear from (12) and (13) that when $h = h_0$ the gap in the spin-wave spectrum becomes zero, i.e., the phase transition is of second order.

1b. Allowance for anharmonicities

We turn to the exact expression (21) and determine first the main corrections, due to the zero-point oscillations, to the thermodynamic characteristics of the system. A simple analysis shows that it is sufficient to calculate the dependence of the equilibrium position of the spin ($\theta = \theta_p(h)$) and the magnetization along the field (M_H) up to terms $\sim O(\delta^{3/2})$ from the following simple conditions:

$$\left.\frac{\partial \tilde{\mathscr{H}}_0}{\partial \theta}\right|_{\theta=\theta_p} = 0, \quad M_H = -\left.\frac{\partial \tilde{\mathscr{H}}_0}{\partial H}\right|_{\theta=\theta_p}. \tag{26}$$

As a result we get

$$\cos\theta(h) = \frac{h}{\beta\delta}[1+P(h)+2Q(h)+O(\delta^2)], \tag{27}$$

$$M_H = \frac{\mu S}{V_0}\frac{h}{\beta\delta}[1+P(h)+Q(h)+O(\delta^2)], \quad h \leqslant h_{cr},$$

$$M_H = \frac{\mu S}{V_0}[1-Q(h)+O(\delta^2)], \quad h \geqslant h_{cr}, \tag{28}$$

where

$$P(h) = (2S)^{-1}\left(1-\frac{1}{2S}\right)^{-1/2}\frac{1}{N}\sum_{\mathbf{k}}\frac{\mathscr{B}_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}} \sim \delta,$$

$$Q(h) = (2S)^{-1}\frac{1}{N}\sum_{\mathbf{k}}\frac{\mathscr{A}_{\mathbf{k}}-\varepsilon_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}} \sim \delta^{3/2}. \tag{29}$$

Equating the right-hand side of (27) to unity we determine the critical field for the orientation phase transition:

$$h_{cr} = \tilde{\beta}\delta[1-P(h)-2Q(h)+O(\delta^2)]. \tag{30}$$

It is now clear from (28) that at the critical point the magnetic moment is continuous, i.e., the order of the transition does not change when one takes anharmonicities into account.

Now differentiating Eq. (28) with respect to the field we determine the longitudinal susceptibility of the ferromagnet:

$$\chi_{HH} = (1/\tilde{\beta})[1+P(h)+hP'(h)+O(\delta^{3/2})], \quad h < h_{cr},$$

$$\chi_{HH} = -\delta Q'(h)+O(\delta^2), \quad h > h_{cr} \tag{31}$$

(the prime indicates here derivatives with respect to h). The quantities $P(h)$, $Q(h)$, and χ_{HH} are calculated in Appendix 4.

It is necessary to note that the terms neglected in Eq. (30) and of higher powers in δ do not contain parts which diverge as one approaches the transition point. This is a consequence of the fact that at $T = 0$ the critical exponents of a three-dimensional quantum system are necessarily determined by Landau's theory[6,23,24] so that the longitudinal susceptibility (corresponding to the specific heat in the standard theory of phase transitions, since $\mu H \Sigma S_l^y$ is quadratic in the order parameter) experiences a finite jump at the transition point.

We now consider the problem of the renormalization of the spin-wave spectrum. Without dwelling on the cumbersome calculation technique (the intermediate calculations are given in Appendix 5) we write down at once the final result: up to $O(\delta^2)$ the renormalization of the spectrum for an arbitrary external field manifests itself only in the renormalization of the coefficient $\tilde{A}_{\mathbf{k}}$ ($\tilde{A}_{\mathbf{k}} \to \tilde{\tilde{A}}_{\mathbf{k}}$), and at small wavevectors $|\mathbf{k}\cdot\mathbf{\Delta}| \ll 1$ we have

$$\tilde{\tilde{A}}_{\mathbf{k}} = \tilde{A}_{\mathbf{k}} - \lambda_1(\mathbf{k})\delta^{3/2} + C\delta^2 + O(\delta^2(\mathbf{k}\Delta)^2), \tag{32}$$

where the quantities λ_1 and C are given, respectively, by (A2.1) and (A5.8). As a result the spin wave spectrum ($\tilde{\varepsilon}_{\mathbf{k}}$) has the following form:

$$\tilde{\varepsilon}_{\mathbf{k}} = [(\tilde{A}_{\mathbf{k}})^2 - (\tilde{B}_{\mathbf{k}})^2]^{1/2}. \tag{33}$$

To conclude this section we give the explicit expresion for $\tilde{\varepsilon}_{\mathbf{k}}$ for fields close to the critical value,

$$\tilde{\varepsilon}_{\mathbf{k}}^2 = (J(0)S)^2(\varepsilon_1+\varepsilon_2+\varepsilon_3), \tag{34}$$

where ε_1 determines the spectrum gap caused by the dipole-dipole interaction:

$$\varepsilon_1 = 32\pi\tilde{\beta}\delta^2\cos^2\theta\sin^2\theta_{\mathbf{k}}\sin^2\varphi_{\mathbf{k}}; \tag{35}$$

ε_2 determines the spectrum gap caused by the deviation from the transition point ($\varepsilon_2 \equiv 0$ at the critical point):

$$\varepsilon_2 = 4\tilde{\beta}\delta(h\cos\theta + h_{cr}(1-2\cos^2\theta)); \tag{36}$$

ε_3 determines the speed of the spin waves for small wavevectors ($|\mathbf{k}\cdot\mathbf{\Delta}| \ll 1$):

$$\varepsilon_3 = (1-\nu(\mathbf{k}))\left\{1-\nu(\mathbf{k})+4h\cos\theta-4\tilde{\beta}\delta\left(1-\frac{3}{2}\cos^2\theta\right)\right.$$

$$+16\pi\delta\sin^2\theta_{\mathbf{k}}+\beta^2\delta^2\left[\frac{\left(1-S+2WS\left(1-\frac{1}{2S}\right)\right)}{S^2\left(1-\frac{1}{2S}\right)}\right.$$

$$+O\left(\frac{4\pi}{\beta}\right)\Big]+O(\delta^2(\mathbf{k}\Delta)^2)\Big\}. \quad (37)$$

In formulae (35) to (37)

$$\cos\theta=h/h_{cr},\quad h\leqslant h_{cr},$$
$$\cos\theta=1,\quad h\geqslant h_{cr},$$

and W is given by (20).

1. UNIAXIAL FERROMAGNET WITH SINGLE ION EASY PLANE TYPE ANISOTROPY

In this case the Hamiltonian of the system has, if we take into account the exchange and dipole-dipole interactions and an external field along the z axis, the following form:

$$\mathscr{H}=\frac{J}{2}\sum_{1\Delta}\mathbf{S}_1\mathbf{S}_{1+\Delta}+\beta V_0^{-1}\mu^2\sum_1(S_1^z)^2+2\mu H\sum_1 S_1^z$$
$$+2\mu^2\sum_{1,1'}R_{1,1'}^{-5}\,[R_{1,1'}^2\mathbf{S}_1\mathbf{S}_{1'}-3(\mathbf{R}_{11'}\mathbf{S}_1)(\mathbf{R}_{11'}\mathbf{S}_{1'})]. \quad (38)$$

It is clear that when there is no external field the magnetic moment lies in the XY plane when the field is switched on the moment is deflected in the direction towards the z axis and in fields $H>H_{cr}$ the ground state of the system corresponds to a maximum z-component of its total spin. Therefore, when $H>H_{cr}$ the system is collinear and has been relatively well studied.[1] In fields $H<H_{cr}$ the equilibrium position of the spin makes an angle θ with the z axis. Choosing the y axis along the direction of the spin when there is no field and performing further transformations completely equivalent to those performed in Sec. 1 we get the following expression for the Hamiltonian (38):

$$\mathscr{H}=\tilde{\mathscr{H}}_0+\sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}C_{\mathbf{k}}^+C_{\mathbf{k}}-(i\mathscr{H}_1(\theta,h)C_0^++\mathscr{H}_D+\text{H.c.}), \quad (39)$$

where the quantities $\mathscr{H}_D$, $\hat{\mathscr{H}}_0$, $\mathscr{H}_1$ and ε_k are, as before, given by Eqs. (A3.1), (22), (23), and (14) (one must change in (A3.1) the sign of $\mathscr{H}_D^{(1)}$ and replace $\cos^2\theta$ by $\sin^2\theta$ in $\mathscr{H}_D^{(2)}$) while the coefficients $A_k, B_k, \tilde{A}_k, \tilde{B}_k$ which determine the spectrum and the classical ground state energy have the following form:

$$\mathscr{H}_0=J(0)S^2N\Big[-{}^1/_2+\tilde{\beta}\delta\cos^2\theta+\beta\delta-2h\cos\theta$$
$$+2\delta V_0\sum_R(R^2-3(z')^2)\Big/R^5\Big], \quad (40)$$

$$\tilde{A}_{\mathbf{k}}=A_{\mathbf{k}}^{(0)}+A_{\mathbf{k}}^{(1)}\delta+A_{\mathbf{k}}^{(2)}\delta^{3/2}+A_{\mathbf{k}}^{(3)}\delta^2+O(\delta^{5/2}), \quad (41)$$

$$\tilde{B}_{\mathbf{k}}=B_{\mathbf{k}}^{(1)}\delta+B_{\mathbf{k}}^{(2)}\delta^{3/2}+B_{\mathbf{k}}^{(3)}\delta^2+O(\delta^{5/2}). \quad (42)$$

Here $A_k^{(0)}$ determines the spin wave spectrum when there is only the exchange interaction

$$A_{\mathbf{k}}^{(0)}=J(0)S(1-\nu(\mathbf{k})), \quad (43)$$

while $A_k^{(1)}$ and $B_k^{(1)}$ determine to first order in δ the spectrum corrections due to the anisotropy, to the dipole interaction, and to the external magnetic field:

$$A_{\mathbf{k}}^{(1)}=J(0)S[2(h/\delta)\cos\theta+\tilde{\beta}(1-3\cos^2\theta)+8\pi\sin^2\theta_{\mathbf{k}}],$$

$$B_{\mathbf{k}}^{(1)}=-J(0)S\tilde{\beta}\,[(1+\gamma(S)(1-\nu(\mathbf{k}))]\sin^2\theta+(8\pi/\tilde{\beta})\times\sin^2\theta_{\mathbf{k}}e^{-2i\varphi_{\mathbf{k}}}\}. \quad (44)$$

Moreover,

$$A_{\mathbf{k}}=A_{\mathbf{k}}^{(0)}+A_{\mathbf{k}}^{(1)}\delta, \quad (45)$$

$$B_{\mathbf{k}}=-J(0)S\tilde{\beta}\delta\left[\left(1-\frac{1}{2S}\right)^{-1/2}\sin^2\theta+(8\pi/\tilde{\beta})\sin^2\theta_{\mathbf{k}}e^{-2i\varphi_{\mathbf{k}}}\right].$$

The remaining coefficients $A^{(i)}$ and $B^{(i)}$ are given explicitly in Appendix 2.

One easily gets from the Hamiltonian (39) equations for the equilibrium position of the spin **S** as a function of the external field:

$$\cos\theta=(h/\tilde{\beta}\delta)\,[1-P(h)+2Q(h)+O(\delta^2)], \quad (46)$$

as well as for the magnetization along the z axis:

$$M_H=\frac{\mu S}{V_0}\frac{h}{\tilde{\beta}\delta}[1-P(h)+Q(h)+O(\delta^2)],\quad h\leqslant h_{cr},$$
$$M_H=\frac{\mu S}{V_0}[1-Q(h)+O(\delta^2)],\quad h>h_{cr} \quad (47)$$

and the longitudinal susceptibility

$$\chi_{HH}=(1/\tilde{\beta})\,[1-P(h)-hP'(h)+O(\delta^{3/2})],\quad h<h_{cr},$$
$$\chi_{HH}=-\delta Q'(h)+O(\delta^2),\quad h>h_{cr} \quad (48)$$

(primes indicate differentiation with respect to the field). The quantities $P(h)$, $Q(h)$, and χ_{HH} are evaluated in Appendix 4. We note that the deflectionof the magnetic moment from its nominal value in fields $h>h_{cr}$ and the shift of the transition point compared to the value determined in the RPA($h_0=\tilde{\beta}\delta$) are caused by the effect of the dipole interactions.

One further shows easily (see Sec. 1) that the renormalization of the spin/wave spectrum up to terms $\sim O(\delta^2)$ manifests itself in the renormalization of the coefficient $\tilde{A}_k$ and for small wave vectors we have

$$\tilde{A}_{\mathbf{k}}\to\tilde{\tilde{A}}_{\mathbf{k}}=\tilde{A}_{\mathbf{k}}-\lambda_1(\mathbf{k})\delta^{3/2}+C\delta^2+O(\delta^2(\mathbf{k}\Delta)^2), \quad (49)$$

where c and λ_1 are defined in (A5.8) and (A2.2), (A2.1). Finally, the spin wave spectrum has the form

$$\tilde{\varepsilon}_{\mathbf{k}}=[(\tilde{A}_{\mathbf{k}})^2-|\tilde{B}_{\mathbf{k}}|^2]^{1/2}, \quad (50)$$

and one checks easily that the gap in the spin-wave spectrum in fields $j<h_{cr}$ is caused exclusively by the dipole interactions. We give now the explicit expression for the spin-wave spectrum in a zero field ($|\mathbf{k}\cdot\Delta|\ll1$):

$$\tilde{\varepsilon}_{\mathbf{k}}^2=(J(0)S)^2(\varepsilon_1+\varepsilon_2), \quad (51)$$

where ε_1 determines the spectrum gap caused by the dipole interaction:

$$\varepsilon_1=32\pi\tilde{\beta}\delta^2\sin^2\theta_{\mathbf{k}}\sin^2\varphi_{\mathbf{k}}, \quad (52)$$

and ε_2 determines the speed of the spin waves for small wave vectors:

$$\varepsilon_2=(1-\nu(\mathbf{k}))\left\{1-\nu(\mathbf{k})+2\beta\delta+16\pi\delta\sin^2\theta_{\mathbf{k}}+\beta^2\delta^2\left[\left(1-S+2WS\left(1-\frac{1}{2S}\right)\right)\Big/S^2\left(1-\frac{1}{2S}\right)+O\left(\frac{4\pi}{\beta}\right)\right]+O(\delta^2(\mathbf{k}\Delta)^2)\right\}. \tag{53}$$

Equations (51) and (52) generalize the results of Refs. (9) and (17) in which the coefficient of the term quadratic in δ was evaluated up to terms of order $1/S$ and $1/S^2$, respectively.

CONCLUSION

In our analysis we limited ourselves to the case $T=0$, i.e., we studied only the zero-point oscillations of the ferromagnet. It is, however, well known[6,23–25] that at arbitrarily low but finite temperatures the system will behave classically in the vicinity of the transition point, i.e., for $h=h_{\rm cr}$ one will observe divergences of the thermodynamic characteristics with exponents corresponding to the Ising model for EAFM and XY model for EPFM.[6)] At the same time outside the vicinity of the critical point the behavior of the system is determined by the quantum-mechanical zero-point oscillations, so that the critical exponents correspond to the Landau theory (see Ref. 24). This means that when one approaches the transition field ($h=h_{\rm cr}(T)$) one shoudl observe a cross-over from the Landau exponents to the corresponding model exponents,[23] and the change in behavior occurs when

$$|h-h_{\rm cr}(T)|\approx(T/J(0)S)^{2/z},$$

where z determines the ratio of the power exponents of the wave vector and frequency in the effective classical Hamiltonian that leads to the same partition function as the initial quantum-mechanical Hamiltonian.[6,23] For EPFM in fields above critical the system is collinear and the construction of the corresponding Hamiltonian proceeds by well known methods[6]; as a result we get $z=2$. For EAFM the effective classical Hamiltonian must correspond to the four-vertex Ising model[24,25]; the frequency and wavevector then enter in the Hamiltonian in identical manner, so that $z=1$.

In conclusion the author wants to express his deep gratitude to M. I. Kaganov for his constant support of the work and for many conversations, and also thank all with whom he has discussed this paper.

APPENDIX 1

The operators $\mathscr{H}_s^{(n)}$ have the following form:

$$\mathscr{H}_s^{(3)}=\sum_{123}\Phi(1,2,3)\,a_1^+a_2^+a_3\Delta(1+2-3)+\text{H.c.},$$

$$\mathscr{H}_s^{(4)}=\mathscr{H}_{s,ex}^{(4)}+\mathscr{H}_{s,rel}^{(4)}, \tag{A1.1}$$

where $\mathscr{H}_{s,ex}^{(4)}$ was given in the main text and

$$\mathscr{H}_{s\,rel}^{(4)}=\sum_{1234}\Phi^{rel}(12,34)\,a_1^+a_2^+a_3a_4\Delta(1+2-3-4)$$
$$+\Phi(123,4)\,a_1^+a_2^+a_3^+a_4\Delta(1+2+3-4)+\text{H.c.}, \tag{A1.2}$$

$$\mathscr{H}_s^{6}=\Sigma\,\Phi(123,456)\,a_1^+a_2^+a_3^+a_4a_5a_6\Delta(1+2+3-4-5-6), \tag{A1.3}$$

where the $\Phi(\dots)$ are some functions of the wave vectors $\mathbf{k}$ (we use the notation $\mathbf{1}\equiv\mathbf{k}_1$, and so on).

Explicitly these functions look as follows:

$$\Phi(12,3)=-2iJ(0)S\left(\frac{S}{2N}\right)^{1/2}\left\{\left[\beta\delta\cos\theta\left(1-\left(1-\frac{1}{2S}\right)^{1/2}+S^{-1}\left(1-\frac{1}{2S}\right)^{-1/2}\right)-h\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)\right]\sin\theta\right.$$
$$\left.+i\frac{3\delta}{S}V_0\sum\frac{(z')R^+}{R^5}(e^{i\mathbf{k}_1\mathbf{R}}+e^{i\mathbf{k}_2\mathbf{R}})\right\},$$

$$\Phi^{rel}(12,34)=\frac{J(0)\delta}{2N}\left\{\beta(3\cos^2\theta-2)+V_0\sum_{R}\frac{R^2-3(z')^2}{R^5}\right.$$
$$\times\left[[e^{i(\mathbf{k}_1-\mathbf{k}_3)\mathbf{R}}+e^{i(\mathbf{k}_1-\mathbf{k}_4)\mathbf{R}}+e^{i(\mathbf{k}_2-\mathbf{k}_4)\mathbf{R}}+e^{i(\mathbf{k}_2-\mathbf{k}_3)\mathbf{R}}]\right.$$
$$\left.\left.+2S\left[1-\left(1-\frac{1}{2S}\right)^{1/2}\right][e^{i\mathbf{k}_1\mathbf{R}}+e^{i\mathbf{k}_2\mathbf{R}}+e^{i\mathbf{k}_3\mathbf{R}}+e^{i\mathbf{k}_4\mathbf{R}}]\right]\right\},$$

$$\Phi(123,4)$$
$$=-\frac{J(0)\delta}{2N}\left[\beta S\left(1-\frac{1}{2S}\right)^{-1/2}\left(1-\left(1-\frac{1}{S}\right)^{1/2}\right)\cos^2\theta\right.$$
$$\left.-4S\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)V_0\sum_{R}\frac{(R^+)^2}{R^5}(e^{i\mathbf{k}_1\mathbf{R}}+e^{i\mathbf{k}_2\mathbf{R}}+e^{i\mathbf{k}_3\mathbf{R}})\right],$$

$$\Phi(123,456)$$
$$=\frac{J(0)S}{N^2}\left\{\left[\frac{1}{2}\left(1-\left(1-\frac{1}{S}\right)^{1/2}\right)+\left(1-\frac{1}{2S}\right)^{1/2}-1\right]\right.$$
$$\times\left(\frac{1}{3}\sum_{p=1}^{6}\nu(\mathbf{k}_p)\right)-\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)^2$$
$$\left.\times\left(\frac{1}{18}\sum_{i=1,2,3}\nu(\mathbf{k}_i-\mathbf{k}_p-\mathbf{k}_m)\right)\right\},$$

$p,\ m=4,\ 5,\ 6,\quad p\neq m.$

We give here the explicit expressions for the form in sixth order in the operators $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$, due to the exchange terms in the Hamiltonian, as this operator determines the spin wave spectrum to the same order as the operators which are fourth-order expressions in $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$.

APPENDIX 2

The coefficients determing the solution of the set (12), (13) have the form

(a) EAFM:

$$A_{\mathbf{k}}^{(2)}=\lambda_1(1-{}^1/_2(1-\nu(\mathbf{k})))\equiv\lambda_1(\mathbf{k}),$$
$$A_{\mathbf{k}}^{(3)}=(\qquad-\nu(\mathbf{k})))\cos^4\theta,$$
$$B_{\mathbf{k}}^{(2)}=\lambda_2(1-\nu(\mathbf{k}))\cos^2\theta, \tag{A2.1}$$
$$B_{\mathbf{k}}^{(3)}=(\lambda_5+\lambda_6(1-\nu(\mathbf{k})))\cos^2\theta,$$

where

$$\lambda_1=2J(0)S\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)^2\frac{\delta^{-3/2}}{N}\sum_{\mathbf{p}}\frac{(B_{\mathbf{p}})^2}{\varepsilon_{\mathbf{p}}(A_{\mathbf{p}}+\varepsilon_{\mathbf{p}})},\quad \text{(A2.2)}$$

$$\lambda_2=J(0)S\tilde{\beta}\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)\frac{\delta^{-1/2}}{N}\sum_{\mathbf{p}}\left(\frac{1}{1-\nu(\mathbf{p})}-\frac{J(0)S}{\varepsilon_{\mathbf{p}}}\right),\quad \text{(A2.3)}$$

$$\lambda_3=\frac{J(0)S\tilde{\beta}}{4}\left\{\left(1-\left(1-\frac{1}{S}\right)^{1/2}\right)\left[\frac{3}{S(1-1/2S)}+9W^2-4W+12\gamma W-6\gamma+3\gamma^2\right]+\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)^2[\gamma^2+2\gamma W-9W^2+10W-4]-6\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)[3W^2-2W+4\gamma W-2\gamma+\gamma^2]\right\},\quad \text{(A2.4)}$$

$$\lambda_5=\frac{J(0)S\tilde{\beta}^2}{4S^2(1-1/2S)}\left\{\left(1+2S\left(1-\frac{1}{2S}\right)W\right)\times\left[\left(\frac{2h}{\tilde{\beta}\delta}\cos\theta+2\left(1-\frac{3}{2}\cos^2\theta\right)\right)\times 2S\left(1-\frac{1}{2S}\right)^{1/2}\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)-(3\cos^2\theta-2)\right]\right\}.\quad \text{(A2.5)}$$

The expressions for the coefficients λ_4 and λ_6 lare not used in what follows and we do not give them. The quantities γ and W are given in (20);

(b) EPFM:

$$\begin{aligned}A_{\mathbf{k}}^{(2)}&=\lambda_1[1-{}^1/_2(1-\nu(\mathbf{k}))]\doteq\lambda_1(\mathbf{k}),\\ A_{\mathbf{k}}^{(3)}&=[\lambda_3+\lambda_4(1-\nu(\mathbf{k}))]\sin^4\theta,\\ B_{\mathbf{k}}^{(2)}&=\lambda_2(1-\nu(\mathbf{k}))\sin^2\theta,\\ B_{\mathbf{k}}^{(3)}&=[\lambda_5+\lambda_6(1-\nu(k))]\sin^2\theta,\end{aligned}\quad \text{(A2.6)}$$

where λ_1, λ_2, and λ_3 are given by Eqs. (A2.2) and (A2.4) (using (43), (44)) while the coefficient λ_5 turns out to equal:

$$\lambda_5=-\frac{J(0)S\tilde{\beta}^2}{4S^2(1-1/2S)}\left\{\left[1+2SW\left(1-\frac{1}{2S}\right)\right]\times\left[\left(\frac{2h}{\tilde{\beta}\delta}\cos\theta+1-3\cos^2\theta\right)\times 2S\left(1-\frac{1}{2S}\right)^{1/2}\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)+(3\sin^2\theta-2)\right]\right\}.\quad \text{(A2.7)}$$

APPENDIX 3

The operator $\mathscr{H}_D$ determining the contribution from the anharmonic terms in the Hamiltonian has the form

$$\mathscr{H}_D=\mathscr{H}_D^{(1)}+\mathscr{H}_D^{(2)}+\mathscr{H}_D^{(3)}+\dots,\quad \text{(A3.1)}$$

where the term $\mathscr{H}_D^{(1)}$ is caused by the effect of the exchange interaction:

$$\mathscr{H}_D^{(1)}=-\frac{J(0)}{8N}\sum_{1234}(\nu(1-3)+\nu(1-4)+\nu(2-3)+\nu(2-4)-4S\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)(\nu(1)+\nu(2)+\nu(3)+\nu(4)))\{\alpha_1\alpha_2\beta_{-3}^*\beta_{-4}^*C_1^+C_2^+C_{-3}^*C_{-4}^+ + 2\alpha_1\alpha_2\beta_{-3}^*\alpha_4C_1^+C_2^+C_{-3}C_4 + {}^1/_2\,\alpha_1\alpha_2\alpha_3\alpha_4C_1^+C_2^+C_3C_4\}\,\Delta(1+2-3-4),\quad \text{(A3.2)}$$

while the operators $\mathscr{H}_D^{(2)}$ and $\mathscr{H}_D^{(3)}$ determine the effect of the anisotropy, the dipole interaction, and the external field:

$$\mathscr{H}_D^{(2)}=-\frac{J(0)\delta}{2N}\sum_{1234}(\alpha_1\alpha_2\alpha_3\alpha_4C_1^+C_2^+C_3^+C_4+\alpha_1\alpha_2\alpha_3\beta_{-4}^*C_1^+C_2^+C_3^+C_{-4}^+)\times\left[\tilde{\beta}S\left(1-\frac{1}{2S}\right)^{-1/2}\left(1-\left(1-\frac{1}{S}\right)^{1/2}\right)\times\cos^2\theta+4S\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)V_0\sum_R\frac{(R^+)^2}{R^5}(e^{i\mathbf{k}_1\mathbf{R}}+e^{i\mathbf{k}_2\mathbf{R}}+e^{i\mathbf{k}_3\mathbf{R}})\right]\Delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4),\quad \text{(A3.3)}$$

$$\times\mathscr{H}_D^{(3)}=iJ(0)S\left(\frac{2S}{N}\right)^{1/2}\sum_{123}(\alpha_1\alpha_2\beta_{-3}^*C_1^+C_2^+C_{-3}^+ +\alpha_1\alpha_2\alpha_3C_1^+C_2^+C_3)\left\{\left[h\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)-\tilde{\beta}\delta\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)+S^{-1}\left(1-\frac{1}{2S}\right)^{-1/2}\right)\cos\theta\right]\sin\theta-3iV_0\sum_R\frac{(z')(R^+)}{R^5}(e^{i\mathbf{k}_1\mathbf{R}}+e^{i\mathbf{k}_2\mathbf{R}})\right\}\times\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3)\quad \text{(A3.4)}$$

(we have written down here only the operators which turn out to affect the renormalization of the spin wave spectrum up to $O(\delta^2)$). In expressions (A3.2) and (A3.4)

$$\alpha_{\mathbf{k}}=\left(\frac{\tilde{A}_{\mathbf{k}}+\varepsilon_{\mathbf{k}}}{2\varepsilon_{\mathbf{k}}}\right)^{1/2},\quad \beta_{\mathbf{k}}=-\frac{\tilde{B}_{\mathbf{k}}}{|B_{\mathbf{k}}|}\left(\frac{\tilde{A}_{\mathbf{k}}-\varepsilon_{\mathbf{k}}}{2\varepsilon_{\mathbf{k}}}\right)^{1/2}.$$

APPENDIX 4

We give analytical expressions for the functions $P(h)$ and $Q(h)$ which determine the behavior of the magnetic characteristics in various fields.

a) EAFM:

$$\begin{aligned}P(h)&=\tilde{\beta}\delta\eta(S)h^2/\tilde{\beta}^2\delta^2+O(\delta^{3/2}),\\ Q(h)&=(2S)^{-1}\delta^{3/2}\psi(h)+O(\delta^2),\end{aligned}\quad \text{(A4.1)}$$

while one can only obtain analytical expressions for the function $\psi(h)$ in limiting cases:

$$\psi(h)=Z^{3/2}\frac{1}{\sqrt{2\pi}}\left(\frac{\delta}{h}\right)^{1/2}\left[\frac{\tilde{\beta}^2}{16}+\frac{32\pi^2}{15}\right],\quad h\gg h_{cr},$$

$$\psi(h)=Z^{1/2}\frac{\beta^{1/2}}{3\sqrt{2}\pi^2}, \quad \tilde{\beta}\gg 4\pi, \quad h\approx h_{cr}, \tag{A4.2}$$

$$\psi(h)=Z^{1/2}\pi^{1/2}, \quad \tilde{\beta}\ll 4\pi, \quad h\approx h_{cr},$$

where $Z = J(0)$; $\eta(S) = (2S)^{-1}(1 - 1/2S)(W + \gamma(S))$;

b) EPFM:

$$P(h)=\tilde{\beta}\delta\eta(S)(1-h^2/\tilde{\beta}^2\delta^2)+O(\delta^{3/2}),$$
$$Q(h)=(2S)^{-1}\delta^{3/2}\psi(h), \tag{A4.3}$$

where

$$\psi(h)=Z^{1/2}\frac{32\pi}{15\sqrt{2}}\left(\frac{\delta}{h}\right)^{1/2}. \quad h\gg h_{cr}, \quad \psi(h)=Z^{1/2}\pi^{1/2}. \quad h\approx h_{cr}. \tag{A4.4}$$

One easily obtains from Eqs. (A4.1) to (A4.4) an expression for the longitudinal susceptibility.

a) EAFM:

$$\chi_{HH}=\frac{1}{64\sqrt{2}}\frac{1}{\pi S}(\delta Z)^{1/2}\left(\frac{\delta}{h}\right)^{3/2}\left(\beta^2+\frac{512\pi^2}{15}\right)+O(\delta^2), \ h\gg h_{cr},$$

$$\chi_{HH}=\frac{(\delta Z)^{1/2}}{\pi S\beta^{1/2}}\left(\frac{4}{3\sqrt{2}}+\frac{\beta^{3/2}}{32\sqrt{\pi}}\right)+O(\delta^2), \quad h\gtrsim h_{cr},$$

$$\chi_{HH}=\frac{1}{\beta}\left[1+3\beta\delta\eta(S)\frac{h^2}{(\beta\delta)^2}+O(\delta^{3/2})\right], \quad h<h_{cr}; \tag{A4.5}$$

b) EPFM:

$$\chi_{HH}=\frac{4\sqrt{2}}{15}\frac{\pi}{S}(\delta Z)^{1/2}\left(\frac{\delta}{h}\right)^{3/2}+O(\delta^2), \quad h\gg h_{cr},$$

$$\chi_{HH}=\frac{4}{3\sqrt{2}\pi S}\frac{(\delta Z)^{1/2}}{\beta^{1/2}}+O(\delta^2), \quad h\gtrsim h_{cr}, \tag{A4.6}$$

$$\chi_{HH}=\frac{1}{\beta}\left[1+\beta\delta\eta(S)\left(3\frac{h^2}{(\beta\delta)^2}-1\right)+O(\delta^{3/2})\right], \quad h<h_{cr}.$$

It is clear from (A4.5), (A4.6) that at the transition point the longitudinal susceptibility shows a finite jump.

APPENDIX 5

For the sake of argument we consider EAFM in fields $h > h_{cr}$. We note first of all that the last term in $\mathscr{H}_D^{(1)}$ corresponds to the normal exchange between two magnons and does not have any degree of smallness, so that when determining any term in the perturbation theory series (PTS) in powers of δ we must completely sum the "exchange ladder" (*vide infra*). Moreover, since

$$\frac{1}{N}\sum_{\mathbf{k}}\alpha_{\mathbf{k}}\beta_{\mathbf{k}}\sim\delta, \text{ and } \frac{1}{N}\sum_{\mathbf{k}}|\beta_{\mathbf{k}}|^2\sim\delta^{3/2}$$

(see (A4.1)) we can directly check that the main correction to the spectrum (MCS) is determined by the PTS caused by the operator $\mathscr{H}_D^{(1)}$ and containing magnon creation and annihilation processes with the same indexes $\beta_{\mathbf{p}_j}$ and magnon exchange processes with indexes $\alpha_{\mathbf{p}_i}$. Since $\Sigma|B\mathbf{p}_j|^2$ is determined by small $\mathbf{p}_j$ ($\mathbf{p}_j^2\lesssim\delta$) and $\Sigma\alpha^2_{\mathbf{p}_i}$ by large $\mathbf{p}_i\approx 1$, we can to first approximation integrate independently over $\mathbf{p}_i$ and $\mathbf{p}_j$ putting $\mathbf{p}_j\equiv 0$ in all functions of $\mathbf{p}_i$ and $\mathbf{p}_i$ in the numerator. Moreover, a simple analysis shows that the PTS for the MCS $(\Delta\varepsilon_{\mathbf{k}}^{(1)}$ can be constructed from just four blocks (see the figure, *a* to *d*) and since any combination of blocks is realized through an interemediate state with energy $\approx 2\varepsilon_{\mathbf{k}}$, each succeeding term in the series will, apart from a numerical factor, differ from the preceding one by the quantity

$$R(h,\mathbf{k})=q(\mathbf{k})\tilde{A}_{\mathbf{k}}/\varepsilon_{\mathbf{k}}^2,$$
$$R(h,\ k)\ll 1 \text{ when } h>h_{cr},$$
$$R(h,\ k)\ll 1 \text{ when } h\approx h_{cr}, \quad |\mathbf{k}\Delta|\lesssim 1, \tag{A5.1}$$
$$R(h,\ k)\approx 1 \text{ when } h\approx h_{cr}, \quad |k\Delta|\ll 1,$$

where $\tilde{A}_{\mathbf{k}}$ and $\varepsilon_{\mathbf{k}}$ are given by Eqs. (12) and (14) while the quantity $q(\mathbf{k})$ is determined by integrating over the internal wavevectors in the block:

$$q(\mathbf{k})=-4J(0)S\left[1-\left(1-\frac{1}{2S}\right)^{1/2}\right]^2 \times\sum_{\mathbf{p}}|\beta_{\mathbf{p}}|^2\left(1-\frac{1}{2}(1-\nu(\mathbf{k}))\right). \tag{A5.2}$$

In deriving Eq. (A5.2) we succeeded in summing by induction completely the exchange ladder (dark triangle in the figure).

It follows from (A5.1) that outside the immediate vicinity of the transition point we can restrict ourselves in the determination of the MCS to second order perturbation theory, i.e., take only processes *a* and *b* (see the figure) into account, whereas when $h\approx h_{cr}$ (and small $\mathbf{k}$: $|\mathbf{k}\cdot\mathbf{\Delta}|\ll 1$) it is necessary to find the total sum of the series. This can be done relatively simply if we note that the PTS of interest to us corresponds to the following Hamiltonian $\mathscr{H}_{\rm int}$:

$$\mathscr{H}_{int}=\sum_{\mathbf{k}}q(\mathbf{k})\left[(\alpha_{\mathbf{k}}^2+\beta_{\mathbf{k}}^2)C_{\mathbf{k}}^+C_{\mathbf{k}}+\alpha_{\mathbf{k}}\beta_{\mathbf{k}}(C_{\mathbf{k}}C_{-\mathbf{k}}+C_{\mathbf{k}}^+C_{-\mathbf{k}}^+)\right], \tag{A5.3}$$

so that for the determination of the MCS it is sufficient to perform an elementary UV transformation. As a result we find that taking anharmonicity into account to first aproximation leads to a renormalization of the coefficient $A_{\mathbf{k}}$: $\tilde{A}_{\mathbf{k}}\rightarrow\tilde{A}_{\mathbf{k}}+\tilde{A}_{\mathbf{k}}^{(1)}$, where

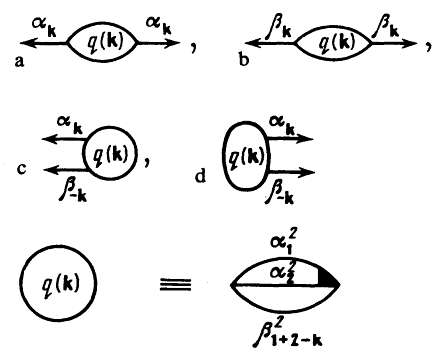

FIG. 1. Virtual processes determining the main contribution to the elementary excitation spectrum; the arrows correspond to creation and annihilation operators with well defined indexes ($\alpha_{\mathbf{k}}$ or $\beta_{\mathbf{k}}$); the dark triangle corresponds to the summation of the exchange ladder (see the text).

$$\tilde{A}_{\mathbf{k}}^{(1)}=q(\mathbf{k})\equiv-\lambda_1(\mathbf{k})\delta^{3/2}, \qquad (A5.4)$$

while $\lambda_1(\mathbf{k})$ is given in (A2.2), (A2.1).

The same result (renormalizationof only the coefficient $\tilde{A}_{\mathbf{k}}$) is obtained also when we consider the next order correction to the spectrum ($\Delta\varepsilon_{\mathbf{k}}^{(2)}\sim\delta^2\tilde{A}_{\mathbf{k}}/\varepsilon_{\mathbf{k}}$) and it then turns out that the corresponding integrals are determined by the large wave vector region so that the quantity $\tilde{A}_{\mathbf{k}}^{(2)}\propto\delta^2$ is independent of the field and hence can be determined elementarily from the condition that the gap in the spin wave spectrum vanish at the phase transition point ($|\mathbf{k}\cdot\Delta|\ll 1$):

$$\tilde{A}_{\mathbf{k}}^{(2)}=\left\{J(0)\beta^2\left[1-S\left(1-\left(1-\frac{1}{2S}\right)^{1/2}\right)\right]\left[W+(2S)^{-1}\cdot\times\left(1-\frac{1}{2S}\right)^{-1}\right]-\lambda_3+O(|\mathbf{k}\Delta|^2)\right\}\delta^2 \qquad (A5.5)$$

(see (A2.4)). We can similarly consider the region below the transition point; the additional renormalization of the coefficient $\tilde{A}_{\mathbf{k}}$ due to the term $\mathcal{H}_D^{(3)}$ (see (A3.4)) can be evaluated exactly:

$$\tilde{A}_{\mathbf{k}}^{(3)}=-J(0)S\beta^2\delta^2\cos^2\theta\sin^2\theta\left(\frac{2W}{S}+\frac{1}{S^2}\frac{1}{(1-1/2S)}\right)+O(\delta^2|\mathbf{k}\Delta|^2). \qquad (A5.6)$$

The final result is the following: taking anharmonicity into account leads to a renormalization of the coefficient $\tilde{A}_{\mathbf{k}}$ ($\tilde{A}_{\mathbf{k}}\rightarrow\tilde{\tilde{A}}_{\mathbf{k}}$) with

$$\tilde{\tilde{A}}_{\mathbf{k}}=\tilde{A}_{\mathbf{k}}-\lambda_1(\mathbf{k})\delta^{3/2}+C\delta^2+O(\delta^2|\mathbf{k}\Delta|^2), \qquad (A5.7)$$

where for EAFM

$$C=\tilde{A}_{\mathbf{k}}^{(2)}\cos^4\theta+\tilde{A}_{\mathbf{k}}^{(3)}, \qquad (A5.8)$$

while for EPFM one must replace $\cos\theta$ in (A5.8) by $\sin\theta$.

1) In the present paper the term "collinear structure" and "non-collinear structure" indicate well defined kinds of symmetry of a spin system (see the following text).

2) We have in mind structures described by the Heisenberg model; in what follows we shall be dealing with just such systems.

3) In NFS with noncollinearity due to the external field the quantity $h/\mu V_0^{-1}$ may emerge as the small noncollinearity parameter. It is interesting that in that case the spin-wave spectrum to second order in h can be determined exactly without recourse to second quantization.[11]

4) Generally speaking, when one uses the HP transformation one must bear in mind the presence of unphysical states[1]; however, a comparison of the HP transformation with the recently proposed exact boson expansion for arbitrary spin[16] shows that in calculations with the chosen accuracy the problem of unphysical states does not arise.

5) A similar "generalized" Bogolyubov transformation leading to integral equations for the quantities $\tilde{A}_{\mathbf{k}}$ and $\tilde{B}_{\mathbf{k}}$ has already been used in a number of papers.[26,27] The authors of these papers considered, however, only the case $S\gg 1$ and restricted themselves even in the initial Hamiltonian to terms $\sim O(1/S)$. In such an approach (choice of solutions accurate to terms $\sim O(1/S)$) the equations for the quantities $\tilde{A}_{\mathbf{k}}$ and $\tilde{B}_{\mathbf{k}}$ are in fact algebraic.

6) In this section we do not consider the complications introduced by dipole interactions.

[1] A. I. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminskiĭ, Spinovye volny (Spin waves) Moscow, Nauka, 1967 [English translation published by North Holland, Amsterdam].
[2] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[3] F. Dyson, Phys. Rev. **102**, 1217 (1956); S. V. Maleev, Zh. Eksp. Teor. Fiz. **33**, 1010 (1957) [Sov. Phys. JETP **6**, 776 (1958)].
[4] P. A. Lindgaard and A. Kovalska, J. Phys. **C9**, 2081 (1976).
[5] P. A. Lindgaard and O. Danielson, J. Phys. **C7**, 1523 (1974).
[6] I. Goldhirch, E. Levich, and V. Yakhot, Phys. Rev. **B19**, 4780 (1979).
[7] L. D. Filatova and V. M. Tsukernik, Zh. Eksp. Teor. Fiz. **53**, 2203 (1967) [Sov. Phys. JETP **26**, 137 (1968)].
[8] J. Feder and E. Pytte, Phys. Rev. **168**, 640 (1968).
[9] E. Rastelli and P. A. Lindgaard, J. Phys. **C12**, 1899 (1979).
[10] M. Steiner and J. Kjems, J. Phys. **C10**, 2655 (1977).
[11] I. G. Gochev and V. M. Tsukernik, Fiz. Tverd. Tela (Leningrad) **15**, 1963 (1973) [Sov. Phys. Solid State 15, 1315 (1974)].
[12] L. D. Filatova and V. M. Tsukernik, Phys. Stat. Sol. **70**, 45 (1975).
[13] L. M. Noskova, Fiz. Tverd. Tela. (Leningrad) **18**, 1669 (1976) [Sov. Phys. Solid State **18**, 969 (1976)].
[14] V. M. Loktev and V. S. Ostrovskiĭ, Ukr. Fiz. Zh. **23**, 1708 (1978).
[15] F. P. Onufrieva, Zh. Eksp. Teor. Fiz. **80**, 2372 (1981) [Sov. Phys. JETP **53**, 1241 (1981)].
[16] I. Goldhirch, J. Phys. **A13**, 453 (1980).
[17] E. Rastelli and A. Tassi, J. Appl. Phys. **53**, 3(II) (1982).
[18] E. Rastelli and A. Tassi, J. Phys. **C13**, 4377 (1980).
[19] E. Rastelli and A. Tassi, J. Phys. **C15**, 509 (1982).
[20] U. Balucani, M. G. Pini, A. Rettori, and V. Tognetti, J. Phys. **C13**, 3895 (1980).
[21] U. Balucani and V. Togmetti, J. Phys. **C12**, 5513 (1980).
[22] L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum mechanics) Moscow, Nauka, 1973 [English translation published by Pergamon, Oxford].
[23] J. Hertz, Phys. Rev. **B14**, 1165 (1976).
[24] M. Suzuki, Progr. Theor. Phys. **56**, 1454 (1976).
[25] P. Pfeuty, J. Phys. **C9**, 3993 (1976); A. P. Young, J. Phys. **C8**, L309 (1975).
[26] V. G. Bar'yakhtar, E. V. Zarochentsev, and V. A. Popov, Fiz. Met. Metalloved. **25**, 3 (1968).
[27] V. G. Kukharenko, Zh. Eksp. Teor. Fiz. **69**, 632 (1975) [Sov. Phys. JETP **42**, 321 (1975)].

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