Theory of "current states" in metals

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Rectification of a current in metals located in a parallel magnetic field h_0 and irradiated by radio waves of sufficiently high amplitude $\mathscr{H}$ is theoretically studied. An equation for the induced magnetic moment of the specimen is obtained on the basis of a simple and physically lucid model. It is shown that at wave amplitudes greater than the critical value $\mathscr{H}_{cr}$ the induced moment has hysteresis as a function of h_0. Initiation of the hysteresis loops is demonstrated and the dynamics of the variation of their shape with increasing amplitude $\mathscr{H}$ is studied. At sufficiently high values of $\mathscr{H}$, the hysteresis loop broadens and asymptotically approaches the limiting universal curve obtained by Makarov and Yampol'skiĭ [JETP Lett. **35**, 520 (1982)].

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1. INTRODUCTION

In recent years, theoretical and experimental studies have been (and continue to be) reported on different nonlinear electromagnetic properties of pure metallic specimens at low temperatures. A list of these studies may be found in the survey of Dolgopolov[1] and in the dissertation of Demikhovskiĭ.[2] Dolgopolov discussed nonlinearity under the conditions of anomalous skin effect.

Unlike other conductors, metals possess a high degree of electrical conductivity, i.e., low surface impedance. Consequently, the magnetic component of the electromagnetic wave in metals is always much greater than the electrical component. The distinguishing facture of metallic nonlinearity is the decisive role of the magnetic component of the electromagnetic field; under skin-effect conditions this component can alter substantially the electron path. The change of the electron motion under the influence of an alternating and nonuniform magnetic field is the principal reason for a host of nonlinear effects in metallic specimens. Most of the nonlinear effects recently discovered in metals therefore exhibit properties that are not found in such patently nonlinear objects as semiconductors and gas-discharge plasmas.

What are known as "current states," experimentally found in a number of metals[3–7] are typical examples of such a nonlinear response to electromagnetic perturbation. In these studies, under the conditions of an anomalous skin effect, a rectified current was produced in pure specimens irradiated by radio waves of sufficiently high amplitude and induced a constant magnetic field **h** and, consequently, an observable magnetic moment. To excite the moment it was necessary to apply a constant magnetic field $\mathbf{h}_0$ parallel to the boundary of the metal. However, this magnetic moment continued to exist even after $\mathbf{h}_0$ had been removed, and exhibited hysteresis as a function of $\mathbf{h}_0$. Similar hysteresis is exhibited by the magnetic-field-dependent kinetic coefficients. Excitation of current states has a threshold, the hysteresis loops appearing only after the amplitude $\mathscr{H}$ of the incident wave exceeds some critical value $\mathscr{H}_{cr}$.

Babkin and Dolgopolov[8] have found the physical cause of these current states. According to them nonlinear current rectification in metals is due to the fact that the nonuniform magnetic field, which is the sum of $\mathbf{h}_0$ and of the wave magnetic field $\mathbf{H}(x, t)$, shapes the electron trajectory (the x axis is directed into the specimen and t is the time). The electron motion depends essentially on whether there is a plane $x = x_0(t)$ in the specimen on which $H(x_0, t) + h_0 = 0$. During those time intervals during which such a plane is absent, the electrons move along paths similar to those shown in Fig. 1a. These paths are virtually indistinguishable from a closed Larmor orbit. If $|h_0| \leqslant 2\mathscr{H}$, there exists during the wave period $2\pi/\omega$ a time interval in which the spatial distribution of $H(x, t) + h_0$ is of alternating sign. In this case, the effective electrons which land in the skin layer move along the surface of the metal in paths which twist about the plane $x = x_0(t)$

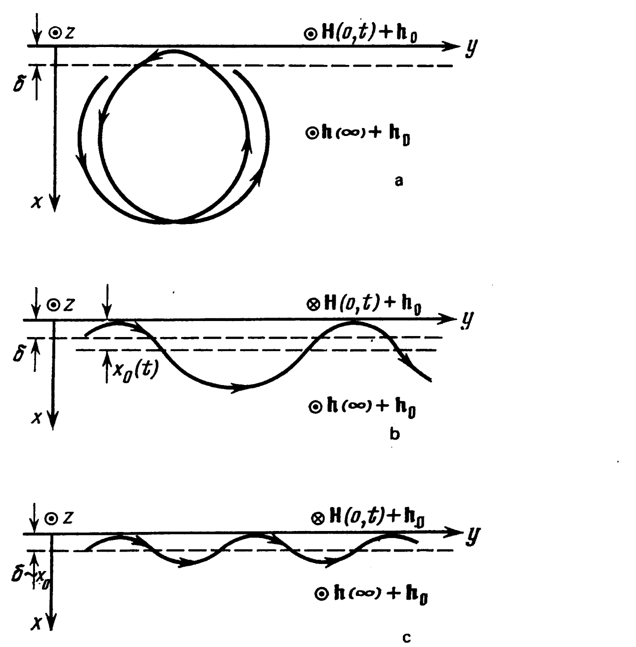

FIG. 1. Paths of effective electrons in a nonuniform magnetic field. (a) Path in a field of constant sign ($T_a \approx \pi mc/e|h(\infty) + h_0|$); (b) path in field of alternating sign ($T_b \approx (mc\delta\,|H(0,\ t)|/ev_F)^{1/2}/|h(\infty) + h_0|$) when $|h(\infty) + h_0| \ll 2\mathscr{H}$; (c) path in field of alternating sign ($T_c \approx \pi(mc/ev_F|H'(x_0, t)|)^{1/2}$ when $|h(\infty) + h_0| \sim 2\mathscr{H}$.

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(see Figs. 1b, c). Hence it is clear that the conductivity will differ at different times. Sharp variations in the conductivity during the wave period $2\pi/\omega$ will lead to excitation of a dc current component which induces a constant but nonuniform field $\mathbf{h}(x)$.

Note that the dc current flowing around the surface of a specimen is closed and attenuates in the interior of the metal at a distance on the order of the thickness of the skin layer δ; at the center of the specimen the current is zero. This means that over the same these distances the magnetic field $h(x)$ varies from zero at the boundary of the metal to a value $h(\infty)$ at the middle of the current loop, i.e., at the center of the bulky specimen. The induced field $h(x)$ constitutes the value of $H(x, t)$ averaged over the period $2\pi/\omega$ of the incident wave. Because of the skin effect, the alternating component of $H(x, t)$ oscillates and rapidly attenuates over a distance of the order of δ.

Such a rectification mechanism will be present if the path of the effective electron in the skin layer $L\sim(4cp_F\delta/e\mathscr{H})^{1/2}$ is much less than the mean free path $l = v/\nu$; in addition, the wave phase should remain invariant during the free path time, i.e.,

$$4cp_F\delta/el^2\ll\mathscr{H},\quad \omega\ll\nu. \tag{1.1}$$

The inequality (1.1) means that the characteristic value of the alternating magnetic field $2\mathscr{H}$ in a metal must exceed the field $\mathscr{H}_0$ at which $L = l$. In other words, the rectification effect is characterized by a small parameter b equal to the ratio L/l, or

$$b=(\mathscr{H}_0/2\mathscr{H})^{1/2}<1,\quad \mathscr{H}_0=8mc\delta\nu^2/ev. \tag{1.2}$$

Here c is the speed of light, e the absolute value of the charge, p_F and v are the Fermi momentum and velocity, and m and ν are the mass and relaxation frequency of the electron.

In the study we already mentioned,[8] excitation of current states was considered by perturbation theory. It was assumed that the "twisting" electrons move along the asymmetric paths shown in Fig. 1b. The threshold nature of the effect and the dependence of the critical amplitude $\mathscr{H}_{cr}$ on the frequency ω and on the mean free path l were established. These results are qualitatively correct for the case of not too high values of the nonlinear parameter b. In our previous article,[9] we studied the regime of advanced nonlinearity ($b\ll 1$), in which the plane $x = x_0(t)$ is near the surface of the metal ($x_0\sim\delta$) and the path of the twisting electrons become symmetric relative to $x = x_0(t)$ in the magnetic field $H'(x_0, t)\delta$ (Fig. 1c). It was shown that the hysteresis loop of the magnetic moment, as a function of h_0, is within the range $(-2\mathscr{H}, 2\mathscr{H})$ and has a universal form, whereas the values of the induced field $h(\infty)$ are comparable with the amplitude $\mathscr{H}$ of the incident wave.

In the present article, we use a relatively simple, though physically lucid model to construct a perturbation theory for current states, combining the previous results[8,9] and valid over the entire range of variation of the parameters $|h_0|\leqslant 2\mathscr{H}$ and $b\lesssim 1$. An equation for $h(\infty)$ is obtained and analyzed; by means of this equation, generation of hysteresis loops and the variation of their shape with increasing amplitude $\mathscr{H}$ of the incident wave (with decreasing parameter b) are investigated. The conclusions of the theory are in good agreement with the experimental results.

2. DISTRIBUTION OF ELECTROMAGNETIC FIELD

Let us consider a solid metallic specimen placed in a constant and uniform magnetic field $\mathbf{h}_0$ parallel to its surface. A plane monochromatic wave of amplitude $\mathscr{H}$ and frequency ω, whose magnetic vector is collinear with $\mathbf{h}_0$, is incident on the interface (the yz plane). The x axis is directed along the inward normal to the metal, the z axis is parallel to the vectors of the magnetic fields (Fig. 1). The electrical and magnetic components of the electromagnetic field in the metal are parallel to the y and z axes:

$$\mathbf{E}(x, t)=\{0, E(x, t), 0\},\quad \mathbf{H}(x, t)=\{0, 0, H(x, t)\}. \tag{2.1}$$

To find $E(x, t)$ and $H(x, t)$ in explicit form, it is necessary to solve the Maxwell equations

$$\frac{\partial H(x,t)}{\partial x}=-\frac{4\pi}{c}j(x,t),\quad \frac{\partial E(x,t)}{\partial x}=-\frac{1}{c}\frac{\partial H(x,t)}{\partial t} \tag{2.2}$$

with the boundary conditions

$$H(0, t)=2\mathscr{H}\cos\omega t,\quad H(\infty, t)=h(\infty). \tag{2.3}$$

Note that the first equation in (2.3) is accurate to terms of order $\omega\delta/c\ll 1$. This equation expresses the fact that to the extent that the impedance is small an electromagnetic wave of frequency ω impinging on the metal is reflected basically at the same frequency.

The current density $j(x, t)$ is not a monochromatic function and depends in a complex way on the time t. Therefore we represented the solutions of the Maxwell equations in the form

$$\begin{aligned} H(x,t)&=\sum_{n=-\infty}^{\infty} H_n\exp\left[-\frac{x}{\delta_n}-in\xi(\omega t)\right],\\ E(x,t)&=\sum_{n=-\infty}^{\infty} E_n(\omega t)\exp\left[-\frac{x}{\delta_n}-in\xi(\omega t)\right]. \end{aligned} \tag{2.4}$$

Here $E_n(\phi)$, δ_n, and $\xi(\phi)$ are determined from (2.2), and the coefficients H_n are found from the boundary conditions (2.3). Since the solutions of (2.4) must be periodic in the time t with the period $2\pi/\omega$ of the incident wave, the function $\xi(\phi)$ must satisfy the condition $\xi(\phi+2\pi)=\xi(\phi)+2\pi$ and vary from 0 to 2π in the interval $0\leqslant\phi\leqslant 2\pi$. Moreover, $\xi(\phi)$ must be continuous, monotonic, and single-valued if the set of functions $\exp[-in\xi(\phi)]$ is to be complete.

In the representation (2.4), the current density of the conduction electrons may be written in the form

$$j(x,t)=\sum_{n=-\infty}^{\infty}\sigma_n(\omega t)E_n(\omega t)\exp\left[-\frac{x}{\delta_n}-in\xi(\omega t)\right]. \tag{2.5}$$

It follows from (2.2) and from the fact that H_n is independent of ϕ that the product $\sigma_n(\omega t)E_n(\omega t)$ is independent of the time t.

The conductivity of the n-th current-density component (under the conditions of the anomalous skin effect) may be represented in the form

$$\sigma_n(\varphi)=\frac{\sigma_0}{1-\exp(-2\nu T_a)}\frac{\delta_n}{l}\tilde{\sigma}(\varphi),$$

$$\tilde{\sigma}(\varphi)=\Theta\left(\frac{2\mathscr{H}\cos\varphi+h_0}{h(\infty)+h_0}\right)+\alpha(\varphi)\Theta\left(-\frac{2\mathscr{H}\cos\varphi+h_0}{h(\infty)+h_0}\right),\quad(2.6)$$

$$\alpha(\varphi)=\frac{4}{\pi}[1-\exp(-2\nu T_a)]\int_0^v\frac{v_\perp dv_z}{v^2}[1-\exp(-2\nu T(\varphi))]^{-1}.$$

Here σ_0 is the static conductivity of the metal, $\Theta(x)$ is the Heaviside function ($\Theta(x)=0$ if $x<0$ and $\Theta(x)=1$ if $x>0$), $2T_a$ is the electron period in a constant-sign magnetic field (Fig. 1a) and $2T(\phi)$ the electron period in an alternating-sign magnetic field (Figs. 1b, c), and $v_\perp$ is the electron velocity in the xy-plane perpendicular to the total magnetic field vector. Equation (2.6) for $\alpha(\phi)$ is written for a metal with a spherical Fermi surface, with

$$T_a=\frac{\pi mc}{e|h(\infty)+h_0|},\quad v_\perp=(v^2-v_z^2)^{1/2}.\quad(2.7)$$

That (2.5) and (2.6) are valid may be demonstrated by means of an exact derivation based on the kinetic equation for the electron distribution function. For lack of space we do not present this derivation, all the more since from the point of view of physics, the structure of (2.6) is sufficiently clear by itself. The spatial dispersion in (2.4)–(2.6) is taken into account in the ineffectiveness-concept model (see the Appendix), which yields correct results to within real constant factors. In this model, the factor δ_n/l in (2.6) is the usual one for the anomalous skin conductivity. This factor reflects the fact that only some of the effective electrons impinging on the skin layer δ_n, and not all of the electrons, interact with the electromagnetic field in the metal. The denominators $(1-\exp(-2\nu T))$ take into account the probability that the effective electron will repeatedly return to the skin layer in a magnetic field parallel to the boundary of the specimen. In writing out $\tilde{\sigma}(\omega t)$, it was assumed that the total field $H(x,t)+h_0$ in the metal changes sign not more than once. In this case, the path of the effective electrons depends on the orientation of the total field on the metal surface

$$H(0,t)+h_0=2\mathscr{H}\cos\omega t+h_0$$

relative to the field within the metal

$$H(\infty,t)+h_0=h(\infty)+h_0.$$

During these time intervals when the field has the same sign at the boundary and within the specimen it vanishes nowhere, $\tilde{\sigma}(\omega t)=1$, and the conductivity (2.6) is equal to the well-known conductivity of the "Larmor" electrons (Fig. 1a). But if $H(0,t)+h_0$ and $H(\infty,t)+h_0$ are of different sign, the spatial distribution of $H(x,t)+h_0$ is an alternating-sign function, $\tilde{\sigma}(\omega t)=\alpha(\omega t)$, and the conductivity will be completely determined by the twisting electrons (Figs. 1b, 1c). The quantity $\alpha(\phi)$ is the ratio of the conductivity of the twisting electrons to the conductivity of the metal in a field of constant sign without such electrons. The period $2T(\phi)$ of the twisting electrons depends on $v_\perp$, so that the formula for $\alpha(\phi)$ contains averaging with respect to v_z. Still without presenting the explicit form for $T(\phi)$, we may show that $T(\phi)\ll T_a$ as long as the skin effect is anomalous with respect to the variable field ($\delta\ll cp_F/2e\mathscr{H}$). This means that $\alpha(\phi)$ is always greater than unity.

Substituting (2.4)–(2.6) in the Maxwell equations (2.2), we easily find that

$$\delta_n=\left(\frac{c^2l[1-\exp(-2\nu T_a)]}{4\pi\omega\sigma_0}\frac{\rho}{|n|}\right)^{1/3}\exp\left(\frac{i\pi}{6}\frac{n}{|n|}\right)$$

$$\xi(\varphi)=\frac{1}{\rho}\int_0^\varphi\frac{d\varphi'}{\tilde{\sigma}(\varphi')},\quad\rho=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{\tilde{\sigma}(\varphi)},\quad(2.8)$$

$$E_n(\varphi)=-\frac{in\omega\delta_n}{c\rho\tilde{\sigma}(\varphi)}H_n.$$

To determine the coefficients H_n, we turn to the boundary conditions (2.3). As $x\to\infty$, there remains only a single term with $n=0$ in formula (2.4) for $H(x,t)$. Consequently, it follows from the boundary condition at infinity that $h(\infty)=H_0$. From the boundary condition $x=0$ at the metal surface, we find that

$$\sum_{n=-\infty}H_n\exp[-in\xi(\varphi)]=2\mathscr{H}\cos\varphi.\quad(2.9)$$

This equation is an expansion of the right side of $2\mathscr{H}\cos\phi$ in a series in the complete set of functions $\exp[-in\xi(\phi)]$. These functions are orthogonal on the interval $0\leqslant\phi\leqslant2\pi$ with weight $\xi(\phi)=1/\rho\tilde{\sigma}(\phi)$. Therefore, the coefficients of the expansion (2.9) are

$$H_n=\frac{\mathscr{H}}{\pi\rho}\int_0^{2\pi}\frac{d\varphi}{\tilde{\sigma}(\varphi)}\cos\varphi\exp[in\xi(\varphi)].\quad(2.10)$$

From (2.10) there follows, in particular, an equation for the induced constant magnetic field in the interior of the specimen:

$$h(\infty)=\frac{\mathscr{H}}{\pi\rho}\int_0^{2\pi}\frac{d\varphi}{\tilde{\sigma}(\varphi)}\cos\varphi.\quad(2.11)$$

Equations (2.4)–(2.11) describe a nonlinear distribution of an electromagnetic field in the case $|h_0|\leqslant2\mathscr{H}$, when twisting electron orbits appear. Let us emphasize that the representation (2.4) is not an expansion in a harmonic Fourier series in ϕ, as each term in (2.4) contains, in general, all the field harmonics at frequencies which are multiples of the frequency ω of the incident wave. The linear situation is reached by making $\alpha(\phi)$ tend to unity. When $\alpha(\phi)=1$, we have

$$\tilde{\sigma}(\varphi)=1,\quad\rho=1,\quad\xi(\varphi)=\varphi,$$

and $H_n=\mathscr{H}$ if $n=\pm1$ and $H_n=0$ if $n=0,\pm2,\pm3,\ldots$. In other words, only two terms with $n=\pm1$ are left in the sums (2.4).

3. ANALYSIS OF EQUATION FOR $h(\infty)$

Equation (2.11) for the induced magnetic field are best written in dimensionless variables by introducing the new notation

$$a=\frac{h_0}{2\mathscr{H}},\quad\varkappa=\frac{h(\infty)}{2\mathscr{H}},\quad\tilde{\varkappa}=\frac{h(\infty)+h_0}{2\mathscr{H}}=\varkappa+a;$$

$$\beta=\arccos(a\,\mathrm{sign}\,\tilde{\varkappa}),\quad0\leqslant\beta\leqslant\pi.\quad(3.1)$$

After a number of algebraic manipulations we obtain

$$\varkappa=\frac{1}{2\pi}\int\limits_{-\beta}^{\beta}d\varphi\,(\varkappa+\operatorname{sign}\tilde{\varkappa}\cos\varphi)\,[1-\alpha^{-1}(\varphi)\,]. \qquad (3.2)$$

Here we have taken into account the explicit form of ρ and also the fact that $\alpha(\phi\pm\pi)=\alpha(\phi)$; sign $x=1$ if $x>0$, sign $x=-1$ if $x<0$, and sign $x=0$ if $x=0$.

Equation (3.2) which is a rather complicated integral equation, determines the functional $\varkappa(a)$ dependence. It has one trivial solution $\varkappa=0$ at $a=0$. It is not possible to find its nontrivial solutions analytically in the general case, since the period $2T(\phi)$ of the twisting electrons is a functional of the total magnetic field $H(x,t)+h_0$ and, depending on the quantity b as well as the relations between h_0, $h(\infty)$, and $2\mathscr{H}$, assumes different values within the range $T_c\leqslant T(\phi)\leqslant T_b$ (see Fig. 1). Nevertheless, Eq. (3.2) can be analyzed in the two important limiting cases, and the general laws governing the solution $\varkappa(a)$ found by means of such an analysis. Next, (3.2) may be simplified to permit a computer solution. This demonstrates the generation of hysteresis loops and the dynamics of the variation of their shapes with decreasing b (increasing $\mathscr{H}$).

1. Let us first consider the case of weak nonlinearity when $\alpha(\phi)-1\ll 1$, i.e., $2\nu T(\phi)\ll 1$. As will be seen below, this case corresponds to the inequality

$$|h(\infty)+h_0|/2\mathscr{H}\ll b. \qquad (3.3)$$

Since $\alpha(\phi)$ is nearly equal to unity, the alterating component $H(x,t)$ is mainly the field of the first harmonic (terms with $n=\pm 1$ in (2.4)). All the other harmonics, including the zeroth $h(x)$, are much less than the first ($|\varkappa|\ll 1$), and therefore $H(x,t)$ may be given in the form

$$H(x,\ t)=2\mathscr{H}\operatorname{Re}\exp\left(-x/\delta_1-i\omega t\right)+h(x). \qquad (3.4)$$

Hence it follows that the point $x_0(t)$ at which the total magnetic field vanishes is located for a considerable fraction of the time within the metal at a depth greater than δ below the surface. The effective electrons, moving in the skin layer into the strong field $2\mathscr{H}$, penetrate far into the specimen, where the alternating component is zero, and become twisted by the weak constant field $h(\infty)+h_0$ ($|\tilde{\varkappa}|\ll 1$).[8] The asymmetric paths of these electrons are shown in Fig. 1b. Their period may be computed by means of (3.4). As in Ref. 1, we find that

$$2\nu T_b(\varphi)=\left(\frac{v}{v_\perp}\right)^{1/2}\frac{b}{|\tilde{\varkappa}|}\left|\cos\left(\varphi-\frac{\pi}{6}\right)\right|^{1/2}. \qquad (3.5)$$

Here and below, the depth of the skin layer δ in the definition (1.2) of the parameter b should be taken to mean $|\delta_1|$. Condition (3.3) from (3.5) follows in the case of weak nonlinearity.

We substitute (3.5) in the formula (2.6) for $\alpha(\phi)$, and then expand the right side of (3.2) in powers of the small quantity $\alpha(\phi)-1$ and compute the integrals. We bear in mind that $\beta\approx\pi/2$ and that in the integral with respect to ϕ the major contribution is made by the point $\phi=-\pi/3$ at which $\cos(\phi-\pi/6)$ vanishes. Ignoring the numerical real factors, we find that

$$\pi b^2\varkappa=(\varkappa+a)^2\operatorname{sign}\,(\varkappa+a). \qquad (3.6)$$

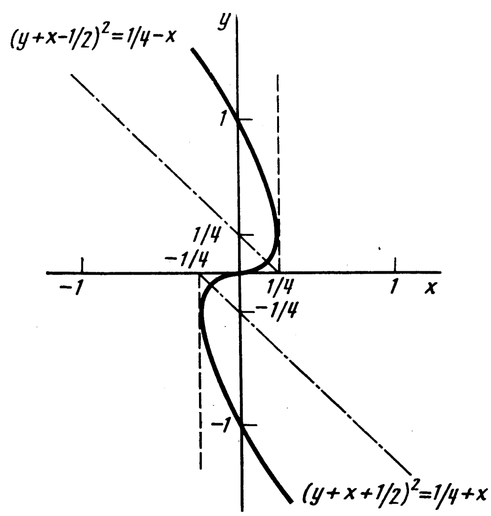

FIG. 2. Solution $y(x)$ of Eq. (3.7).

This equation leads to an interesting result: the variables a and $\varkappa$ may be normalized in a natural way with respect to πb^2 so that the equation subsequently no longer depends on any of the parameters

$$a=\pi b^2x,\quad \varkappa=\pi b^2y,\quad y=(y+x)^2\operatorname{sign}\,(y+x). \qquad (3.7)$$

That such a normalization is possible means that the induced field $h(\infty)$ in the region (3.3) is determined by the characteristic field $\mathscr{H}_0$ and is practically independent of the amplitude of the incident wave. In fact, according to (3.7), we have

$$h(\infty)=\pi\mathscr{H}_0y\,(h_0/\pi\mathscr{H}_0). \qquad (3.8)$$

The function $y(x)$ is determined from (3.7), which may be solved quite simply (it is a quadratic equation); the function is plotted in Fig. 2. Note that $y(-x)=-y(x)$.

Recall that the point $\phi=-\pi/3$ made the major contribution to Eq. (3.2) upon averaging over the phase, while the integration limits β and $-\beta$ did not "work." At these limits, there is a transition from the ordinary conduction to the conduction by the twisting electrons. Thus, the results obtained in this section are independent of the law that governs the transition from one type of conduction to the other.

2. As $|\tilde{\varkappa}|$ increases, inequality (3.3) breaks down, and we enter the region of strong nonlinearity, where

$$b\ll|h(\infty)+h_0|/2\mathscr{H},\quad b\ll 1. \qquad (3.9)$$

In this region, the conductivity of the metal in a field of alternating sign greatly exceeds the conductivity in a field of constant sign. Their ratio $\alpha(\phi)\gg 1$, i.e., $2\nu T(\phi)\ll 1$. This means that the period of the twisting electrons attains its minimum $2T_c\sim L/v$ ($2\nu T_c\sim b\ll 1$); here the plane at which the sign of $x=x_0(t)$ changes is near the surface of the metal ($x_0\sim\delta$), and the path followed by the twisting electrons becomes symmetric relative to $x=x_0(t)$ in the magnetic field $H'(x_0,t)\delta$ (Fig. 1c).

Neglecting $\alpha^{-1}(\phi)$ in (3.2) by comparison with unity, we find two solutions[9] $\varkappa(a)$:

$$\varkappa=\pm(1-a^2)^{1/2}[\arccos(\mp a)]^{-1}, \quad |a|\leq 1. \qquad (3.10)$$

By (3.10), under the conditions of advanced nonlinearity ($b \ll 1$), $\varkappa(a)$ is a universal function without any parameters. In particular, $\varkappa(a)$ is independent of the electrodynamics of the metal (i.e., of b). For this reason, the result (3.10) is not affected by the choice of the model for $\sigma_n(\phi)$. The induced field $h(\infty)$ is determined by the amplitude $\mathscr{H}$ of the incident wave and is independent of $\mathscr{H}_0$:

$$h(\infty)=2\mathscr{H}\varkappa(h_0/2\mathscr{H}). \qquad (3.11)$$

In our analysis of (3.2), we have until now not given a specific value for the product $2\nu T_a$. Note here that if $2\nu T_a \ll 1$, only the strong nonlinearity condition (3.9) is realized, since the period $2T(\phi)$ of the twisting electrons is always much less than the Larmor period $2T_a$.

To derive the results (3.10) and (3.11), we used only the fact that the ratio $\alpha(\phi)$ of conductivities increased significantly. The exact value of $\alpha(\phi)$ did not play any role here. In other words, $2\nu T(\phi) \ll 1$ must hold in the region of advanced nonlinearity (3.9), and it makes no difference what the period $2T(\phi)$ is actually equal to. At the same time, Eq. (3.5) for $T_b(\phi)$ is such that the product $2\nu T_b(\phi) \ll 1$ in the case (3.9). This means that the solutions (3.10) and (3.11) are obtained also when $T_b(\phi)$ from (3.5) is substituted for $T(\phi)$ in the formula (2.6) for $\alpha(\phi)$, despite the fact that, formally speaking, Eq. (3.5) is not applicable in the case of strong nonlinearity (for which $T(\phi) = T_c$). Thus, Eq. (3.2) for $\varkappa(a)$ with $\alpha(\phi)$ containing $2\nu T(\phi)$ in the form (3.5) is closed and yields correct results in the case of both weak and strong nonlinearity. There is no reason for doubting that such an equation describes, at least qualitatively, also the intermediate situation.

3. Let us substitute (3.5) in the Eq. (2.6) for $\alpha(\phi)$ and assume that $2\nu T_a \gg 1$. Allowance for the actual electron dispersion law only the real constants in (3.2) change. For the sake of simplicity, therefore, we will consider the case of a cylindrical Fermi surface with symmetry axis along the magnetic-field vector (z axis). In this case,

$$v_\perp \to v, \quad \frac{4}{\pi}\int\limits_0 \frac{v_\perp dv_z}{v^2} \to 1.$$

As a result, (3.2) assumes the simpler form

$$\varkappa=\frac{1}{2\pi}\int\limits_{-\beta}^{\beta} d\varphi(\varkappa+\operatorname{sign}\tilde{\varkappa}\cos\varphi)\exp\left[-\frac{b}{|\tilde{\varkappa}|}\left|\cos\left(\varphi-\frac{\pi}{6}\right)\right|^{1/2}\right]. \qquad (3.12)$$

Figure 3 shows plots of the computer solutions $\varkappa(a)$ of Eq. (3.12) for different values of the parameter b. The curves in this figure demonstrate that the excitation of the current states has a threshold and show the dynamics of the variation of the shape of the hysteresis loops with decreasing b (increasing wave amplitude $\mathscr{H}$).

If $b \geqslant 0.3$, no hysteresis is present and the dimensionless value $\varkappa$ of the induced field is a single-valued function of a. As the amplitude $\mathscr{H}$ increases, b decreases, and at some $b = b_{cr}$ the plot of $\varkappa(a)$ has two singularities at which $\varkappa'(a)$ becomes infinite. For (3.12), the b_{cr} lies in the range $0.2 < b_{cr} < 0.3$. At a fixed frequency ω of the external signal and at

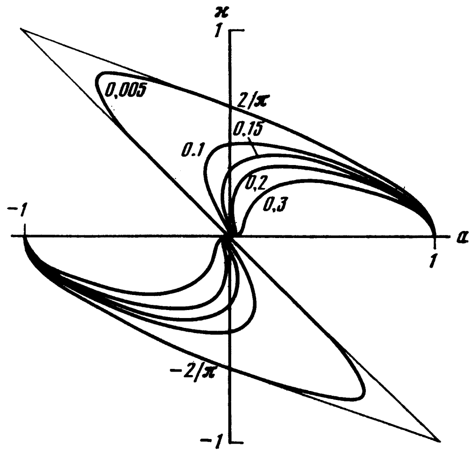

FIG. 3. Plots of solutions $\varkappa(a)$ of Eq. (3.12) for different values of b (the values of b are written next to the curves).

fixed temperature, the value of b_{cr} for a given metal determines the threshold value $\mathscr{H}_{cr}$ of the amplitude of the incident wave. In accordance with (1.2), we find that

$$\mathscr{H}_{cr}=\mathscr{H}_0/2b_{cr}{}^2\infty\omega^{-1/2}l^{-2}.$$

For $b < b_{cr}$, $\varkappa(a)$ is no longer single-valued; the derivative $\varkappa'(a)$ has four pairwise symmetric singularities at which the experimental induced field $\varkappa$ is discontinuous. The discontinuous behavior of $\varkappa(a)$ also means the appearance of a hysteresis loop.

If $b \leqslant b_{cr}$, but is large enough, Eq. (3.12) has no nontrivial solutions if $a = 0$ and the $\varkappa(a)$ curve does not cross the y axis. In this case, a constant field h_0 is required to excite the current states and the hysteresis is in the form of two centrosymmetric loops. The function $\varkappa(a)$ with $b = 0.2$ is an example of such a situation.

At some supercritical value $b = b^0_{cr}$, the $\varkappa(a)$ curve is tangent to the vertical axis ($0.15 < b^0_{cr} < 0.2$ in Fig. 3). Finite solutions $\varkappa(0) \neq 0$ appear beginning with this point, i.e., the induced field $h(\infty)$ acquires the ability to maintain itself in a zero external magnetic field $h_0 = 0$. The loops which exist in the interval $b^0_{cr} < b < b_{cr}$ merge (when $b = b^0_{cr}$) into a single loop with four discontinuites, which form pairwise two steps. After "some time" (as b decreases), the steps vanish and two discontinuities remain.

Finally, as b decreases further, the hysteresis loop broadens, asymptotically approaching the limiting curve described by (3.10) (outer curve in Fig. 3). From an analysis of Eq. (3.12) it is clear that the approach to the limiting curve is linear in b along the ordinate axis but follows a $b^{2/5}$ law along the abscissa axis. In other words, $\varkappa(a)$ reaches the limiting curve faster along the ordinate axis than along the abscissa axis.

All the hysteresis loops of the induced field $h(\infty)$ lie (as functions of h_0) within the range $(-2\mathscr{H}, 2\mathscr{H})$. At amplitudes $\mathscr{H}$ far from the threshold amplitude ($b \ll 1$), the width of the loop is equal to $4\mathscr{H}$ accurate to $b^{2/5}$.

Note that even under the conditions of advanced hysteresis ($b \ll 1$), when there is only a single loop, there is a stable-state section in a neighborhood of the origin $a = 0$ and

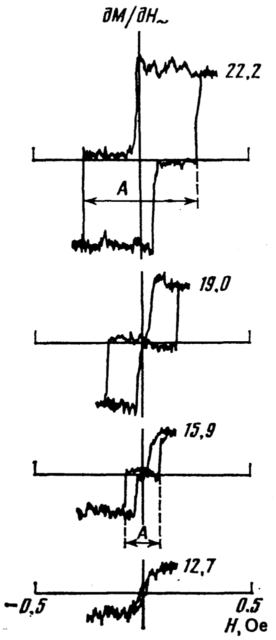

FIG. 4. Dependence $\partial M/\partial\mathscr{H}$ on h_0 in tin for different values of the amplitude $\mathscr{H}$ of the alternating field.[10] Thickness of specimen 0.6 mm, $T = 4.15$ K, **H**‖[100], $\omega/2\pi = 1.7$ MHz. The numbers near the curves denote the values of $\mathscr{H}$ in oersteds.

$\varkappa = 0$ (by way of example, see the curves with $b = 0.15$ and $b = 0.1$ in Fig. 3). The width of this section is $\pi b^2/2$ for Eq. (3.12), and the $\varkappa(a)$ dependence is described by (3.7) (see also Fig. 2). As shown in Sec. 3.2, in this state $h(\infty)$ is determined by the field $\mathscr{H}_0$ and is independent of the amplitude $\mathscr{H}$ of the incident wave.

In Fig. 4, which we have borrowed from Murzin's dissertation,[10] we present typical experimental curves for the derivative $dM/d\mathscr{H}$ of the magnetic moment as a function of the external field h_0. This figure demonstrates that the experimental results are in good agreement with some of the conclusions we have arrived at here. Moreover, a direct proportionality has been observed[5] between the critical value $\mathscr{H}_{cr}$ of the amplitude and $\omega^{-1/3}$. It should be noted that all the experiments known to us pertain to the region of not "very high" amplitudes $\mathscr{H}$ ($b\sim 1$–0.1), where the conclusions we have drawn from an analysis of (3.2) and 3.12) agree in the main with the conclusions which follow from a previously proposed model.[8,6] There are as yet no experiments that show that the hysteresis loop reaches the limiting universal curve (3.10) as $b\to 0$. A measurement of the value of $h(\infty) = \pm 4\mathscr{H}/\pi$ at $h_0 = 0$ would be one way of experimentally verifying (3.10).

Equation (3.12) together with its analysis above pertains to the case $2\nu T_a \gg 1$. Even here, values $b \ll 1$ may be attained and the limiting curve (3.10) approached. If the amplitude $\mathscr{H}$ continues to increase, we enter the region where $2\nu T_a \sim cp_F/2e\mathscr{H}l \ll 1$. In this region, the ratio of the conductivities is

$$\alpha(\varphi)\sim T_a/T_c\sim(cp_F/2e\mathscr{H}\delta)^{1/2}, \qquad (3.13)$$

meaning that in (3.2) for $\varkappa(a)$ the factor in the square brackets is of the same order as

$$1-(2e\mathscr{H}\delta/cp_F)^{1/2}. \qquad (3.14)$$

As a result, the induced magnetic moment decreases with increasing amplitude $\mathscr{H}$, and when $2\mathscr{H}\gtrsim cp_F/e\delta$ the hysteresis loop vanishes.

Studies of current states in the case of a "nearly normal skin effect" have recently been reported.[11] The observed irreversible magnetic-moment hysteresis and the dynamics of the variation of its shape resemble that which occurs under the conditions of the maximally anomalous skin effect.[3–6] Despite certain differences between the results of Ref. 11 and other results,[3–6] we believe that the cause of the excitation of current states in Ref. 11 is the same as in the case of an anomalous skin effect. This cause is during the external-wave period $2\pi/\omega$ there exists a time interval when the total magnetic field is of constant sign, and in the remainder of the period there is a plane in the metal at which total field vanishes: $x = x_0(t)$. This is the necessary condition for the existence of current states, as is clear from the following simple reasoning. If $h(\infty) + h_0 = 0$, the plane $x = x_0(t)$ exists throughout the entire wave period $2\pi/\omega$. Electromagnetic field harmonics are generated as a consequence of the time dependence of the conductivity which, being a functional of $|H(x,t)|$, has in this case a period π/ω and contains only even harmonics. Consequently, at $h_0 = 0$ an incident wave of frequency ω generates only odd harmonics in the metal, and no constant magnetic field component $h(x)$ is induced. To excite current states the conduction must be rid of this periodicity. This may be accomplished by "switching on " the external field h_0, but then the plane $x = x_0(t)$ will be absent during a definite part of the period. In conclusion, note that under the conditions of the "almost normal skin effect" current states can be induced only in compensated metals. In uncompensated metals, Hall conductivity prevents the excitation of the current states.

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APPENDIX

In solving the Maxwell equations (2.2) we used an expression for the current density written in the ineffectiveness-concept model (2.5), (2.6). The ineffectiveness concept was first proposed by Pippard in the linear theory of the anomalous skin effect. It has since been successfully used in studying different high-frequency properties of metals and is known to give correct results to within constant real factors on the order of unity (see Ref. 12). Its use in nonlinear problems, however, is as yet not self-evident and requires some justification and generalization. The present Appendix deals with just this subject.

The current density $j(x,t)$ is found by solving a kinetic equation, which is linearized with respect to the electric field $E(x,t)$, but contains the Lorentz force of the total magnetic field $H(x,t) + h_0$. We then compute the asymptote $j(x,t)$ which is valid under the conditions of the anomalous skin effect. The asymptote has the structure

$$j(x,t)=\int_0^\infty dx'Q(x,x')\tilde{\sigma}(\omega t)E(x',t). \quad (A.1)$$

Here $Q(x,x')$ is a time-independent kernel of the conductivity operator of the "Larmor" electrons. It describes the spatial dispersion in the case of the anomalous skin effect. The factor $\tilde{\sigma}(\omega t)$ describes the time dependence of the conductivity; its form can be found in (2.6).

In accord with (A.1), the solutions of the Maxwell equations (2.2) will be found in the form of an expansion in the complete set of functions $\exp[-in\xi(\omega t)]$:

$$H(x,t)=\sum_{n=-\infty}^{\infty} H_n(x)\exp[-in\xi(\omega t)],$$
$$E(x,t)=\sum_{n=-\infty}^{\infty} E_n(x,\omega t)\exp[-in\xi(\omega t)]. \quad (A.2)$$

From the fact that $H_n(x)$ is independent of time and from the first equation in (2.2) it follows that the product $\tilde{\sigma}(\phi)E_n(x,\phi)$ is independent of ϕ $(\phi=\omega t)$. Moreover,

$$H_n'(x)=-\frac{4\pi}{c}\int_0^\infty dx'Q(x,x')\tilde{\sigma}(\varphi)E_n(x',\varphi), \quad (A.3)$$

where the prime denotes the derivative with respect to x.

From the second Maxwell equation (2.2), we obtain the equation

$$E_n(x,\varphi)=\frac{in\omega}{c\rho\tilde{\sigma}(\varphi)}H_n(x), \quad (A.4)$$

and also that the product $\tilde{\sigma}(\phi)\dot{\xi}(\phi)$ must be constant (the dot denotes the derivative with respect to ϕ). This constant, denoted ρ^{-1} in (A.4), is determined from the condition $\xi(\phi+2\pi)=\xi(\phi)+2\pi$ (see the text following (2.4)). Hence we have the expressions for $\xi(\phi)$ and ρ presented in (2.8).

Let us differentiate (A.4) with respect to x and substitute $H_n'(x)$ from (A.3). As a result, we obtain an equation for $E_n(x,\phi)$:

$$E_n''(x,\varphi)=-\frac{4\pi in\omega}{c^2\rho}\int_0^\infty dx'Q(x,x')E_n(x',\varphi). \quad (A.5)$$

Equation (A.5), together with the boundary conditions

$$E_n'(0,\varphi)=\frac{in\omega}{c\rho\tilde{\sigma}(\varphi)}H_n(0),\quad E_n(\infty,\varphi)=0 \quad (A.6)$$

determines uniquely the n-th component of the electric field.

Let us now justify the ineffectiveness concept. As in the linear theory, the gist of the concept is the following. Instead of solving (A.5) we estimate it by the simplest method wherein that the solution $E_n(x,\phi)$ retains its principal features. The estimate is performed by replacing the integral conductivity operator $\hat{Q}$ by an effective multiplication operator $Q_{\rm eff}$. In this replacement it is necessary to take into account the fact that the field $E_n(x,\phi)$ varies sharply and is non-zero in a thin surface layer of thickness σ_n. In the case of the anomalous skin effect, the kernel $Q(x,x')$ is a maximally smooth function of its arguments and varies over distances of the order of l or else $cp_F/2e\mathscr{H}$ $(\sigma_n \ll l, cp_F/2e\mathscr{H})$. The characteristic value of $Q(x,x')$ at $x, x' \lesssim \sigma_n$ has the form

$$Q(x,x')\sim\frac{\sigma_0}{l}[1-\exp(-2\nu T_a)]^{-1}.$$

Therefore, in estimating (A.5) we can make substitution

$$\int_0^\infty dx'Q(x,x')E_n(x',\varphi)\to\frac{\sigma_0}{1-\exp(-2\nu T_a)}\frac{\delta_n}{l}E_n(x,\varphi). \quad (A.7)$$

In the upshot, the integro-differential equation turns into an ordinary differential equation whose solution

$$E_n(x,\varphi)=E_n(0,\varphi)\exp(-x/\delta_n) \quad (A.8)$$

varies exponentially over a distance δ_n determined by (2.8).

Equations (2.4)–(2.6) follow directly from (A.1), (A.2), (A.7), and (A.8).

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