Theoretical treatment of the interactions between the resonant particles of a magnetized plasma and high-frequency wave packets

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A consistent general theory of nonlinear resonance interactions between the particles of a relativistic collisionless plasma and high-frequency waves propagating in an arbitrary direction relative to an external magnetic field is presented. The theory is based on a drift-kinetic description of a plasma and the eikonal approximation for a high-frequency field. The general integral of the characteristic equations of the zeroth-order approximation is found. Cherenkov resonance at the beats of two wave packets is treated. The permittivity tensor of a relativistic weakly-inhomogeneous plasma immersed in a nonuniform magnetic field is calculated without recourse to linearization of the kinetic equation.

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Resonance interactions between particles and high-frequency waves play an important role in a variety of problems of plasma physics under both experimental conditions and conditions found in outer space.[1–3] Methods of simplified (averaged) description of the interaction are used to one degree or another in solving this type of problem. A rather general theory to describe the interaction of plasma particles and a high-frequency wave under cyclotron resonance conditions (with Doppler shift)[4] and semicyclotron resonance conditions[5] has been developed[4,5] using the Bogolyubov-Mitropol'skiĭ method.[6] However, this theory is limited to the case of a uniform plasma immersed in a uniform and steady magnetic field whose resonant particles interact with an isolated plane wave. The influence of relativistic effects that may be of substantial importance under resonance conditions is ignored here.

A consistent theory based on a drift-kinetic description of the plasma and on the eikonal approximation for a high-frequency field has been previously considered.[7] However, the theory is valid only for waves propagating quasilongitudinally relative to an external magnetic field. This corresponds to the case of particles with small cyclotron radius (compared with the wavlength).

In the present article we propose a general theory to describe resonance interactions between the particles of a relativistic collisionless plasma and high-frequency fields in the case of waves propagating in an arbitrary direction relative to the magnetic field (the case of a finite cyclotron radius). In this case, the high-frequency fields are considered in the form of a superposition of a finite number of wave packets.[8] Such waves may be excited in the plasma either by external sources or as a consequence of a variety of instabilities. The small parameter of the drift theory is taken as the expansion parameter. The resulting zeroth-order approximation theory describes cyclotron-resonance interactions. In the first-order approximation, which is quadratic in wave amplitude, resonances are produced in which two wave packets participate in particular, the semicyclotron resonances previously considered[5] in a special case, and cyclotron resonances at the beats of two waves. More complicated resonances involving three, etc., wave packets are possible in the succeeding approximations.

Note that, despite the fact that the effects of higher approximations are minor, they may be of substantial importance due to the large number of plasma particles participating in the resonant interactions.

In this article we obtain a drift-kinetic zeroth-and first-order approximation equation for the resonant particles. The variable part of the distribution function of the first-order approximation is computed and a technique for treating the higher-order approximations is indicated. The general integral of the characteristic equations of the zeroth-order approximation is found. Motion of particles interacting in resonance with the beats of two wave packets is considered. The permittivity tensor is computed for a collisionless weakly-inhomogeneous plasma immersed in a nonuniform magnetic field for a case of a wave packet propagating in an arbitrary three-dimensional direction. The standard technique for linearizing a kinetic equation is not used here.

1. BASIC ASSUMPTIONS AND INITIAL EQUATIONS

We assume that the electromagnetic fields **E** and **B** may be divided into two parts: slowly varying fields $\mathbf{E}_0(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ and high-frequency fields $\mathbf{E}_\sim$ and $\mathbf{B}_\sim$ which may be represented in the form

$$\mathbf{E}_\sim = \sum_{1\leqslant s\leqslant M} \mathbf{E}_s \exp i\theta_s + \text{c.c.}, \quad \mathbf{B}_\sim = \sum_{1\leqslant s\leqslant M} \mathbf{B}_s \exp i\theta_s + \text{c.c.} \tag{1}$$

Here $\mathbf{E}_s(\mathbf{r},t)$ and $\mathbf{B}_s(\mathbf{r},t)$ are slowly varying complex amplitudes and θ_s is the fast phase (eikonal) of the s-th wave packet. If $|\mathbf{B}_\sim| < |\mathbf{B}_0|$, the unit vectors of the drift theory[7,9] may be introduced:

$$\mathbf{e}_1(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r},t)/B_0(\mathbf{r},t), \ \mathbf{e}_2(\mathbf{r},t), \ \mathbf{e}_3(\mathbf{r},t). \tag{2}$$

Then the motion equations of the relativistic particle may be written in cylindrical coordinates in momentum space in the form

$$\frac{d\mathbf{r}}{dt}=v_\parallel \mathbf{e}_1+\frac{v_\perp}{2}(\mathbf{e}_-e^{i\theta_0}+\mathbf{e}_+e^{-i\theta_0})=\mathbf{v},$$

$$\frac{dp_\parallel}{dt}=a_0+\Big\{a_1e^{i\theta_0}+a_2e^{2i\theta_0}+\sum_{1\le s\le M}(a_{3s}e^{i\theta_s}+a_{4s}e^{i(\theta_s+\theta_0)}+a_{5s}e^{i(\theta_s-\theta_0)})+\text{c.c.}\Big\}\equiv a_\parallel, \tag{3}$$

$$\frac{dp_\perp}{dt}=a_\perp,\quad \frac{d\theta_0}{dt}=\omega_0+A_0.$$

The expressions for $a_\perp$ and A_0 are similar to those for $a_\parallel$ with the coefficients a_i replaced by b_i and c_i, respectively. These coefficients are

$$a_0=eE_{0\parallel}+\frac{v_\perp p_\perp}{2}\nabla\mathbf{e}_1,\quad a_1=\frac{p_\perp}{2}\mathbf{e}_-\mathbf{e}_1',\quad a_2=\frac{v_\perp p_\perp}{4}(\mathbf{e}_-\mathbf{e}_-:\nabla\mathbf{e}_1),$$
$$a_{3s}=e\mathbf{E}_s\mathbf{e}_1,\quad a_{4s}=\frac{ip_\perp e\mathbf{B}_s\mathbf{e}_-}{2mc},\quad a_{5s}=-\frac{ip_\perp e\mathbf{B}_s\mathbf{e}_+}{2mc},$$
$$b_0=-\frac{p_\perp v_\parallel}{2}\nabla\mathbf{e}_1,\quad b_1=\frac{\mathbf{e}_-}{2}(e\mathbf{E}_0-p_\parallel\mathbf{e}_1'),\quad b_2=-\frac{p_\parallel}{p_\perp}a_2, \tag{4}$$
$$b_{3s}=0,\quad b_{4s}=\frac{e}{2}\mathbf{e}_-\left(\mathbf{E}_s-i\frac{v_\parallel}{c}\mathbf{B}_s\right),\quad b_{5s}=\frac{e}{2}\mathbf{e}_+\left(\mathbf{E}_s+i\frac{v_\parallel}{c}\mathbf{B}_s\right),$$
$$c_0=-\frac{v_\parallel}{2}\mathbf{e}_1\operatorname{rot}\mathbf{e}_1+\frac{1}{2i}\mathbf{e}_-\mathbf{e}_+',\quad c_1=\frac{v_\perp}{4i}(\mathbf{e}_-\mathbf{e}_-:\nabla\mathbf{e}_+)-\frac{b_1}{ip_\perp},$$
$$c_2=-\frac{b_2}{ip_\perp},\quad c_{3s}=-\frac{e\mathbf{B}_s\mathbf{e}_1}{mc},\quad c_{4s}=-\frac{b_{4s}}{ip_\perp},\quad c_{5s}=\frac{b_{5s}}{ip_\perp},$$

where the colons denote the double scalar products of the dyads (tensors).

Here θ_0 is the phase of the particle cyclotron rotation in the field $\mathbf{B}_0$; $\omega_0=-e\mathbf{B}_0/m_0c\gamma\equiv\Omega_0/\gamma$ is the relativistic cyclotron frequency:

$$v_{\parallel,\perp}=\frac{p_{\parallel,\perp}}{m_0\gamma};\quad \gamma=\left(1+\frac{p_\parallel^2+p_\perp^2}{m_0^2c^2}\right)^{1/2};$$

and m_0 is the rest mass of the particle:

$$\mathbf{e}_\pm=\mathbf{e}_2\pm i\mathbf{e}_3;\quad (\dots)'=\left(\frac{\partial}{\partial t}+v_\parallel\mathbf{e}_1\nabla\right)(\dots).$$

The fast phases of the wave pockets (1) are described by the equations

$$\frac{d\theta_s}{dt}=-\omega_s+\mathbf{k}_s\frac{d\mathbf{r}}{dt}=\nu_s+\frac{v_\perp}{2}\mathbf{k}_s(\mathbf{e}_-e^{i\theta_0}+\mathbf{e}_+e^{-i\theta_0})\equiv\nu_s+A_s, \tag{5}$$

where

$$\omega_s(\mathbf{r},t)=-\partial\theta_s/\partial t,\quad k_s(\mathbf{r},t)=\nabla\theta_s \tag{5a}$$

are the local frequency and wave vector of the s-th packet, and

$$\nu_s=-\omega_s+v_\parallel k_{s\parallel},\quad k_{s\parallel}=\mathbf{e}_1\mathbf{k}_s. \tag{5b}$$

Equations (3) and (5) constitute what is known as a multiperiodic system[9] of type

$$dx/dt=a(t,x,\theta),\quad d\theta/dt=\omega(t,x)+A(t,x,\theta). \tag{6}$$

In our problem,

$$x=(\mathbf{r},p_\parallel,p_\perp),\quad a=(\mathbf{v},a_\parallel,a_\perp),\quad \omega=(\omega_0,\nu_1,\dots,\nu_M),$$
$$A=(A_0,A_1,\dots,A_M),\quad \theta=(\theta_0,\theta_1,\dots,\theta_M).$$

If L is the characteristic linear scale of the field inhomogeneities and of the parameters of the plasma and if T is the corresponding time scale, then in the case of a strong magnetic field $\mathbf{B}_0$ the small parameter of the drift theory

$$\varepsilon=r_c/L \tag{7}$$

may be introduced, where r_c is the cyclotron radius of the particles. If the phases θ vary rapidly, this will mean that the corresponding frequencies ω are high, i.e.,

$$1/|\omega|T\sim\varepsilon. \tag{8}$$

Condition (8) applied to the wave packet (1), (5) breaks up into two conditions:

$$1/\omega_sT\sim\varepsilon,\quad 1/k_{s\parallel}v_\parallel T\sim 1/k_{s\parallel}L\sim\varepsilon, \tag{9}$$

which constitute the approximation conditions of geometric optics for packets propagating along a magnetic field.

Equations (5) are valid only if the wave propagates quasilongitudinally, in which case

$$k_\perp r_c\sim\varepsilon. \tag{10}$$

This corresponds to the case of a small cyclotron radius. If the cyclotron radius is finite, i.e.,

$$k_\perp r_c\sim 1, \tag{11}$$

it is best to switch from the phases θ_s to new phases ψ_s by means of the formula

$$\psi_s=\theta_s-\frac{v_\perp}{\omega_0}\mathbf{k}_s(\mathbf{e}_2\sin\theta_0-\mathbf{e}_3\cos\theta_0). \tag{12}$$

Then the "high-frequency" parts of the kinetic equations (3) of the particle become more complicated, for example,

$$(a_\parallel)_{\text{HF}}=\sum_{1\le s\le M}\sum_{-\infty\le n\le\infty}a_{3s}^{\,n}e^{i(\psi_s+n\psi_0)}+\text{c.c.}, \tag{13}$$

where $\psi_0=\theta_0$,

$$a_{3s}^{\,n}=e^{-in\varphi_s}\{a_{3s}J_n+a_{4s}e^{i\varphi_s}J_{n-1}+a_{5s}e^{-i\varphi_s}J_{n+1}\},\quad \varphi_s=\operatorname{arctg}(\mathbf{k}_s\mathbf{e}_3/\mathbf{k}_s\mathbf{e}_2), \tag{13a}$$

$J_n\equiv J_n(\rho_s)$ are Bessel functions, and

$$\rho_s=\frac{v_\perp}{\omega_0}[(\mathbf{k}_s\mathbf{e}_2)^2+(\mathbf{k}_s\mathbf{e}_3)^2]^{1/2}.$$

The high-frequency parts of the expressions $a_\perp$ and A_0 are analogous to (13) with the coefficient a replaced by b and c, respectively.

The new phases ψ_s are described by the equations

$$\frac{d\psi_s}{dt}=\nu_s+c_{0s}+\Big\{c_{1s}e^{i\psi_0}+c_{2s}e^{2i\psi_0}+\sum_{1\le s_1\le M}\sum_{-\infty\le n\le\infty}c_{3ss_1}^{\,n}\exp\{i(\psi_{s_1}+n\psi_0)\}+\text{c.c.}\Big\}, \tag{14}$$

where

$$c_{0s}=-\frac{p_\perp v_\perp}{4im_0}\mathbf{e}_+\nabla\frac{(\mathbf{k}_s\mathbf{e}_-)}{\Omega_0}+\frac{b_1-ip_\perp c_1}{2im_0\Omega_0}\mathbf{k}_s\mathbf{e}_++\text{c.c.},$$

$$c_{1s}=-\frac{p_\perp}{2im_0}\left(\frac{\mathbf{k}_s\mathbf{e}_-}{\Omega_0}\right)'-\frac{\mathbf{k}_s}{2im_0\Omega_0}\{(b_0+ip_\perp c_0)\mathbf{e}_--(b_2-ip_\perp c_2)\mathbf{e}_+\},$$

$$c_{2s}=-\frac{p_\perp v_\perp}{4im_0}\mathbf{e}_-\nabla\frac{(\mathbf{k}_s\mathbf{e}_-)}{\Omega_0}-\frac{\mathbf{k}_s\mathbf{e}_-}{2im_0\Omega_0}(b_1+ip_\perp c_1),$$

$$c^{n}_{3ss_1}=-\frac{\mathbf{k}_s}{2im_0\Omega_0}\{\mathbf{e}_-(b^{n-1}_{3s_1}+ip_\perp c^{n-1}_{3s_1})-\mathbf{e}_+(b^{n+1}_{3s_1}-ip_\perp c^{n+1}_{3s_1})\}.$$

The Vlasov kinetic equation for a relativistic plasma is best used in the form[7]

$$-\left(\omega,\frac{\partial f}{\partial\psi}\right)=\varepsilon\left\{\frac{\partial f}{\partial t}+\left(a,\frac{\partial f}{\partial x}\right)+\left(A,\frac{\partial f}{\partial\psi}\right)\right\}. \tag{15}$$

Here the equations of the characteristics coincide with the kinetic equations (6) of the particle subject to the substitution (12); the symbol (a,b) denotes the scalar product of the vectors a and b; the small parameter ε is introduced explicitly in accordance with the procedure for ordering physical quantities.

If there are resonance relations between the frequencies,

$$\left|\sum_{0\leqslant i\leqslant M}N_i\omega_i\right|\ll\min\omega, \tag{16}$$

where N_i are certain integers, the corresponding resonant phase combination

$$\psi_r=(N,\psi)\equiv N_0\psi_0+N_1\psi_1+\ldots+N_M\psi_M \tag{16a}$$

should pertain to the number of additional slow variables of the distribution function. It is then necessary to write in the braces of (15) the additional term $(A_r,\partial f/\partial\psi_r)$, whose explicit form depends on the particular resonance.

2. DRIFT-KINETIC EQUATION FOR RESONANT PARTICLES OF A COLLISIONLESS PLASMA

According to the Bogolyubov-Mitropol'skiĭ technique,[6,7] the smoothened quantities are defined by the formula

$$\bar f=(2\pi)^{-(M+1)}\int_0^{2\pi}\ldots\int d\psi_0\,d\psi_1\ldots d\psi_M f, \tag{17}$$

so that $\bar f$ depends on the slow variables (including the resonance phase difference). In the general case,

$$f=\bar f+f_\sim, \tag{18}$$

where the variable part $f_\sim$ depends on the fast phases ψ.

Expanding the distribution function in powers of the parameters ε,

$$f=f_0+\varepsilon f_1+\varepsilon^2f_2+\ldots, \tag{19}$$

we easily see from (15) that f_0 is independent of the fast phases:

$$f_0=f_0(t,\mathbf{r},p_\parallel,p_\perp,\psi_{lN}). \tag{20}$$

Here

$$\psi_{lN}=\psi_l+N\psi_0 \tag{21}$$

is the resonant phase difference of the zeroth-order approximation and corresponds to the resonance relations between the frequencies,

$$|\nu_l+N\omega_0|\ll\min|\nu_l|,\ \omega_0. \tag{21a}$$

The subscript "l" denotes one of the M resonating wave packets, and N is an arbitrary integer.

In the first approximation in the parameter ε, equation (15) has the form

$$-\left(\omega,\frac{\partial f_1}{\partial\psi}\right)=\frac{\partial f_0}{\partial t}+\left(a,\frac{\partial f_0}{\partial x}\right)+\left(A_{lN},\frac{\partial f_0}{\partial\psi_{lN}}\right). \tag{22}$$

The drift-kinetic equation for the distribution function f_0 follows hence after averaging over all fast phases in accordance with (17):

$$\frac{\partial f_0}{\partial t}+\left(\bar a,\frac{\partial f_0}{\partial x}\right)+\left(\bar A_{lN},\frac{\partial f_0}{\partial\psi_{lN}}\right)=0. \tag{23}$$

The characteristics of this equation are determined by the smoothened kinetic equations of the particle in the zeroth-order approximation:

$$\frac{d\mathbf{r}}{dt}=v_\parallel\mathbf{e}_1,\qquad\frac{dp_\parallel}{dt}=a_0+(a_{3l}{}^N\exp i\psi_{lN}+\text{c.c.})\equiv\bar a_\parallel,$$

$$\frac{dp_\perp}{dt}=b_0+(b_{3l}{}^N\exp i\psi_{lN}+\text{c.c.})\equiv\bar a_\perp,$$

$$\frac{d\psi_{lN}}{dt}=\nu_l+N\omega_0+c_{0l}+Nc_0+\{(c^{N}_{3ll}+Nc_{3l}{}^N)\exp i\psi_{lN}+\text{c.c.}\}\equiv\bar A_{lN}. \tag{24}$$

Equations (24) have a common formal integral

$$I=I_0+2\,\mathrm{Re}\,P_0{}^*G+|G|^2, \tag{25}$$

where

$$I=\left|P\exp\int_{t_0}^{t}dt'\,A(t')\right|^2,\qquad I_0=|P|_{t=t_0}|^2,$$

$$P=\varphi(p_\perp)\exp i\psi_{lN},\ A(t)=-b_0\frac{d\ln\varphi}{dp_\perp}-i(\nu_l+N\omega_0+c_{0l}+Nc_0),$$

$$G(t)=\int^{t}dt'Q(t')\exp\int^{t'}dt''A(t''),$$

$$Q(t)=\frac{d\varphi}{dp_\perp}b_{3l}{}^{N*}+i\varphi(c^{N}_{3ll}+Nc_{3l}{}^N)^*.$$

Here the function $\varphi(p_\perp)$ satisfies the equation

$$\frac{d\varphi}{dp_\perp}b_{3l}{}^N+i\varphi(c^{N}_{3ll}+Nc_{3l}{}^N)=0. \tag{25a}$$

In the case of a small cyclotron radius, the function $\varphi\to p_\perp$, in which case $Q\to1$. The quantity $I=p_\perp{}^2/B_0$ constitutes then a "transverse invariant" which is not preserved in the case of a high-frequency resonant field. Equation (25) is a generalization of the well-known result of Canobbio,[10] which is valid in the case of a cyclotron wave propagating in the longitudinal direction, and may be useful in problems of cyclotron heating of a plasma in toroidal magnetic traps. If the resonance relations (21a) do not hold, i.e., if the phases ψ_{lN} are fast, averaging with respect to them is required. The drift-kinetic equation (23) is then entirely independent of the high-frequency field, in agreement with previous results.[7]

From Eq. (22) it follows that the variable part of the distribution function of the first-order approximation satisfies the equation

$$-\left(\omega,\frac{\partial f_{1\sim}}{\partial\psi}\right)=\left(a_{\sim},\frac{\partial f_0}{\partial x}\right)+\left(A_{lN\sim},\frac{\partial f_0}{\partial\psi_{lN}}\right),\qquad(26)$$

where $a_{\sim}$ and A_{lN} are periodic (nonresonant) parts of the functions a and A_{lN}. Hence, it is easily found that

$$f_{1\sim}=-\left(\check{a},\frac{\partial f_0}{\partial x}\right)-\left(\check{A}_{lN},\frac{\partial f_0}{\partial\psi_{lN}}\right).\qquad(27)$$

Here the quantities $\hat{a}$ are defined by formulas of the form

$$\check{a}_{\parallel}\equiv\frac{a_1e^{i\psi_0}}{i\omega_0}+\frac{a_2e^{2i\psi_0}}{2i\omega_0}+\sum{}'\frac{a_{s_s}{}^{n}e^{i(\psi_s+n\psi_0)}}{i(\nu_s+n\omega_0)}+\text{c.c.}\qquad(27a)$$

The prime at the summation sign denotes that terms with resonant combinations of phases (and frequencies must be omitted, and that summation must be over all $1<s<M$ and $-\infty<n\leqslant\infty$.

The equation for the smoothened part of the distribution function follows from the second approximation of Eq. (15):

$$-\left(\omega,\frac{\partial f_2}{\partial\psi}\right)=\frac{\partial f_1}{\partial t}+\left(a,\frac{\partial f}{\partial x}\right)_1+\left(A,\frac{\partial f}{\partial\psi}\right)_1+\left(A_r,\frac{\partial f}{\partial\psi_r}\right),$$

where the resonances ψ_r are, in general, different from the resonances ψ_{lN}. Averaging over the fast phases, we thus obtain the drift-kinetic equation of the first approximation:

$$\frac{\partial\bar{f}_1}{\partial t}+\left(\bar{a},\frac{\partial\bar{f}_1}{\partial x}\right)+\left(\bar{A}_r,\frac{\partial\bar{f}_1}{\partial\psi_r}\right)+\overline{\left(a_{\sim},\frac{\partial f_{1\sim}}{\partial x}\right)}+\overline{\left(A_{\sim},\frac{\partial f_{1\sim}}{\partial\psi}\right)}+\overline{\left(A_{r\sim},\frac{\partial f_{1\sim}}{\partial\psi_r}\right)}=0.\qquad(29)$$

Its explicit form depends on the particular resonance (or resonances) under consideration.

Terms occur in this equation that are independent of the high-frequency fields, along with terms that are determined entirely by the high-frequency fields; the high-frequency parts contain both nonresonant and resonant terms. If the waves propagate in the longitudinal direction, the nonresonant terms in (29) vanish, so that if there are no resonances the high-frequency field does not occur in the drift-kinetic equation of the first-order approximation. This is in agreement with previous conclusions[7] that the averaged action of a nonresonant high-frequency field on a plasma is a second-order effect. At first glance it would seem that, according to (27), Eq. (29) contains the second derivatives of the distribution function. However, rather cumbersome computations demonstrate that terms with second derivatives drop out, and the characteristics of the equation coincide with the corresponding averaged kinetic equations of the particle.

The resonances (21) are the strongest resonances, since they appear even in the zeroth-order approximation. If these resonances do not occur in the zeroth-order approximation, according to (27) they will not occur in any of the succeeding approximations. Other, weaker resonances may then appear. It follows from (27) and (29) that resonances with the phase combinations

$$\psi_{lsN}^{\pm}=\psi_l\pm\psi_s+N\psi_0\qquad(30)$$

may occur in the first-order approximation. Here the subscripts "l" and "s" denote the two resonating wave packets and N is an integer. *If* $l=s$, and if N is odd and if there are only plus signs in the equation, the resonant phase combination in (30) will correspond to the previously considered semi-cyclotron resonances.[5] If $l\neq s$, then if there is a minus sign in (30) the phases of (30) will correspond to cyclotron resonances at the beats of the two waves. Both types of resonance are quadratic in the wave amplitude.

By means of eq. (28) it is possible to compute the function $f_{2\sim}$, which has a very cumbersome form. In the third-order approximation, it is then possible to find from Eq. (15) an equation for $\bar{f}_2$, etc. Complex resonances with three or more wave packets then arise.

3. CHERENKOV RESONANCE AT BEATS OF TWO WAVES

Let us consider the Cherenkov resonance ($N=0$) at the beats of a longitudinal and a transverse wave as a simple example of the resonances of the first-order approximation (30). We will assume that the transverse wave with phase ψ_1 is a whistler propagating at some angle to a steady magnetic field $\mathbf{B}_0$, while the longitudinal wave with phase ψ_2, which travels along the magnetic field, is a Langmuir wave. Then the equations for the characteristics of the drift-kinetic equation (29) (averaged kinetic equations of the resonant particle) have the form (at $\mathbf{E}_0=0$)

$$\frac{dz}{dt}=v_{\parallel},\quad\frac{dp_{\parallel}}{dt}=\frac{2\Delta kJ_0(\rho)(1-v_{\parallel}{}^2/c^2)}{m_0\gamma\omega_1\omega_2}e^2E_1E_2\sin\varphi,\qquad(31)$$

$$\frac{dp_{\perp}}{dt}\approx0,\quad\frac{d\psi}{dt}=\nu_1-\nu_2\equiv-\Delta\omega+v_{\parallel}\Delta k.$$

Here

$$\Delta\omega=\omega_1-\omega_2,\quad\Delta k=k_1-k_2,\quad\psi=\psi_1-\psi_2,$$

$$\rho=\frac{k_{\perp1}p_{\perp}}{m_0\Omega_0},\quad E_{\parallel1,2}=E_{1,2}e^{i\varphi_{1,2}},\quad\varphi=\psi+\varphi_1-\varphi_2.$$

It is assumed that the velocity of the particle is much less than the phase velocities of the waves under consideration, so that the relativistic effects in equations (31) are negligible. In the classical limit, an approximate "energy" integral in the space of the dephasing of φ may be obtained from a uniform plasma from the sytem of equations (31). This system is analogous to the corresponding integral under the conditions of Cherenkov resonance with a separate Langmur wave[2,11]:

$$\tfrac{1}{2}\dot{\varphi}^2+U(\varphi)=\text{const}. \tag{32}$$

Here the "potential energy" is determined by the formula

$$U(\varphi)=2\left(\frac{e\Delta k}{m}\right)^2\frac{J_0(\rho)E_1E_2}{\omega_1\omega_2}\cos\varphi\equiv\Omega^2\cos\varphi, \tag{33}$$

where the potential barrier is quadratic in the wave amplitude; by contrast, in the case of Cherenkov and cyclotron resonances with a separate wave the potential barrier is linear in wave amplitude.

The quantity

$$\tau=\frac{1}{\Omega}=\frac{m}{e|\Delta k|}\left(\frac{\omega_1\omega_2}{2J_0E_1E_2}\right)^{1/2} \tag{34}$$

characterizes the time in which the resonant particles interact nonlinearly with beats of two wave packets.

The motion of a charged particle may be computed by means of (32) and (33) using the well-known technique for the treatment of a nonlinear Cherenkov resonance in the field of a Langmuir wave.[2,11]

4. PERMITTIVITY TENSOR FOR A COLLISIONLESS RELATIVISTIC NON-UNIFORM PLASMA IMMERSED IN A NON-UNIFORM MAGNETIC FIELD

Usually the permittivity tensor is computed by assuming that the given wave propagates without leaving some plane that passes through the force line of the magneitc field.[8,12] This assumption in general is not justified for a non-uniform plasma and a non-uniform magnetic field.

Below we give a general derivation of the permittivity tensor without making use of the standard procedure for linearizing the Vlasov kinetic equation. Bearing in mind the ordering (9), it follows from Maxwell's equations that the amplitudes of the wave packets (1) in the lowest approximation are related as

$$\mathbf{B}_s=\frac{c}{\omega_s}[\mathbf{k}_s\times\mathbf{E}_s], \tag{35}$$

$$-\frac{c}{\omega_s}[\mathbf{k}_s\times\mathbf{B}_s]=\mathbf{E}_s+\frac{4\pi i}{\omega_s}\int_0^{2\pi}\frac{d\theta_s}{2\pi}e^{-i\theta_s}\mathbf{j}_s\equiv(\hat{I}+\hat{\chi}_s)\mathbf{E}_s\equiv\hat{\varepsilon}_s\mathbf{E}_s.$$

Here $\hat{\varepsilon}_s=\hat{I}+\hat{\chi}_s$ is the permittivity tensor, $\hat{I}$ a unit tensor, and $\mathbf{j}_s$ the density of the current induced by the wave

$$\mathbf{j}_s=\sum_a e_a\int d\mathbf{p}\,(\mathbf{v}f_{a1})_\sim. \tag{36}$$

The subscript "a" indicates the type of particle, and $(\ldots)_\sim$ denotes the alternating part of the expression. The alternating part of the distribution function f_l is determinee by (27). When integrating with respect to $\psi_0\equiv\theta_0$, the inverse transformation $\psi_s\to\theta_s$ must be performed in the high-frequency terms of the expression for $f_{l\sim}$ in accordance with (12), since the high-frequency fields (1) are determined by the phases θ_s, and not by the phases ψ_s. Thus we obtain

$$\hat{\chi}_s=\sum_a\sum_{-\infty\leqslant n\leqslant\infty}\int d\mathbf{p}\,\mathbf{G}_{an}\mathbf{T}_{an}. \tag{37}$$

Here

$$\begin{aligned}
\mathbf{T}_{an}&=\mathbf{e}_1T_{n\parallel}^{a}+\tfrac{1}{2}(\mathbf{e}_+T_{n-}^{a}+\mathbf{e}_-T_{n+}^{a}),\\
T_{n\parallel}^{a}&=J_n\left[\left(1-\frac{n\omega_0}{\omega_s}\right)\frac{\partial F_a}{\partial p_\parallel}+\frac{n\omega_0}{\omega_s}\frac{p_\parallel}{p_\perp}\frac{\partial F_a}{\partial p_\perp}\right],\\
T_{n\pm}^{a}&=\frac{J_{n\mp1}}{\omega_s}e^{\pm i\varphi_s}\left(k_{s\parallel}v_\perp\frac{\partial F_a}{\partial p_\parallel}-\nu_s\frac{\partial F_a}{\partial p_\perp}\right),
\end{aligned} \tag{38}$$

$$\begin{aligned}
\mathbf{G}_{an}&=\mathbf{e}_1G_{n\parallel}^{a}+\frac{1}{2}(\mathbf{e}_+G_{n-}^{a}+\mathbf{e}_-G_{n+}^{a})\\
&\equiv-\frac{\omega_{pa}^2}{\gamma\omega_s(\nu_s+n\omega_0)}\{\mathbf{e}_1p_\parallel J_n+p_\perp(\mathbf{e}_+J_{n-1}e^{-i\varphi_s}+\mathbf{e}_-J_{n+1}e^{i\varphi_s})\}
\end{aligned} \tag{39}$$

$$f_{a0}=n_aF_a,\quad \omega_{pa}^2=4\pi e_a^2n_a/m_{0a},\quad J_n=J_n(\rho_s).$$

In particular cases, Eqs. (37)–(39) turn into well-known expressions.[12–14]

[1] A. A. Ivanov, in: Voprosy teoril plazmy (Problems of Plasma Theory), Vol. 6, Atomizdat, Moscow, 1972, p. 139.
[2] A. A. Galeev and R. Z. Sagdeev, *ibid.*, Vol. 7, Atomizdat, Moscow, 1973, p. 3.
[3] N. I. Budko, V. I. Karpman, and O. A. Pokhotelov, Cosmic Electrodynamics, **3**, 165, 1972.
[4] A. B. Kitsenko, I. M. Pankratov, and K. N. Stepanov, Zh. Eksp. Teor. Fiz. **66**, 166, 1974 [Sov. Physics. JETP, **39**, 77 (1974)].
[5] A. B. Kitsenko, I. M. Pankratov, and K. N. Stepanov, Zh. Eksp. Tekh. Fiz. **67**, 1728, 1974 [Sov. Physics. JETP, **40**, 860 (1975)].
[6] N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, Asimptoticheskie metody v teorii nelineĭnykh kolebaniĭ (Asymptotic Methods in the Theory of Nonlinear Oscillations), Nauka, Moscow, 1974, p. 202.
[7] V. P. Milant'ev, Zh. Eksp. Teor. Fiz., **72**, 159, 1977 [Sov. Physics. JETP, **45**, 84 (1977)].
[8] A. B. Mikhaĭlovskiĭ, Teoriya plazmennykh neustoĭchivosteĭ (Theory of Plasma Instabilities), Vol. 1, Atomizdat, Moscow, 1975, 77.
[9] A. I. Morozov and L. S. Solev'ev, In: Voprosy teorii plazmy (Problems of Plasma Theory), Vol. 2, Gosatomizdat, Moscow, 1963, 194.
[10] E. Canobbio, Nucl. Fusion, **9**, 27, 1969.
[11] G. Pocobelli, Phys. Fluids, **24**, 2173, 1981.
[12] B. A. Trubnikov, In: Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (Plasma Physics and the Problem of Controllable Thermonuclear Reactions), Vol. 3, Izd, AN SSSR, 1958, 104.
[13] I. Fidone, G. Granata, R. L. Meyer, E. H. Jornada, R. S. Schneider, and L. F. Zeibell, Phys. Fluids, **23**, 1980, 1336.
[14] H. Weitzner and D. B. Batchelor, Phys. Fluids, **23**, 1980, 1359.

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