Threshold magnetic fields for long-lived modes in the spin dynamics in ^{3}He-B

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The temperature dependence of the threshold values of the magnetic fields turned off in WP-mode experiments is investigated. It is shown that the threshold values can be related to the characteristics of the attractor regimes of magnetization relaxation in ^{3}He-*B* in a turned-off field. The regime in which the external field is partially turned off and the subsequent relaxation of the magnetization occurs in a residual field of the order of the dipole field is also considered. It is concluded that the fields turned off can have threshold values in this case as well. The existence of the threshold fields is ascribed to the existence of instability and chaotic-state regimes in the spin dynamics of superfluid ^{3}He-*B*. The form of the order parameter for the initial phase of the spatially inhomogeneous relaxation in a threshold magnetic field is discussed.

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INTRODUCTION

The progress that has been made in the understanding of the nature of the superfluid phases of ^{3}He is in many respects due to their investigation with the aid of nuclear magnetic resonance. The first experiments were performed by the pulsed NMR method (see Ref. 1), which allowed the detection in ^{3}He-*B* of a distinctive regime of nonlinear magnetization ringing called the Brinkman-Smith mode.[1] Later experiments were devoted to the study of regimes in which the external field $\mathbf{H}_i$ was changed abruptly to some steady field of intensity $\mathbf{H}$ parallel to the initial field $\mathbf{H}_i$. Nonlinear ringing modes different from the Brinkman-Smith mode were discovered in these experiments,[1] and the analysis of the nonlinear magnetization ringing modes occurring in different NMR regimes is an interesting problem.

In the present paper we consider the relaxation of the magnetization in superfluid ^{3}He-*B* in two regimes: 1) in the absence of an external field; 2) in a constant external magnetic field whose strength is, in order of magnitude, close to that of the dipolar field. It is assumed that in the initial equilibrium state of the system the spin vector $\mathbf{S}$ is perpendicular to the rotation axis $\mathbf{c}$ of the orthogonal order-parameter matrix $R(\theta,\mathbf{c})$ for ^{3}He-*B* (here θ is the rotation angle of the matrix $R(\theta,\mathbf{c})$); in actual experiments the direction of the vector $\mathbf{c}$ is fixed by the walls of the rectangular cavity containing the ^{3}He-*B*, while the direction of the spin vector $\mathbf{S}$ is determined by the initial external field (see Refs. 1–3). After the system has relaxed to the equilibrium state, the field $\mathbf{H}_i$ is turned off abruptly (i.e., in a time much shorter than the relaxation time) either completely or partially to some field $\mathbf{H}$ (which in the present paper will be called the residual field) parallel or antiparallel to the initial field $\mathbf{H}_i$. After the initial field has been turned off, the system relaxes from the resulting nonequilibrium state, which is characterized by the perpendicularity of the initial spin $\mathbf{S}$ and the initial axis $\mathbf{c}$ of the order parameter.

The relaxation regimes described above have been experimentally investigated.[1–3] The relaxation in zero field, i.e., after the initial magnetic field has been switched off completely, has been investigated in great detail. A long-lived magnetization ringing mode—the so-called WP (or wall-pinned) mode—has been found to occur in this regime, and the theoretical explanation of this mode is one of the triumphs of the Leggett-Takagi (LT) theory[4] of spin relaxation in superfluid ^{3}He (Refs. 4–7). In particular, a very good experimental confirmation of the linear time dependence of the square of the period of the WP mode has been obtained.[3,4]

Webb *et al.*[3] have reported the discovery of an interesting phenomenon: the dependence of the WP mode on the strength of the initial field $\mathbf{H}_i$. They found that, if the strength of the field $\mathbf{H}_i$ turned off is increased at a fixed temperature, then there exists a critical field for which the WP mode is no longer observed, and that the normal nonlinear magnetization ringing appears on going through the critical field. Thus, in the case of the WP mode there exists a threshold value for the external field turned off, which, as has been established by Webb *et al.*,[3] is temperature dependent (see Fig. 1). In their paper[3] Webb *et al.* attempt to explain the observed temperature dependence by suggesting that a textural transition that destroys the ringing of the WP mode occurs in the ^{3}He-*B* sample at the threshold value of the magnetic field. But there exists for the textural transitions occurring in ^{3}He-*B* located in an external magnetic field a theory, developed in Ref. 8, which shows that the corresponding critical field is temperature dependent. It is significant that this temperature dependence does not agree, even qualitatively, with the dependence reported by Webb *et al.*[3] Thus, it can be assumed that the threshold fields for the WP mode do not owe their existence to the textural transitions.

In that case what is the cause of the disappearance of the WP mode for the threshold magnetic fields?

The answer to this question is offered in the present paper on the basis of the phenomenon of instability that we

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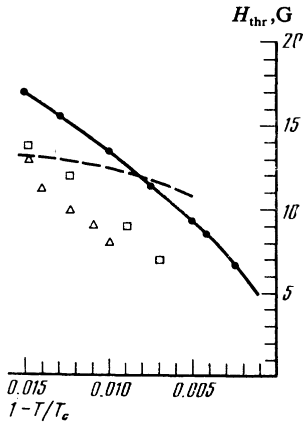

FIG. 1. Dependence on the reduced temperature $1 - T/T_c$ of the critical field H_{thr} above which the WP mode is no longer observed in ^{3}He-B under a pressure of 20.7 bar. The dashed curve depicts the temperature dependence of the critical field for the textural transition; the continuous curve, the temperature dependence of the threshold value computed in the present paper from the condition for certain passage through the pinch point P; actually, the threshold is apparently lower; △ and □ denote two series of experimental points.

earlier found[9] to occur in the spin dynamics in ^{3}He-B. Indeed, we can consider the more general situation, corresponding to the mode found by Maki and Hu,[6] in which the magnetic field is not turned off completely, but there remains some residual field several times (e.g., 2–3 times) weaker than the dipolar field. It turns out that in this regime also there exist threshold fields for which the corresponding modes cease to exist.

The theoretical investigation of the magnetization relaxation in the field range studied in the present paper meets with certain difficulties, since the highly developed asymptotic methods for strong fields[10] are of limited applicability. Therefore, it is justified to use the methods of qualitative and numerical analyses, which have been successfully used in a number of recent investigations.[11–14]

THE THRESHOLD VALUES FOR FIELDS TURNED OFF COMPLETELY

The order parameter for ^{3}He-B has the form

$$A_{ij}=(\Delta/\sqrt{3})\,e^{i\varphi}R_{ij},$$

where R_{ij} is a three-dimensional rotation matrix given by the angle θ and the unit vector $\mathbf{c}$ of the axis of rotation:

$$R_{ij}=\delta_{ij}\cos\theta+(1-\cos\theta)\,c_ic_j-\varepsilon_{ijk}c_k\sin\theta.$$

It should be noted that R_{ij} is invariant under the transformation

$$\mathbf{c}\to-\mathbf{c},\qquad \theta\to2\pi-\theta. \tag{1}$$

In the region of fields studied in the present paper the characteristic frequencies satisfy the condition $\omega\tau \ll 1$, where τ is the quasiparticle lifetime at the Fermi surface, and, consequently, the LT theory[4] can be considered in the hydrodynamic approximation. Further, assuming spatial homogeneity with respect to the spin and the order parameter, we can represent it as a Hamiltonian system with a dissipative function.[15] With the aid of the dimensionless variables

$$\mathbf{S}_R=\omega_0^{-1}\gamma^2\chi^{-1}\mathbf{S},\qquad t_R=\omega_0 t,$$

where γ is the gyromagnetic ratio, χ is the susceptibility, and ω_0 is a characteristic frequency, which is taken in the present paper to be equal to 10^6 rad/sec, since we consider fields close in strength to the dipolar field and the corresponding frequencies are close to Ω_L, we can write the LT equations in the form (the index R is dropped everywhere below)

$$\begin{aligned}
&\frac{d}{dt}\mathbf{S}=[\mathbf{S}\times\mathbf{H}]+\frac{16}{15}\Omega^2\omega_0^{-2}\sin\theta\left(\cos\theta+\frac{1}{4}\right)\mathbf{c},\\
&\frac{d}{dt}\mathbf{c}=\frac{1}{2}[(\mathbf{S}-\mathbf{H})\times\mathbf{c}]+\frac{1}{2}(\mathbf{S}-\mathbf{H})\operatorname{ctg}\frac{1}{2}\theta\\
&\qquad-\frac{1}{2}((\mathbf{S}-\mathbf{H})\mathbf{c})\,\mathbf{c}\operatorname{ctg}\frac{1}{2}\theta,\\
&\frac{d}{dt}\theta=(\mathbf{S}-\mathbf{H})\mathbf{c}+\frac{16}{15}\Gamma_\parallel\omega_0^{-1}\sin\theta\left(\cos\theta+\frac{1}{4}\right).
\end{aligned}\tag{2}$$

Here $\mathbf{H}$ is the external field, $\Omega=\Omega_L$ is the Leggett frequency, and $\Gamma_\parallel$ is the width of the longitudinal NMR lines. It should be noted that the equations (2) are invariant under the transformation (1). In this section we assume that $\mathbf{H}=0$, i.e., that the initial field is suddenly turned off.

In Ref. 13 one of us (V.L.) introduced the scalar variables

$$S_\parallel=\mathbf{S}\mathbf{c},\quad S_\perp=(S^2-S_\parallel^2)^{1/2},\ \theta,$$

which, in the case of zero external field, satisfy the following system of equations[11]:

$$\begin{aligned}
&\frac{d}{dt}S_\parallel=\frac{1}{2}S_\perp^2\operatorname{ctg}\frac{\theta}{2}+\frac{16}{15}\left(\frac{\Omega}{\omega_0}\right)^2\sin\theta\left(\cos\theta+\frac{1}{4}\right),\\
&\frac{d}{dt}S_\perp=-\frac{1}{2}S_\parallel S_\perp\operatorname{ctg}\frac{\theta}{2},\\
&\frac{d}{dt}\theta=S_\parallel+\frac{16}{15}\Gamma_\parallel\omega_0^{-1}\sin\theta\left(\sin\theta+\frac{1}{4}\right).
\end{aligned}\tag{3}$$

It should be noted that the equations for the WP mode, which is equivalent in form to the mode found by Brinkman,[5] are obtained by discarding the relaxation term proportional to $\Gamma_\parallel\omega_0^{-1}$, and equating the right-hand sides of the system (3) to zero. As shown in Refs. 11 and 9, the space of the variables $S_\parallel$, $S_\perp$, and θ has a surface that attracts the solutions to the equations (3) (see Fig. 2), i.e., the system of

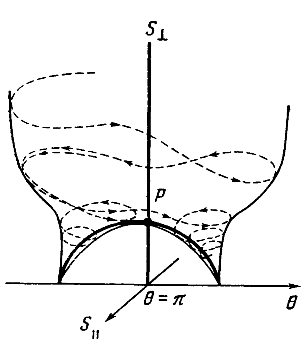

FIG. 2. Phase picture of the system of equations (3). The heavy curves depict the trident formed by the WP mode and the unstable solution $\theta=\pi$, $S_\parallel=0$. The thin lines depict the attractor; P is the pinch point of the attractor; the dashed curves depict typical trajectories.

equations (3) possesses an attractor. Since the equations (3) are a consequence of the equations (2), the latter also should possess an attractor, which, however, is less visualizable. In fact it is located in the seven-dimensional space of the variables **S**, **c**, and θ. Thus, the scalar variables $S_\parallel$, $S_\perp$, and θ are, as it were, a three-dimensional window in the seven-dimensional world of the spin dynamics.

As shown in Ref. 9, the legs of the attractor (see Fig. 2) come very close to the prongs of the WP mode, which, together with the unstable solution[11] $\theta = \pi$, $S_\parallel = 0$, form a figure resembling a trident. At the trident's cross-piece, which corresponds to magnetic fields of the order of the dipolar field, the attractor possesses a pinch point, which determines the region of instability,[9] in the sense that two close trajectories, having come fairly close to the pinch point, diverge in the order-parameter space so rapidly that the distance between them become of the order of unity after one period of rotation of the vector **c**.[9] The dispersal of the trajectories in parameter space entails the dispersal of the corresponding values in spin space.[9]

The central point is the observation that the threshold fields discussed in the Introduction can be estimated as a function of the nature of the arrival of the solutions to the system (3) at the attractor. If the initial field is sufficiently weak, a trajectory is drawn fairly quickly (with characteristic time of the order of 1 msec) to an attractor leg, and is identified in experiments as the WP mode.[9] If the initial field is too high, the trajectory is drawn to the attractor in the region above the pinch point, and is identified as a nonlinear magnetization ringing.[4,11] If the initial field has a value at which the trajectory is drawn to the neighborhood of the pinch point, then the system falls within the instability region,[9] where the spatial inhomogeneities of the spin and the order parameter should be taken into consideration. Actually, inhomogeneities of this kind always exist in a real system; the question is how strong they are. Owing to the instability, they can develop to such an extent that a dephasing of the spin occurs, and coherent ringing becomes impossible.

A possible procedure for computing the upper threshold fields with the aid of a computer follows from the observations made above. In the present investigation the solutions to the equations were computed with the aid of the Runge-Kutta algorithm; the double-precision regime was used to monitor the accuracy of the solutions.

The experimental data, which correspond to a pressure of 20.7 bar, were taken from Refs. 3 and 16. The temperature dependence for Ω has the form[16]

$$\Omega = 2\pi\sqrt{12}\cdot 10^5 (1 - T/T_c)^{1/2}.$$

The temperature dependence for $\Gamma_\parallel$ can be extracted from Ref. 3 in the following manner. It follows from the Leggett-Tagaki theory[4] [see Eqs. (6.13) and (6.31) in Ref. 4] that

$$\Gamma_\parallel = {}^4/_3 (2\pi)^{-2} \Omega^2 \alpha,$$

where α is the coefficient of proportionality in the time dependence of the mode frequency: $f^{-2} = f_0^{-2} + \alpha t$. The values of α appreciably depend on the initial fields (see Fig. 7 in Ref. 3). If we extrapolate α to zero initial fields, then the temperature dependence for α has the form $\alpha^{-2} = A \times 10^{10} (1 - T/T_c)$. A value of 5.8 is given for A in Ref. 3.

As T_c is approached, A varies within the limits 5.8–6.0, but, as the computations carried out in the present investigation show, this does not affect the computed threshold-magnetic-field values to within 0.01 G.

Figure 1 shows the computed threshold field values. The constant A was taken to be equal to 5.8. It can be seen that the computed threshold field curve qualitatively agrees with the experimental data.

THE THRESHOLD VALUES FOR PARTIALLY TURNED OFF FIELDS

In the present section we assume that the field is not turned off completely, and that the system relaxes in a residual field that is parallel or antiparallel to the initial field and is of the same order of magnitude as, but lower than, the dipolar field. At the initial moment of the relaxation the vector **c** is oriented by the walls of the container perpendicularly to the field, while the angle θ has the equilibrium value $\arccos(-1/4)$.

The behavior of the solutions to the system (2) is investigated by constructing an auxiliary system for the orthogonal invariants of the initial vectorial variables **S** and **c**. Constructions of this kind were first introduced by Pohlmeyer[17] in the chiral theory of fields. In the present paper we consider the three-dimensional rotation invariants **Hc**, $(\mathbf{S}-\mathbf{H})\mathbf{H}$, $(\mathbf{S}-\mathbf{H})^2$, $(\mathbf{S}-\mathbf{H})\mathbf{c}$, $[\mathbf{S}\times\mathbf{H}]\mathbf{c}$, θ, for which the following equations, which follow from the system (2), are satisfied:

$$\frac{d}{dt}\mathbf{Hc} = -\frac{1}{2}[\mathbf{S}\times\mathbf{H}]\mathbf{c} + \frac{1}{2}\operatorname{ctg}\frac{\theta}{2}\left[(\mathbf{S}-\mathbf{H})\mathbf{H} - ((\mathbf{S}-\mathbf{H})\mathbf{c})(\mathbf{Hc})\right],$$

$$\frac{d}{dt}(\mathbf{S}-\mathbf{H})\mathbf{c} = [\mathbf{S}\times\mathbf{H}]\mathbf{c} + \frac{1}{2}\left[(\mathbf{S}-\mathbf{H})^2 - ((\mathbf{S}-\mathbf{H})\mathbf{c})^2\right]\operatorname{ctg}\frac{\theta}{2} + \frac{16}{15}\left(\frac{\Omega}{\omega_0}\right)^2 \sin\theta\left(\cos\theta + \frac{1}{4}\right),$$

$$\frac{d}{dt}(\mathbf{S}-\mathbf{H})\mathbf{H} = \frac{16}{15}\left(\frac{\Omega}{\omega_0}\right)^2 \sin\theta\left(\cos\theta + \frac{1}{4}\right)\mathbf{Hc},$$

$$\frac{d}{dt}(\mathbf{S}-\mathbf{H})^2 = \frac{8}{15}\left(\frac{\Omega}{\omega_0}\right)^2 \sin\theta\left(\cos\theta + \frac{1}{4}\right)(\mathbf{S}-\mathbf{H})\mathbf{c},$$

$$\frac{d}{dt}[\mathbf{S}\times\mathbf{H}]\mathbf{c} = (\mathbf{Hc})\left[(\mathbf{S}-\mathbf{H})\mathbf{H} + \frac{1}{2}(\mathbf{S}-\mathbf{H})^2\right],$$

$$\frac{d}{dt}\theta = (\mathbf{S}-\mathbf{H})\mathbf{c} + \frac{16}{15}\Gamma_\parallel\omega_0^{-1}\sin\theta\left(\cos\theta + \frac{1}{4}\right). \tag{4}$$

It should be noted here that the general behavior of a conservative dynamical system is in many respects determined by the steady-state solutions for the system. These solutions are of no less importance in the description of the relaxation effects. All the steady-state solutions to the Leggett equations for ^{3}He-B were recently found by Fomin[18] with the aid of the Euler angles and rotating systems of coordinates. For the purposes of the present investigation it is necessary to find an expression for the steady-state solutions with the aid of the variables introduced above. To do this, it is sufficient to set the right-hand sides of the equations in (4) to zero, and discard the relaxation term in the equation for θ. As a result, we obtain the following expressions:

1) the Brinkman-Smith solution[19]

$$\theta=\arccos(-{}^1/_4),\qquad (\mathbf{S}-\mathbf{H})\mathbf{c}=0,$$

$$(\mathbf{S}-\mathbf{H})\mathbf{H}=-2\sin^2\frac{\theta}{2}[\mathbf{H}^2-(\mathbf{Hc})^2],\quad |\mathbf{H}|=|\mathbf{S}|;$$

2) the constant solution

$$\mathbf{S}=\mathbf{H},\qquad \mathbf{c}=\mathrm{const},\qquad \theta=\arccos(-{}^1/_4),$$

3) a solution that is a generalization of the solution for the WP mode in the presence of a field[18]

$$\arccos(-{}^1/_4)\leqslant\theta\leqslant 2\pi-\arccos(-{}^1/_4),$$

$$(\mathbf{S}-\mathbf{H})\mathbf{c}=0,\quad (\mathbf{S}-\mathbf{H})^2=[\mathbf{H}\times(\mathbf{S}-\mathbf{H})]^2\Big/\left(\mathbf{H}\sin\frac{\theta}{2}\right)^2$$

$$(\mathbf{S}-\mathbf{H})\mathbf{H}$$
$$=-\mathbf{H}^2\sin^2\frac{\theta}{2}\left[1\mp\left(1-\frac{64}{15}\left(\frac{\Omega}{\omega_0}\right)^2\mathbf{H}^{-2}\left(\cos\theta+\frac{1}{4}\right)\right)^{1/2}\right];$$

4) solutions that have been destabilized on account of the presence of relaxation,

$$\text{A. } \theta<\arccos(-{}^1/_4)\quad\text{or}\quad \theta>2\pi-\arccos(-{}^1/_4),$$

the expressions for $(\mathbf{S}-\mathbf{H})^2$ and $(\mathbf{S}-\mathbf{H})\cdot\mathbf{H}$ are the same as in 3) above;

$$\text{B. } \theta=\pi,\quad (\mathbf{S}-\mathbf{H})\mathbf{c}=0,\quad \mathbf{Hc}=0,\quad \mathbf{S}\|\mathbf{H};$$
$$\text{C. } \theta=\pi,\quad (\mathbf{S}-\mathbf{H})\mathbf{c}=0,\quad (\mathbf{S}-\mathbf{H})\mathbf{H}=2(\mathbf{Hc})^2-2\mathbf{H}^2,$$
$$(\mathbf{S}-\mathbf{H})^2=4\mathbf{H}^2-4(\mathbf{Hc})^2;$$
$$\text{D. } \theta=\pi,\quad \mathbf{H}=\mathbf{S},\quad \mathbf{c}=\mathrm{const}.$$

The indicated formulas describe the relative configuration of the spin and the order parameter; for the resonance frequency shifts, see Refs. 4 and 18.

It is noteworthy that all the steady-state solutions are given by equations for only $\mathbf{H}\cdot\mathbf{c}$, $(\mathbf{S}-\mathbf{H})\cdot\mathbf{H}$, and θ, i.e., they can be seen all at once through the same three-dimensional window defined by the indicated scalars.

Besides the steady-state solutions, the solution first found by Maki and Hu[6] is of great importance for the understanding of the nature of the relaxation. It was obtained by Maki and Hu under the assumption that there was no dissipation, and that the angle θ varied little. Setting $\theta=\mathrm{const}$, Maki and Hu obtained an expression for this solution by means of elliptic integrals. Below we shall call the mode corresponding to the Maki-Hu solution the MH mode. In the presence of relaxation it is highly improbable that we shall obtain a handy formula for the MH mode, and in the present paper we study this mode numerically.

The form of the MH mode as seen through the constructed three-dimensional window is shown in Fig. 3; the

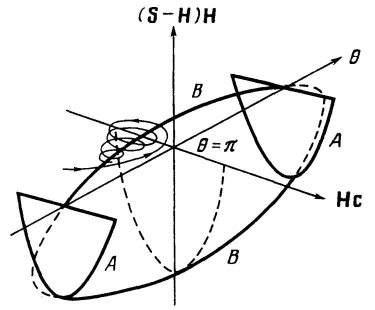

FIG. 3. Picture of the three-dimensional window defined by the system of equations (4). The heavy lines represent the stable steady-state solutions: A) the Brinkman-Smith mode; B) the WP mode. The dashed lines represent unstable steady-state solutions. The lines in the $\theta=\pi$ plane are also unstable steady-state solutions. The thin curve represents a typical MH mode. The transformation (1) maps the right half (with reference to the $\theta=\pi$ plane) into the left half of the picture.

mode is long-lived—its lifetime can be as long as several hundred milliseconds—and its turns also number in the hundreds.

In studying the long-lived modes, we should take into account the fact that the experimental conditions existing at the moment when the external field is switched off can assign to the spin and the order parameter a configuration that does not belong to any of the long-lived modes. Generally speaking, the system quickly relaxes into one of such modes, but the specific dependence of a mode on the initial conditions is difficult to predict. In the present investigation it was assumed that at zero time the spin vector $\mathbf{S}$ is perpendicular to the axis $\mathbf{c}$ of the order parameter, that the residual external field is parallel or antiparallel to the spin vector, and that the angle of turn of the order parameter has the equilibrium value $\arccos(-\frac{1}{4})$. A numerical analysis shows that the only modes that can be obtained from the indicated initial configurations are the MH mode and oscillations about the constant solution $\mathbf{S}=\mathbf{H}$, $\mathbf{c}=\mathrm{const}$, $\theta=\arccos(-\frac{1}{4})$. It is very significant that, for the MH mode to exist in the course of the relaxation, it is necessary that the strengths of the turned-off and residual fields lie within certain limits. This phenomenon can be analyzed as follows.

If the residual field is not equal to zero, then we cannot use such distinct structures as the attractor in the preceding section, but we can still use the most important part of the argument: the existence of instability bands. Here we should again use the steady-state solutions. Using the usual perturbation-theory technique, we can verify[18] that the only steady-state solutions that are stable in the presence of relaxation are the Brinkman-Smith mode, the constant solutions $\mathbf{S}=\mathbf{H}$, $\mathbf{c}=\mathrm{const}$, $\theta=\arccos(-\frac{1}{4})$, and the WP mode, in the sense that in the presence of relaxation it goes over into oscillations about the initial steady-state solution. But the most important circumstance is the fact that all the steady-state solutions corresponding to the fixed value $\theta=\pi$ of the order-parameter turn angle are unstable.

We can use for the purpose of deducing from the indicated instability a criterion for the determination of the upper threshold value of the magnetic field the following property of the solutions to the equations (4) in respect of the fields turned off. If for a fixed residual field $\mathbf{H}$ the increment $\Delta\mathbf{H} = \mathbf{H}_i - \mathbf{H}$, where $\mathbf{H}_i$ is the initial field, is sufficiently small, then the system will quickly (with a characteristic time of the order of 1 msec) relax into a regime of oscillations about the constant solution $\mathbf{S} = \mathbf{H}$, $\mathbf{c} = \text{const}$, $\theta = \arccos(-\frac{1}{4})$, and the MH mode is not realized. If we continue to increase the increment $\Delta\mathbf{H}$ for a fixed residual field, after the turned-off field exceeds a certain value, which we can call the lower threshold, the system begins to relax into the MH mode, which in turn relaxes slowly into the indicated oscillations about the constant solution.

It should be noted here that damping of the magnetization ringing can occur in the case when the strength of the field turned off is close to the lower threshold value. Indeed, a numerical analysis shows that, for fields weaker than the lower threshold field, the typical situation is characterized by the presence of narrow bands (of widths smaller in order of magnitude than 0.1 G), such that, for fields that assume values in the various bands, the system relaxes into the indicated constant solutions, the $\mathbf{H}\cdot\mathbf{c}$ values for which can differ by an amount of the order of the residual field $\mathbf{H}$ itself. We can assume on these grounds that the weak spatial inhomogeneities in the spin and the order parameter that obtain at the initial moment can begin to grow and destroy the coherent ringing in the system. This observation should be taken into consideration when estimating the lower threshold field for the MH mode.

If after reaching the lower threshold value we continue to increase the strength of the field turned off, then, as a numerical analysis shows, the following picture will be observed. The system will continue to relax into the MH mode, and, as the strength of the field turned off is increased, the trajectories will, in the initial fast period of the relaxation, come closer and closer to the region defined by the condition $\theta \approx \pi$ (see Fig. 4). For the same value $\mathbf{H}_i = \mathbf{H}_{ex}$, they will begin to enter the neighborhood of the vertical $\theta = \pi$, $\mathbf{H}\cdot\mathbf{c} = 0$, and at that moment the following can happen. Two close trajectories will come close to the axis $\theta = \pi$, $\mathbf{H}\cdot\mathbf{c} = 0$, and one of them will immediately begin to coil along the spiral of the MH mode, while the other will bypass the axis $\theta = \pi$, $\mathbf{H}\cdot\mathbf{c} = 0$ and arrive at the MH-mode spiral from the right side. It should be borne in mind here that the transformation (1) maps the right half of the picture in Fig. 4 into the left half, and, consequently, the right spiral will be mapped into the spiral in the left half of the figure; since the relaxation is small, the two spirals will be located close to each other. In order to verify that the trajectories have diverged, we must consider the pattern of motion in the order-parameter space.

Topologically, the order-parameter space for ^{3}He-B is a three-dimensional sphere with radius π and with the antipodal points of the bounding spherical surface identified in accordance with the transformation (1). The points of the sphere correspond to the vectors $\theta\mathbf{c}$ drawn from the coordinate origin of the three-dimensional space. As usual, θ and $\mathbf{c}$ are the angle and axis of the order parameter. The trajectories discussed above give rise to two trajectories in the order-parameter space. As the trajectories approach the region $\theta \approx \pi$ in the space of the variables $\mathbf{H}\cdot\mathbf{c}$, $(\mathbf{S}-\mathbf{H})\cdot\mathbf{H}$, and θ, the trajectories in the order-parameter space approach the boundary of the sphere (see Fig. 5), and in such a way that the trajectory corresponding to the left spiral does not go as far as the boundary of the sphere, whereas the trajectory corresponding to the right spiral pierces the boundary of the sphere and emerges from the antipodal point, which corresponds to a change in sign of the vector $\mathbf{c}$ under the transformation (1).

The numerical analysis performed shows that the motion in the order-parameter space is such that two close trajectories diverge so quickly that the distance between them after one rotation period of the vector $\mathbf{c}$ is of the same order of magnitude as the values of the dynamical variables themselves (see Fig. 5), and entails spin divergence. Thus, $\theta \approx \pi$ corresponds to a region of instability of the relaxation.

The above-described picture allows us to suggest a procedure for estimating the upper threshold value as functions of the temperature and the residual magnetic field. Figure 6 shows the results of the computations carried out for residual fields of 1 and 2.5 G. It should be noted that there are two upper threshold values (just as there are two lower threshold values) corresponding to the fact that the increment in the

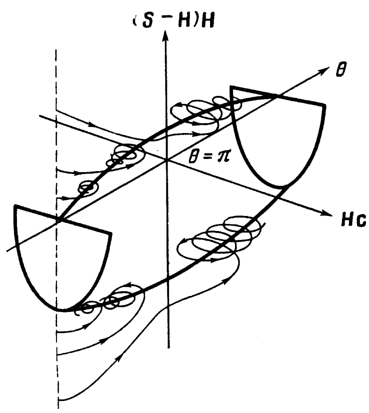

FIG. 4. A series of trajectories for the MH mode as a function of the intensity of the magnetic field turned off; the residual field is fixed. For the threshold field values the trajectories approach the line $\theta = \pi$, $\mathbf{H}\cdot\mathbf{c} = 0$.

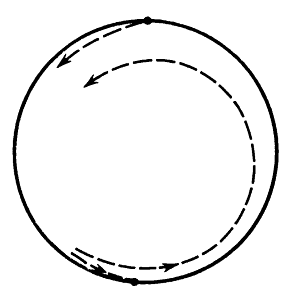

FIG. 5. Two close trajectories A and B in the order-parameter space; A corresponds to a field value higher than the threshold value; B, to a field value lower than the threshold value. The trajectory A pierces the bounding spherical surface and emerges from the antipodal point. To the threshold field correspond those solutions of the equations (3) which go to the pinch point of the attractor. In the order-parameter space they possess limit cycles: arcs of great circles on the bounding spherical surface of radius π. The disposition of the great circles is given by the values of the Brinkman-Cross vector $\mathbf{L}$.

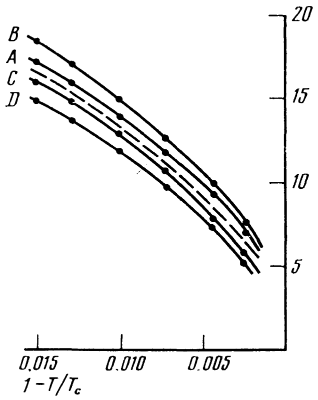

FIG. 6. Possible dependence of the upper threshold field on the temperature $1 - T/T_c$ for a pressure of 20.7 bar. The curves A and B correspond to residual fields parallel to the initial field and equal 1 and 2.5 G; the curves C and D correspond to antiparallel residual fields of intensities 1 and 2.5 G. The dashed curve corresponds to the threshold values in the absence of a residual field.

field turned off can be parallel or antiparallel to the residual field, which corresponds to one or the other sign of $(\mathbf{S}-\mathbf{H})\cdot\mathbf{H}$ (see Fig. 4).

CONCLUSIONS

The existence of threshold values for the magnetic fields turned off in WP-mode experiments is important for the understanding of the general character of the magnetization-relaxation process in ^{3}He-B. The qualitative agreement of the experimental data with the theoretical curve indicates that the instability investigated in the present paper is indeed significant: it can lead to the development of spatial inhomogeneities, whose role in the relaxation processes has already been pointed out by Fomin.[20]

The phenomena investigated in the present paper are first and foremost connected with the complex structure of the order-parameter space for ^{3}He-B. In this respect, some analogy can be drawn here with the electro-hydrodynamic instabilities in the theory of liquid crystals.[21] We can, however, assume that the situation in ^{3}He-B is simpler, since the equations of spin dynamics are less unwieldy than the hydrodynamic equations for liquid crystals, and, furthermore, the greatly simplified spatial-homogeneity approximation in the case of spin dynamics turns out now to be entirely viable for a broad class of experimental situations.

The instability as a consequence of the complex topological structure of the order-parameter space is the cause of the existence of the chaotic spatially homogeneous relaxation regimes in ^{3}He-A (Ref. 22) and ^{3}He-B (Refs. 9 and 23). It should be noted here that it has often been suggested in the last few years[24] that the existence of the chaotic relaxation regimes, which approximates the behavior of real systems (e.g., in hydrodynamics), can serve as a model for turbulent behavior in the original system. The best known example of this kind is the Lorenz model, which describes with some limitations the convection of a heavy viscous fluid.[25] In this respect, the importance of the equations of spin dynamics lies in the fact that they admit of a chaotic regime after a simplification that is minimal from the point of view of the Leggett-Takagi theory: the discarding of the gradient terms. But, as the existence of the upper threshold fields shows, the gradient terms should be taken into account in the instability region, where the angle of the order parameter approaches $\theta \gtrsim \pi$.

It can be assumed that the characteristic dimension of the spatial inhomogeneities at the moment of their appearance will be of the same order of magnitude as the dimension L of the system, since they can appear as a result of the effect of the walls of the vessel and the inhomogeneities of the external field. Consequently, their characteristic frequencies are, in order of magnitude, equal to $\omega_\nabla = v_F/L$, where v_F is the velocity at the Fermi surface, equal in order of magnitude to the velocity of the spin waves corresponding to the spatial inhomogeneities. The frequency ω_∇ is significantly lower than Ω_L, the characteristic frequency of the spatially homogeneous spin dynamics for the external-field intensities considered in the present paper. Thus, there obtains an asymptotic separation, consistent with the allowance for the gradient terms, of the fast and slow variables. We can use the standard procedure: describe the slow variables of the spatial perturbations with the aid of the integrals of the spatially homogeneous problem. For this purpose we can use, for example, the Brinkman-Cross vector[1]

$$\mathbf{L} = \frac{\sin(\theta/2)}{(S^2 - S_\parallel^2)^{1/2}} \left[[\mathbf{S}\mathbf{c}] \operatorname{ctg}\frac{\theta}{2} - \mathbf{S} + S_\parallel \mathbf{c} \right]$$

which in the absence of dissipation is an integral of motion for the spatially homogeneous dynamics. At the moment when the gradient terms become important, the vector $\mathbf{L}$ ceases to be an integral of the motion. If we average the equations of the system over the fast variables, i.e., over the spatially homogeneous solution, then the vector $\mathbf{L}$, together with the angle ψ, which describe the dephasing, will give the order parameter of the averaged spatially inhomogeneous motion. It is worth noting that $\mathbf{L}$ and ψ determine the order parameter, which assumes values in the three-dimensional rotation group. This order parameter describes only the initial phase of the development of the spatial inhomogeneities, when the gradients are still small, and we can use the spatially homogeneous approximation as the first approximation. It is of interest in connection with the description of the onset of spin relaxation in an initial external field H_i equal to the threshold field. The characteristic time of this process can be estimated as the time during which we can find the solution to the LT equations without allowance for the gradient terms in the region $\theta \gtrsim \pi$.

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[1] Progress in Low Temp. Phys., Vol. 7, a & b, Chap. 1, 1978.
[2] R. A. Webb, R. L. Kleinberg, and J. C. Wheatley, Phys. Rev. Lett. 33, 145 (1974).

[3]R. A. Webb, R. E. Sager, and J. C. Wheatley, J. Low Temp. Phys. **26**, 439 (1977).
[4]A. J. Leggett and S. Takagi, Ann. Phys. (N.Y.) **106**, 79 (1977).
[5]W. F. Brinkman, Phys. Lett. A **49**, 411 (1974).
[6]K. Maki and C. R. Hu, J. Low Temp. Phys. **18**, 377 (1975).
[7]M. C. Cross, J. Low Temp. Phys. **30**, 481 (1978).
[8]I. A. Fomin and M. Vuorio, J. Low Temp. Phys. **21**, 271 (1975).
[9]V. L. Golo and A. A. Leman, Zh. Eksp. Teor. Fiz. **83**, 1546 (1982) [Sov. Phys. JETP **56**, 891 (1982)].
[10]I. A. Fomin, Sov. Phys. **3**, 275 (1981).
[11]V. L. Golo, Zh. Eksp. Teor. Fiz. **81**, 942 (1981)[Sov. Phys. JETP **54**, 501 (1981)].
[12]V. L.Golo and A. A. Leman, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 227 (1982) [JETP Lett. **35**, 284 (1982)].
[13]V. L. Golo, Lett. Math. Phys. **5**, 155 (1981).
[14]I. A. Fomin, R. Schertler, and W. Schoepe, Phys. Lett. A **92**, 408 (1982).
[15]I. E. Dzyaloshinskii and G. E. Volovik, Ann. Phys. (N.Y.) **125**, 67 (1980).
[16]A. I. Ahonen, M. Krusius, and M. A. Paalanen, J. Low Temp. Phys. **25**, 421 (1976).
[17]K. Pohlmeyer, Commun. Math. Phys. **46**, 207 (1976).
[18]I. A. Fomin, Zh. Eksp. Teor. Fiz. **84**, 2109 (1983) [Sov. Phys. JETP **57**, 1227 (1983)].
[19]W. F. Brinkman and H. Smith, Phys. Lett. A **53**, 43 (1975).
[20]I. A. Fomin, Zh. Eksp. Teor. Fiz. **78**, 2392 (1980) [Sov. Phys. JETP **51**, 1203 (1980)].
[21]S. A. Pitkin, Strukturnye prevrashcheniya v zhidkikh kristallakh (Structural Transformations in Liquid Crystals), Moscow, 1981, Chap. XI.
[22]J. Jamaguchi, T. Katayama, and C. Ishii, Phys. Lett. A **91**, 299 (1982).
[23]J. Jamaguchi, T. Katayama, and C. Ishii, Preprint, Tokyo Univ., 1982.
[24]Ya. G. Sinaya and L. P. Shil'nikova (editors), Strannye atraktory (Strange Attractors), Mir, Moscow, 1981.
[25]E. N. Lorenz, J. Atoms. Sci. **20**, 130 (1963).

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