Natural oscillations on charged dimples on a helium surface

S. S. Nazin and V. B. Shikin

Institute of Solid State Physics, USSR Academy of Sciences
(Submitted 10 February 1983)
Zh. Eksp. Teor. Fiz. **85**, 530–542 (August 1983)

The natural oscillations of charged dimples in a direction perpendicular to a helium surface are investigated. It is shown that emission of ripplons causes the oscillations to be strongly damped. The relaxation phenomena accompanying the restructuring of a charged dimple following a change of the external parameters are investigated.

PACS numbers: 67.40.Fd

1. INTRODUCTION

Some of the interesting objects on a charged helium surface are singly and multiply charged dimples. These polaron-type formations readily appear on the liquid surface and manifest themselves in a great variety of situations. Among the most pronounced effects with participation of charged dimples are the singularities of the spectrum of collective oscillations of an electron crystal on a liquid substrate, oscillations due to the formation, under each of the localized electrons, of corresponding dimples that influence the motion of the electron system, as well as direct observation of multielectron dimples. An investigation of the indicated singularities of deformation origin enabled Grimes and Adams[1] to observe Wigner crystallization in a two-dimensional electron system on the surface of liquid helium, and yield interesting information on the details of this phase transition. As for the multielectron dimples observed by Leiderer and Wanner,[2] these formations are promising from the viewpoint of solving the problem of increase of the critical density of charges on a helium surface, are helpful in the study of the behavior of surface electrons in a magnetic field, and so forth.

It must be noted that theoretical investigations of the properties of charged dimples by a number of workers[3–7] dealt mainly with the statistical characteristics of these objects or with dynamic characteristics along the helium surface. In the present paper we discuss a number of questions concerning the behavior of charged dimples when their motion is excited in a direction normal to the helium surface. This problem is of interest in its own right, for in the case of multiply charged dimples it is possible to observe visually the natural oscillations of the dimple. In addition, information on the dynamic properties of charged dimples in the transverse direction is needed for the description of the singularities of cyclotron resonance on localized electrons, for the study of transient phenomena that take place in the dimple when the external parameters are abruptly changed, and others.

2. SINGLE ELECTRON ABOVE A LIQUID-HELIUM SURFACE

It is known that at sufficiently low temperatures and in strong enough clamping fields an electron above a liquid-helium surface should produce a surface anion (Ref. 7, whose notation is used in the equations that follow). The ground level of such an anion is obtained using the variational principle from the functional of the total energy w

$$w=\int\left\{\frac{\alpha}{2}((\nabla\xi)^2+\varkappa^2\xi^2)+\frac{\hbar^2}{2m}(\nabla\psi)^2+F\xi\psi^2\right\}d^2r, \qquad (1)$$

$$\varkappa^2=\rho g/\alpha, \quad \int\psi^2 d^2r=1.$$

Here $\xi(r)$ is the self-consistent deformation of the helium surface, ψ is the electron wave function, ρ and α are the density and surface tension of the liquid helium, g is the acceleration due to the force of gravity, m is the free-electron mass, $F=eE_\perp$, $E_\perp$ is the strength of the electric clamping field, and r is the two-dimensional radius along the helium surface. Choosing a trial wave function in the form

$$\psi=\pi^{-1/2}L^{-1}\exp(-r^2/2L^2) \qquad (2)$$

and minimizing with respect to w and L, we obtain

$$L^2=\tilde{L}^2=4\pi\alpha\hbar^2/mF^2. \qquad (3)$$

It can be shown that the level obtained is stable to small "quadrupole" deviations of the wave function, i.e., to deviatins of the type

$$\psi(x,\ y)=(\pi ab)^{-1/2}\exp[-x^2/4a^2-y^2/4b^2], \qquad (4)$$

where $a=L+\Delta$, $b=L-\Delta$, $\Delta/L\ll 1$, and x and y are the Cartesian coordinates in the plane of the surface. Indeed, if the wave function is so chosen, the total energy w is given by

$$w=-\frac{F^2}{2\pi\alpha}\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{|\psi_{kq}|^2\,dk\,dq}{k^2+q^2+\varkappa^2}+\frac{\hbar^2}{4m}\left(\frac{1}{a^2}+\frac{1}{b^2}\right), \qquad (5)$$

where

$$\psi(k,q)=\int_{-\infty}^{+\infty}\psi(x,y)\exp[ikx+iqy]\,dx\,dy. \qquad (6)$$

Calculations yield the following result:

$$\left.\frac{\partial^2 w}{\partial a\,\partial b}\right|_{a=b=\tilde{L}}=-\frac{F^2}{16\pi\alpha\tilde{L}^2},$$

$$\left.\frac{\partial^2 w}{\partial a^2}\right|_{a=b=\tilde{L}}=\left.\frac{\partial^2 w}{\partial b^2}\right|_{a=b=\tilde{L}}=\frac{5}{16}\frac{F^2}{\pi\alpha\tilde{L}^2},$$

whence

$$\left|\begin{array}{cc}\partial^2 w/\partial a^2 & \partial^2 w/\partial a\,\partial b\\ \partial^2 w/\partial b\,\partial a & \partial^2 w/\partial b^2\end{array}\right|_{a=b=\tilde{L}}>0.$$

 0038-5646/83/080310-07$04.00

Starting from the foregoing we can assume that when the surface anion is taken out of its equilibrium position it is capable of executing oscillations at a certain frequency. These oscillations will radiate energy in the form of diverging surface waves and should therefore be strongly damped. By virtue of the extremely low inertia of the electron we can assume that its wave function attunes itself adiabatically to the surface relief. In the adiabatic approximation, which will in fact be used below for the calculations, this means that the localization length of an electron whose wave function is chosen in oscillator form (2) is determined by the second derivative $\xi''(r)|_{r=0}$ (we are considering oscillations with cylindrical symmetry). Let $\varphi(r,z)$ be the hydrodynamic potential of the velocity. Then

$$\varphi(r,z,t)=\int_0^\infty J_0(kr)\varphi(k,t)e^{kz}k\,dk, \tag{7}$$

and the equation for $\varphi(k,t)$ is of the form

$$\frac{\partial^2\varphi(k,t)}{\partial t^2}+\omega_k^2\varphi(k,t)=-\frac{1}{\rho}\frac{\partial P(k,t)}{\partial t}, \tag{8}$$

where $P(k,t)$ is the Fourier-Bessel transform of the electron pressure and $\omega_k^2=\alpha k^3/\rho$. Let the electron localization length L^2 execute small oscillations about its equilibrium value L_0^2, which is equal in the harmonic approximation to $\tilde{L}^2/2$ [$\tilde{L}$ is taken from Eq. (3)], so that $L^{-2}=L_0^{-2}+\alpha(t)$. Then

$$P=\frac{F}{\pi L^2}e^{-r^2/L^2}\approx\frac{F}{\pi L_0^2}e^{-r^2/L_0^2}((L_0^2-r^2)\alpha(t)+1). \tag{9}$$

The following equation should hold in the harmonic approximation

$$F\xi''(0)=\frac{\hbar^2}{mL^4}\approx\frac{\hbar^2}{mL_0^4}+\frac{2\hbar^2}{mL_0^2}\alpha(t), \tag{10}$$

whence

$$\frac{\partial\alpha(t)}{\partial t}=\frac{mL_0^2}{2\hbar^2}F\frac{\partial\xi''(0,t)}{\partial t}. \tag{11}$$

Using the connection between the Fourier-Bessel transform of the surface relief $\xi_k(t)$ and the quantity $\varphi_k(t,z)=e^{kz}\varphi(k,t)$ given by

$$\frac{\partial\xi_k(t)}{\partial t}=\frac{\partial\varphi_k(t,z)}{\partial z}\Big|_{z=0},$$

we obtain

$$-\frac{1}{\rho}\frac{\partial P}{\partial t}=C(k)\int_0^\infty\varphi(q,t)q^4\,dq, \tag{12}$$

where

$$C(k)=\frac{\alpha L_0^2}{4\rho}\exp\left\{-\frac{k^2L_0^2}{4}\right\}\frac{k^2L_0^2}{4}, \tag{13}$$

i.e., we have for $\varphi(k,t)$ the equation

$$\ddot{\varphi}(k,t)+\omega_k^2\varphi(k,t)=C(k)\int_0^\infty\varphi(q,t)q^4\,dq. \tag{14}$$

We use now the unilateral Fourier transformation with respect to time:

$$\varphi(k,\omega)=\int_0^\infty e^{i\omega t}\varphi(k,t)\,dt; \tag{15}$$

the complex variable ω is in the upper half-plane. For the Fourier transform of $\varphi(k,t)$ we obtain the expression

$$-\omega^2\varphi(k,\omega)-B(k,\omega),$$

with $B(k,\omega)$ defined by the initial conditions

$$B(k,\omega)=\dot{\varphi}(k,t)|_{t=0}-i\omega\varphi(k,t)|_{t=0}.$$

The function $\varphi(k,\omega)$ satisfies thus the equation

$$\varphi(k,\omega)(\omega_k^2-\omega^2)=C(k)\int_0^\infty\varphi(q,\omega)q^4\,dq+B(k,\omega). \tag{16}$$

The solution of (16) is

$$\varphi(k,\omega)=[C(k)A(\omega)+B(k,\omega)][\omega_k^2-\omega^2]^{-1}, \tag{17}$$

$$A(\omega)=\left[\int\frac{B(k,\omega)k^4\,dk}{\omega_k^2-\omega^2}\right]\left[1-\int\frac{C(k)k^4\,dk}{\omega_k^2-\omega^2}\right]^{-1}. \tag{18}$$

Using the known formula for taking the inverse Fourier transform of (15), we obtain

$$\varphi(k,t)=\frac{1}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma}e^{-i\omega t}\frac{C(k)A(\omega)+B(k,\omega)}{\omega_k^2-\omega^2}d\omega. \tag{19}$$

The integration in (19) is along a line parallel to the real axis and passing above all the singularities of the integrand.

It will be shown below that $A(\omega)$ is not analytic in the entire ω plane. For a unique determination of its analytic continuation (AC) from the upper plane we draw a cut along the negative imaginary axis. Shifting now the integration path into the lower half-plane, we see that the integral in (19) is determined by the residues at the points $\omega=\pm\omega_k$, by the residues at the poles of the function $A(\omega)$, and by the integral along the negative imaginary axis (on both edges of the cut, Fig. 1).

The contribution of the poles $\omega=\pm\omega_k$ to $\varphi(k,t)$ is proportional to $\exp[\pm i\omega_k t]$. At the same time the contribution of the poles of $A(\omega)$ is proportional to $\exp[-i\omega_{1,2}t]$, where $\omega_{1,2}$ is independent of k, i.e., $\omega_{1,2}$ are eigenfrequencies of the system. The poles of the function $A(\omega)$ are determined by the condition $1=f(\omega)$, where $f(\omega)$ is the AC of the function

$$f(\omega)=\int_0^\infty\frac{C(k)k^4\,dk}{\omega_k^2+\omega^2}, \tag{20}$$

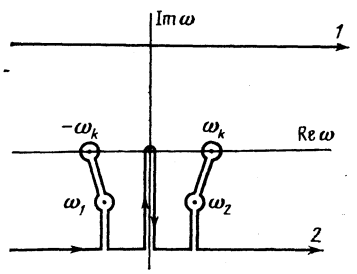

FIG. 1. 1—Unshifted integration path, 2—shifted integration path.

defined by this formula in the upper half-lane on the imaginary axis. Equation (20) reduces in the upshot to the form

$$1=g(\mu)=AC\int_0^\infty \frac{e^{-s^3}s^6\,ds}{s^3+\mu^2}$$

$$=\frac{1}{2}+\frac{\mu^2}{2}\left[-\frac{4\pi}{3^{1/2}}(-i\mu)^{1/3}{}_1F_3\left(\begin{matrix}1;&x\\ \Delta&(3,1)\end{matrix}\right)\right.$$

$$+y^2(\ln y-\psi(2)){}_1F_3\left(\begin{matrix}1;&x\\ \Delta&(3,2)\end{matrix}\right)$$

$$+\frac{2\pi}{3^{1/2}}(-i\mu)^{10/3}{}_1F_3\left(\begin{matrix}1;&x\\ \Delta&(3,3)\end{matrix}\right)$$

$$\left.+\pi^{1/2}{}_1F_3\left(\begin{matrix}1;&x\\ \Delta&(3,1/2)\end{matrix}\right)-y^2H(\Delta(3,2);x)\right], \qquad (21)$$

where

$$x=-\mu^4/27,\ \mu^2=\omega^2/\omega_k^2(2/L_0),\ y=(-i\mu)^{4/3}$$

(the calculations and the explicit expressions for the functions F and H are given in the Appendix).

To solve Eq. (21) we note first that when $\mu=a+ib$ is replaced by $\mu^*=a+ib$ the function $g(\mu)$ is replaced by its complex conjugate. In addition, this equation has no roots in the upper half-plane (their presence would mean instability of the system), since at all points of this half-plane [except the imaginary axis, on which Re $g(\mu)$ decreases monotonically from 0.5 to 0] we have Im $g(\mu)\neq 0$. It suffices therefore to consider only the lower right quadrant of the μ plane.

Computer calculations show that the line on which Im $g(\mu)=0$ and Re $g(\mu)>0$ is given by the function $\varphi(r)$ represented by curve 1 of Fig. 2 ($\mu=r\exp(i\varphi)$). Re $g(\mu)$ increases monotonically along this curve, as seen from Fig. 3 (curve 2), and $g(\mu)=1$ at $r=0.73$ and $\varphi=-0.72$. This means that the sought resonance frequency is

$$\Omega^{(1)}=2.07\omega_k(1/L_0)\,(0.75-0.66i). \qquad (22)$$

It can be shown that when a magnetic field H normal to the surface is applied the equation for the resonance frequency takes the form

$$L_0^2L_t^{-2}=g(\mu), \qquad (23)$$

where L_t^2 is the electron localization length with allowance for the magnetic field, and is given by[7]

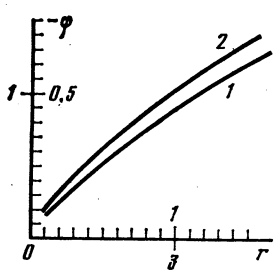

FIG. 2. Behavior of $-\varphi(r)$ as a function of r [$\varphi(r)$ and r are defined in the text]; curve 1—for a one-electron dimple (left-hand y scale and upper x scale), curve 2—for multielectron dimple.

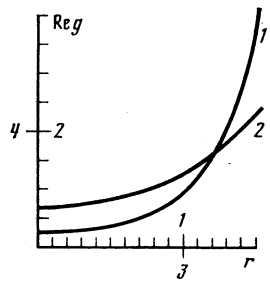

FIG. 3. Behavior of Re $g(\mu)$ as a function r: curve 1—for one-electron dimple (right-hand y scale and upper x scale), curve 2—for multielectron dimple. The sought resonance frequency Ω is determined from the obtained values of r and φ with the aid of the relation $\mu=\omega/\omega_k(2/L_0)$ from (21), which connects ω and μ (in the case of a one-electron dimple).

$$L_t^{-2}=L_0^{-2}+(1/4L_0^4+1/L_H^4)^{1/2},$$

$$L_H^2=2\hbar c/eH, \qquad \mu^2=\omega^2/\omega_k^2\left(\frac{2}{L_t}\right). \qquad (24)$$

It can be seen from Fig. 3 that with increasing H the ratio of the resonance frequency to the characteristic hydrodynamic frequency $\omega_k(2/L_t)$ increases, and the damping becomes stronger.

If the electric field becomes strong enough (in a sense that will be made clear below) the problem can be solved by using another approach. In the equilibrium state[7]

$$\xi(0)=-[F/2\pi\alpha]\ln(2/\gamma^{1/2}\varkappa L_0),$$

where γ is the Euler constant.

This value was obtained for an electron wave function of the type (2), corresponding to a ground state energy level

$$\hbar^2/mL^2=F^2/2\pi\alpha,$$

reckoned from the bottom of the well caused by the deformation of the surface. The depth of the well is

$$F\xi(0)=-[F/2\pi\alpha]\ln(2/\gamma^{1/2}\varkappa L_0). \qquad (25)$$

In a strong enough field, when L_0 so decreases that the logarithm becomes much larger than unity, the succeeding levels also appear in the well. In the same harmonic approximation, the first excited level with radial symmetry is described by the wave function

$$\varphi_1=\frac{1}{\pi^{1/2}L_0}\left(1-\frac{r^2}{L_0^2}\right)\exp\left(-\frac{r^2}{2L_0^2}\right) \qquad (26)$$

and has an energy $3F^2/2\pi\alpha$, i.e., its appearance requires a field at least higher than 100 V/cm. At small oscillations of $\xi(r)$ the ground level of the electron changes little, and we shall therefore approximate it by the wave function $\varphi=\varphi_0+c\varphi_1$, where c is obtained by perturbation theory. During the oscillation time the electron is in the ground state, since the perturbation $eE_\perp\Delta\xi(r,t)$ is adiabatic for it because the characteristic frequency of variation of $\xi(r)$ is of the order of $\omega_k(1/L_0)$, and the latter is much less than the characteristic difference $\sim F^2/2\pi\alpha$ between the electron energy levels (which are mixed by the perturbation with radial symmetry).

In first order perturbation theory we have

$$c=-\frac{\pi\alpha}{F}\int_0^\infty \frac{k^2L_0^2}{4}\exp\left\{-\frac{k^2L_0^2}{4}\right\}\xi(k,t)\,k\,dk. \quad (27)$$

The corresponding change of the electron pressure $P(k,t)$ is

$$-\frac{1}{\rho}\frac{\partial P}{\partial t}=\frac{4C(k)}{L_0^2}\int_0^\infty \exp\left\{-\frac{k^2L_0^2}{4}\right\}\frac{k^2L_0^2}{4}\varphi(k,t)\,k^2\,dk, \quad (28)$$

and calculations similar to the preceding ones lead to the following equation for the eigenfrequency:

$$1={}^1/_4 g(\mu),\ \ \mu=\omega/\omega_k(2^{1/2}/L_0). \quad (29)$$

Its solution is

$$\Omega^{(II)}=2.17\omega_k(1/L_0)\,(0.44-0.9i). \quad (30)$$

Since the first of the considered methods is suitable for arbitrary fields $E_\perp$, the results obtained in both cases should coincide as $E_\perp\to\infty$. The difference between them is a measure of the degree of validity of the approximations employed.

Besides the considered oscillations with radial symmetry, there exist also natural oscillations of the "quadrupole" type, i.e., oscillations in which the electron is described by a wave function of type (4), while a and b execute small oscillations about L_0 in counterphase. To determine the frequency of these oscillations by the first method we write

$$\delta P=\frac{\partial P}{\partial a}\delta a+\frac{\partial P}{\partial b}\delta b \quad (31)$$

and calculations similar to those made for the oscillations with radial symmetry yield the following equations for the natural frequencies:

$$\begin{vmatrix} 1-{}^3/_4 g(\mu) & -{}^1/_4 g(\mu) \\ -{}^1/_4 g(\mu) & 1-{}^3/_4 g(\mu)\end{vmatrix}=0. \quad (32)$$

Equation (32) breaks up into two:

$$g(\mu)=1, \quad (33a)$$

$$g(\mu)=2, \quad (33b)$$

the first describes the previously obtained radial oscillations and the second the oscillations of the "quadrupole" type, when $\delta a(t)=-\delta b(t)$. From (33b) follows

$$\Omega_{qu}^{(I)}=2.94\omega_k(1/L_0)\,(0.59-0.81i). \quad (34)$$

In the presence of a magnetic field the frequency of the considered oscillations is obtained from the equation

$$g(\mu)=2L_t^{-2}L_0^2. \quad (35)$$

To determine the frequency of the "quadrupole" oscillations by the second method it is necessary to use not the wave function φ_1 defined by (26), but

$$\tilde{\varphi}_1=\frac{\cos 2\varphi}{\pi^{1/2}L_0^3}r^2\exp\left\{-\frac{r^2}{L_0^2}\right\},$$

which corresponds to the first excited level connected with the ground-state perturbation $\xi(r,t)$ of the considered symmetry. The sought frequency is then found to be

$$\Omega_{KB}^{(II)}=2.59\omega_k(1/L_0)\,(0.31-0.95i) \quad (36)$$

3. MULTIELECTRON DIMPLE ON LIQUID HELIUM

The problem of transverse oscillations is most vital for multiply charged dimples, for which direct visual observation of the eigenfrequencies is possible even if they are strongly damped.

Just as for a single electron, the characteristics of a multicharged dimple are determined from an energy functional that takes in this case the form[5,6]

$$w=\int d^2r\left(eE_\perp\xi(r)n(r)+\frac{e^2n(r)}{2}\int\frac{n(r')\,d^2r'}{|r-r'|}+\frac{\alpha}{2}\{(\nabla\xi)^2+\varkappa^2\xi^2\}\right),$$

$$\int n(r)\,d^2r=N, \quad (37)$$

where $n(r)$ is the surface density of the number of electrons and n is the total number of charges in the dimple. Variation of w with respect to n and ξ at a constant number N of electron in the dimple yields the following two equations (λ is the Lagrange multiplier):

$$e^2\int\frac{n(r')\,d^2r'}{|r-r'|}=\lambda-eE_\perp\xi(r), \quad (38)$$

$$\Delta\xi-\varkappa^2\xi=eE_\perp n(r)/\alpha. \quad (39)$$

Approximating $\xi(r)$ in (38) by the expression

$$\xi(r)=\xi_0+{}^1/_2Ax^2+{}^1/_2By^2,\ \ A=\xi_x''(0),\ \ B=\xi_y''(0), \quad (40)$$

which is legitimate in the region $E_\perp>E_\perp^{\min}$, where $E_\perp^{\min}$ is the minimum field needed for the dimple to appear, we obtain for $n(r)$ an integral equation that differs only in notation from the equation encountered in the Hertz contact problem [Ref. 8, Eq. (9.7)]. We therefore obtain directly

$$n(r)=\frac{3N}{2\pi ab}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2}, \quad (41)$$

where a and b are connected with A and B from (40) as follows:

$$\frac{3}{4}Ne^2\int_0^\infty\frac{d\xi}{[(a^2+\xi)^3(b^2+\xi)\xi]^{1/2}}=\frac{eE_\perp A}{2}, \quad (42a)$$

$$\frac{3}{4}Ne^2\int_0^\infty\frac{d\xi}{[(a^2+\xi)(b^2+\xi)^3\xi]^{1/2}}=\frac{eE_\perp B}{2}. \quad (42b)$$

Two additional relations between the constants a, b, A, and B stem from the mechanical-equilibrium equation. Thus, in the cylindrical-symmetry case, when $a=b=R$ and $A=B$, the definitions (42a) and (42b) reduce to the equation

$$R^3=3\pi eN/4E_\perp\xi''(0), \quad (43a)$$

while Eq. (39) with allowance for $\varkappa R\ll 1$ assumes at the point $r=0$ the form

$$2\alpha\xi''(0)=eE_\perp n(0), \quad (43b)$$

with $n(0)$ from (41).

From the definitions (43a) and (43b) it follows that

$$R=\pi^2\alpha/E_\perp^2. \quad (44)$$

This result is equal, apart from a numerical factor ~ 1, to the definition of R from Ref. 5 in the limit $\varkappa R \ll 1$, i.e., $E_\perp \gg \pi(\alpha\varkappa)^{1/2} \approx 2.4$ kV/cm. We note also that for $\xi(r) = \xi(0) + \frac{1}{2}Ar^2$ a distribution of the type (41) with $a = b = R$ was obtained for $n(r)$ earlier in Ref. 9.

Proceeding in analogy with the case of the one-electron dimple, we can obtain the following eigenfrequency equation for radially symmetric oscillations:

$$\mathrm{AC}\left(\frac{1}{3}\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{J_{3/2}(z)z^{3/2}}{z^3+\mu^2}dz\right)=1, \quad \mu=\omega/\omega_k\left(\frac{1}{R}\right). \tag{45}$$

Inasmuch as $z\to\infty$ we have $J_{5/2}(z)\sim z^{-1/2}\sin z$, the integral in (45) diverges like cos z at the upper limit. The reason is that the quantity $\partial n(r)/\partial r$, whose Fourier-Bessel transform is contained in the numerator of the integrand of (45), has at $r = R$ a singularity of the type $(R-r)^{1/2}$, so that this Fourier transform does not fall off rapidly enough as $k\to\infty$. However, Eq. (41) for $n(r)$ was obtained by expansion near zero, and is therefore expected to agree well with the exact function $n(r)$ near the center of the dimple, but not at its edge. Replacing in (41) $(1-r^2/R^2)^{1/2}$ by $(1-r^2/R^2)^{1/2+\beta}$, $\beta\to+0$, we obtain an expression that is arbitrarily close to the initial one and has a weaker singularity near $r = R$. As a result of this substitution Eq. (45) takes the form

$$\tilde{g}(\mu)=\mathrm{AC}\left[\lim_{\beta\to+0}\frac{1}{3}\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{J_{3/2}(z)z^{3/2-\beta}}{z^3+\mu^2}dz\right]=1. \tag{46}$$

A numerical analysis shows that the root of this equation is (see Figs. 2 and 3)

$$\Omega=2.35\omega_k(1/R)\,(0.90-0.44i). \tag{47}$$

Just as for a singly charged dimple, we obtain the frequency of the quadrupole oscillations, namely

$$\Omega_{qu}=4.20\omega_k(1/R)\,(0.80-0.60i). \tag{48}$$

Thus, in all the investigated variants of the problem of the natural transverse oscillations of a charged dimple, the fundamental eigenfrequency is of the size of the capillary frequency for wave numbers of the order of the reciprocal radius of the charge distribution in the dimple. The imaginary parts of the natural oscillations of the type considered is comparable with the real one, a fact explained by the emission of surface waves.

4. MULTIELECTRON DIMPLE ON THE INTERFACE BETWEEN LIQUID AND SOLID HELIUM

Leiderer and coworkers have recently performed a number of experiments on the properties of a liquid-solid helium interface to which an external field clamped negative ions produced artificially in the liquid phase.[10,11] Among the results of these experiments were: a) observation of a multiply charged dimple on the solid-helium surface; b) determination of the critical field for dimples with different charges; c) direct visual observations of the dependence of the dimple depth (at the center) on the time following a jumplike change of the clamping field. In the present section we describe the last of the foregoing effects. The starting point is an equation[11] that determines the evolution of the dimple relief $\xi(r,t)$ with time:

$$\left(1-\frac{\rho_1}{\rho_2}\right)\left(\rho_2\frac{\partial\varphi}{\partial t}\Big|_{z=\xi}+\rho_2 g\xi\right)=\frac{\rho_1}{m_4 K}\frac{\partial\xi(r,t)}{\partial t}-\alpha\Delta\xi+P_{el}. \tag{49}$$

Here ρ_1 and ρ_2 are the respective densities of the solid and liquid phases, m_4 is the mass of the He^4 atom, and K is a constant that determines the crystal growth rate as a function of the difference between the chemical potentials of the two phases: $\dot{\xi} = -K(\mu_1-\mu_2)$, P_{el} is the electron pressure equal to $eE_\perp n(r,t)$, and α is the surface tension on the liquid-solid interface (we neglect the anisotropy of α).

Transforming the Fourier-Bessel representation in r and using the connection between $\varphi(k,t)$ and $\xi(k,t)$

$$k\varphi(k,t)=\varepsilon\frac{\partial\xi(k,t)}{\partial t}, \qquad \varepsilon=\frac{\rho_1-\rho_2}{\rho_2},$$

we rewrite (49) in the form

$$-\varepsilon\rho_2\left(\frac{1}{k}\frac{\partial^2\xi(k,t)}{\partial t^2}+g\xi(k,t)\right)$$

$$=\frac{\rho_1}{m_4K}\frac{\partial\xi(k,t)}{\partial t}+\alpha k^2\xi(k,t)+P_{el}(k,t). \tag{50}$$

It will be shown below that owing to the very strong damping of the melting-crystallization waves (the temperature under the conditions of Refs. 10 and 11 was 1.35 K) we can neglect in (50) the second derivative $\ddot{\xi}(k,t)$ compared with the first, i.e., replace (50) by the equation

$$\frac{\partial\xi(k,t)}{\partial t}=-\left(\frac{\alpha}{\rho_1}k^2+\frac{\varepsilon\rho_2}{\rho_1}g\right)\frac{\xi(k,t)}{c}-\frac{P_{el}(k,t)}{c\rho_1}, \tag{51}$$

where $c = 1/m_4K$.

Let now the clamping field change at the instant of time $t = 0$ from E_1 to E_2, and then $\xi(k,t)$ will begin to change gradually from its initial value to a value $\xi^*(k)$ corresponding to the static dimple at the field E_2. If the ion mobility on the interface is large enough, i.e., if $n(r,t)$ at a given $\xi(r,t)$ is determined by the condition that the total potential (38) be constant, it is impossible to solve (51) in the general case. It can be done approximately, however, if the clamping field decreases jumpwise by many times or even drops to zero. Indeed, the electron pressure can then be neglected because, at least for not very large t, we have the inequality

$$P_{el}(k,t)=eE_\perp n(r,t)\ll\left(\frac{\alpha}{\rho_1}k^2+\varepsilon\frac{\rho_2}{\rho_1}g\right)\xi(k,t).$$

From (51) without P_{el} it follows directly that

$$\xi(k,t)=\xi(k,t=0)\exp\left[-\left(\frac{\varepsilon\rho_2}{\rho_1}g+\frac{\alpha}{\rho_1}k^2\right)\frac{t}{c}\right], \tag{52}$$

i.e.,

$$\xi(r=0,t)=\int_0^\infty \xi(k,t=0)\exp\left\{-\left(\frac{\varepsilon\rho_2}{\rho_1}g+\frac{\alpha}{\rho_1}k^2\right)\frac{t}{c}\right\}kdk$$

$$=-\frac{e^{-t/\tau_1}}{\alpha}\int_0^\infty \frac{P_{el}(k,t=-0)}{\varkappa^2+k^2}\exp\left\{-\frac{t}{\tau_2}(kR)^2\right\}k\,dk, \tag{53}$$

where

$$\tau_1=c\rho_1/\varepsilon\rho_2 g,\quad \tau_2=cR^2\rho_1/\alpha,\quad \varkappa^2=\rho_2 g\varepsilon/\alpha,\quad \tau_2/\tau_1=\varkappa^2R^2\ll 1,$$

$R\ll\varkappa^{-1}$ is the radius of the dimple at the instant $t=0$. It follows from (52) that the second derivative with respect to time can be neglected in (50): the characteristic rate of change of ξ is $\xi\varepsilon g/c$, whence

$$\left|\frac{\varepsilon\rho_2 k^{-1}\ddot{\xi}(k,t)}{\rho_1 c\dot{\xi}}\right|\sim\varepsilon^2\frac{Rg}{c^2}\sim 10^{-6},$$

for under the conditions of Refs. 10 and 11 we have $\alpha\sim 0.2$ erg/cm, $\varepsilon\sim 0.1$, and $c\sim 10^3$ cm/sec. Since P_{el} is contained in (53) under an integral sign, it can be assumed that the true value $P(k,t=-0)$ [which should be determined by solving a system of equations of the type (38) and (39)] can be replaced with acceptable accuracy by the function $\tilde{P}(k)$ corresponding to the charge distribution

$$n(r)=(N/\pi R^2)\exp(-r^2/R^2).$$

With this substitution, the integral in (53) is easily evaluated. As a result we have

$$\xi(0,t)=\xi(0,0)e^{-t/\tau_1}G(t/\tau_2)/G(0), \tag{54}$$

where

$$G(x)=\exp[(1/4+x)s^2]E_1[(1/4+x)s^2],$$

$s=\varkappa R$ and E_1 is the integral exponential function.

For large $t\gg\tau_2$ it follows from (54) that

$$\xi(0,t)=\frac{\xi(0,0)}{G(0)}\frac{e^{-t/\tau_1}}{(1/4+t/\tau_2)s^2}. \tag{55}$$

The result (55) for $\xi(0,t)$ can be compared with the experimental data.[11] According to these data, the characteristic relaxation time is of the order of 10–50 sec, and the time dependence of $\xi(0,t)$ is not purely exponential. These two statements agree qualitatively with $\xi(0,t)$ determined from (55). In fact, in accord with the definition (53) of τ_1 the numerical value of this characteristic time is of the order of $\varepsilon\sim 10$ sec at $\varepsilon\sim 0.1$ and $c\sim 10^3$ cm/sec.

5. CONCLUSION

We present some summaries. We obtained the eigenfrequencies of the oscillations of singly and multiply charged dimples on the surface of liquid helium. The simplest modes of such oscillations are radial and quadrupole. The frequencies of the quadrupole oscillations are 1.5–2 times larger than the radial ones. In the case of a singly charged dimple we considered two approaches to the determination of the eigenfrequencies: 1) harmonic approximation, in which the electron is described by a wave function of the oscillator type with a localization length corresponding to a dimple approximated by a parabolid, and 2) perturbation theory for the ground level when the clamping field is strong enough. The first method was used to find also equations for the eigenfrequencies of the dimple in a magnetic field H normal to the helium surface. It was found that the damping of the oscillation increases with increasing H.

The situation in the case of multiply charged dimples is qualitatively the same. The characteristic frequencies of such dimples are of the order of the capillary frequency for wave numbers of the order of the radius of the charged nucleus at the center of the dimple. An interesting modification of the problem of dimple oscillations on a liquid helium surface is the relaxation of a dimple on an interface between liquid and solid helium following an abrupt change of the clamping field. The relaxation time scale predicted by the theory is in qualitative agreement with experiment.[11]

It should be noted that althogh the resonances corresponding to excitation of the radial and quadrupole modes of the charged dimples should be relatively broad, neighboring resonance lines do not overlap strongly, since their excitation calls for an alternating electric field of different polarization (along the normal to the helium or along the helium surface). Besides, the possibility of visual observation of the dynamic properties of multielectron dimples, demonstrated in Ref. 11, can also be used to determine the type of oscillations and the positions of the corresponding resonance frequencies.

APPENDIX

To calculate the integral

$$I=\int_0^\infty\frac{s^6e^{-s^2}\,ds}{s^3+\mu^2}$$

we add and subtract μ^2 in the numerator, make the change of variable $s^2=t$, and replace μ by the variable $y=\mu^{4/3}$. We obtain

$$I=\frac{1}{2}-\frac{\mu^2}{2}\int_0^\infty\frac{te^{-t}\,dt}{t^{3/2}+y^{3/2}}. \tag{56}$$

This integral is a particular case of Eq. 2.3.2.3 of Ref. 12 ($\alpha=2$, $\rho=0$, $r=3/2$, $s=1$). This formula, however, cannot be applied here directly, inasmuch as at the indicated values of the parameters it contains a B function of negative argument. To get around this difficulty we take the limit as $\alpha\to 2$ (this can be done by virtue of a corresponding theorem in mathematical analysis). Putting $\alpha=2+\Delta\alpha$ we obtain

$$I'=\sum_{h=0}^{2}V_{26}^{(h)}+\sum_{j=0}^{1}V_{27}^{(j)},$$

$$I'=\int_0^\infty\frac{e^{-t}t\,dt}{t^{3/2}+y^{3/2}},$$

$$V_{26}^{(0)}=-\frac{4\pi}{3^{3/2}}y^{3/2}{}_1F_2\begin{pmatrix}1;&-\frac{y^3}{27}\\ \Delta&(3;1)\end{pmatrix},$$

$$V_{26}^{(1)}=-\frac{2}{3}B\left(2+\frac{2}{3}\Delta\alpha,\ -1-\frac{2}{3}\Delta\alpha\right)y^{3/2+\Delta\alpha}{}_1F_2\begin{pmatrix}1;&-\frac{y}{27}\\ \Delta&(3,2)\end{pmatrix},$$

$$V_{26}^{(2)}=\frac{2\pi}{3^{3/2}}y^{3/2}{}_1F_2\begin{pmatrix}1;&-\frac{y^3}{27}\\ \Delta&(3,3)\end{pmatrix},$$

$$V_{27}^{(0)}=\sqrt{\pi}\,{}_1F_3\left(\begin{matrix}1; & -\frac{y^3}{27}\\ \Delta & \left(3,\frac{1}{2}\right)\end{matrix}\right),$$

$$V_{27}^{(0)}=-\Gamma(-1+\Delta\alpha)\,y^{1/3}\,{}_1F_3\left(\begin{matrix}1; & -\frac{y^3}{27}\\ \Delta & (3,2-\Delta\alpha)\end{matrix}\right),$$

where

$${}_1F_3\left(\begin{matrix}1;\ x\\ a,b,c\end{matrix}\right)=\sum_{k=0}^{\infty}\frac{x^k}{(a)_k(b)_k(c)_k},\qquad (a)_k=\frac{\Gamma(a+k)}{\Gamma(a)},$$

$\Delta(3,n)=n/3,(n+1)/3,(n+2)/3$, and the transition $\Delta\alpha\to 0$ was made in all expressions that are continuous in $\Delta\alpha$. Using the expansion

$$-\frac{2}{3}B(2+{}^2/_3\Delta\alpha,-1-{}^2/_3\Delta\alpha)=-1/\Delta\alpha+O(\Delta\alpha),$$

$$y^{1/3+\Delta\alpha}=y^{1/3}(1+\Delta\alpha\ln y+O((\Delta\alpha)^2)),$$

$$-\Gamma(-1+\Delta\alpha)=1/\Delta\alpha+\psi(2)+O(\Delta\alpha),$$

and noting also that

$${}_1F_3\left(\begin{matrix}1;\ z\\ \Delta(3,2-\Delta\alpha)\end{matrix}\right)$$

$$={}_1F_3\left(\begin{matrix}1;\ z\\ \Delta(3,2)\end{matrix}\right)+\Delta\alpha H(\Delta(3,2);z)+O((\Delta\alpha)^2),$$

where

$$H(\Delta(p,q);z)$$

$$=\frac{1}{p}\sum_{k=1}^{\infty}z^k\left[\left(\frac{q}{p}\right)_k\left(\frac{q+1}{p}\right)_k\cdots\left(\frac{q+p-1}{p}\right)_k\right]^{-1}$$

$$\times\left(\sum_{n=0}^{k-1}\left(\left[\frac{q}{p}+n\right]^{-1}+\left[\frac{q+1}{p}+n\right]^{-1}\right.\right.$$

$$\left.\left.+\ldots+\left[\frac{q+p-1}{p}+n\right]^{-1}\right)\right),$$

we obtain an expression for I. Replacing in it y by $\mu^{4/3}$ and μ by $-i\mu$, we obtain the final expression for $g(\mu)$. A formula for $\tilde{g}(\mu)$ is also derived with the aid of 2.3.2.3 of Ref. 12, but $J_{5/2}(z)$ is first expressed in terms of $\sin z$ and $\cos z$ which are in turn represented in the form $\operatorname{Im}\exp(-iz)$ and $\operatorname{Re}\exp(-iz)$.

[1]C. C. Grimes and G. Adams, Phys. Rev. Lett. **42**, 795 (1979).
[2]P. Leiderer and M. Wanner, *ibid.* p. 315.
[3]V. B. Shikin and Yu. P. Monarkha, Zh. Eksp. Teor. Fiz. **65**, 751 (1973) [Sov. Phys. JETP **38**, 373 (1974)].
[4]Yu. P. Monarkha, Fiz. Nizk. Temp. **1**, 526 (1975) [Sov. J. Low Temp. Phys. **1**, 258 (1975)].
[5]V. B. Shikin and P. Leiderer, Zh. Eksp. Teor. Fiz. **81**, 184 (1981) [Sov. Phys. JETP **54**, 92 (1981)].
[6]V. I. Mel'nikov and S. V. Meshkov, *ibid.* p. 951 [505].
[7]V. B. Shikin and Yu. P. Monarkha, Fiz. Nizk. Temp. **1**, 957 (1975) [Sov. J. Low Temp. Phys. **1**, 459 (1975)].
[8]L. D. Landau and E. M. Lifshitz, Theory of Elasticity, Pergamon, 1970.
[9]V. I. Mel'nikov and S. V. Meshkov, ITF Preprint, 1981.
[10]D. Savignac and P. Leiderer, Phys. Rev. Lett. **49**, 1869 (1982).
[11]D. Savignac, Diploma Thesis, Tech. Univ. Munich, 1982.
[12]A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integraly i ryady (Integrals and Series), Nauka, 1981, p. 319.

Translated by J. G. Adashko