Numerical modeling of wave propagation in periodic media

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We consider one-dimensional problems in which a plane wave is incident on a layer of a periodically inhomogeneous medium and in which there is emission from a source within the medium. Different boundary conditions are imposed at the boundaries of the layer. Results of an approximate analysis (averaging with respect to fast oscillations, perturbation theory) are compared with exact results obtained by numerical integration of equations of the imbedding method.

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1. Problems of waves in periodic media have traditionally drawn the attention of physicists in view of their importance in virtually every field of physics. A survey of the contemporary status of the theory can be found in Refs. 1 and 2. Researchers have usually limited themselves to analyzing the dispersion relations (existence of transparency and opacity zones), i.e., relationships between the frequency of the monochromatic wave and its wave number that allow for the possibility of wave propagation. In a numer of problems in physics, however, it is of interest to study the propagation of a particular wave (with designated frequency and wave number) in a periodically inhomogeneous medium. An example of such a problem is the propagation of radio waves in the ionosphere, the inhomogeneities in which are created by a powerful pumping wave.[3,4] A variety of approximation methods are utilized in analyzing such problems, chief among which are peturbation theory and the averaging method (passing to truncated equations). The method of truncated equations is not an asymptotic method here, and its principal advantages lies in its simplicity and the physical lucidity of the results. It would be of definite interest to compare the results of approximation methods and the exact solutions. This is done in the present article using as an example the simplest one-dimensional problem of waves in a periodically inhomogeneous medium.

2. Statement of problem. Suppose that a layer of an inhomogeneous medium occupies part of the space $0 \leqslant x \leqslant L$ and that a plane wave of unit amplitude $\exp\{-ik(x-L)\}$ is incident from the right on the layer. Then the wave field within the layer is described by the Helmholtz equation[5]

$$d^2U(x)/dx^2+k^2[1+\varepsilon(x)]U(x)=0. \tag{1}$$

with boundary conditions

$$ik^{-1}\frac{dU}{dx}\Big|_L+U(L)=2,\quad ik^{-1}\frac{dU}{dx}\Big|_0-U(0)=0, \tag{2}$$

where the function $\varepsilon(x)$ describes the deviation of the permittivity from unity. It is supposed that $\varepsilon = 0$ outside the layer. We assume that

$$\varepsilon(x)=-4\mu\cos 2Kx+2i\gamma, \tag{3}$$

where 2γ is the extinction coefficient.

The complex coefficient of reflection of the wave from the layer and the complex wave transmission coefficient are determined by solving the boundary-value problem (1) and (2) using the equalities.

$$R_L=U(L)-1,\quad T_L=U(0). \tag{4}$$

If there is no extinction (i.e., if $\gamma = 0$), conservation of the energy flux density occurs, and consequently,

$$|R_L|^2+|T_L|^2=1. \tag{5}$$

Substituting (3) in (1) and passing to dimensionless distances (i.e., setting $k = 1$), we can rewrite (1) and (2) in the form

$$\frac{d^2U(x)}{dx^2}+[1-4\mu\cos 2(1+\Delta)x+2i\gamma]U(x)=0,\quad \Delta=\frac{K-k}{k},$$
$$i\frac{dU}{dx}\Big|_L+U(L)=2,\quad i\frac{dU}{dx}\Big|_0-U(0)=0. \tag{6}$$

Equation (6) is a Mathieu equation and has been extensively studied.[6] If $\gamma = 0$, there exist regions (in the μ,Δ plane) that correspond to parametric instability (parametric resonance), six of which are depicted in Fig. 1 (shaded area). In our boundary-value problem, these regions correspond to increased reflectivity of the layer. Outside these regions, the wave should pass through the layer of the medium relatively freely.

The solution of the boundary value problem (6) can be written in a form that contains Mathieu functions and their derivatives. However, even though these functions have been extensively studied and tabulated in detail, because of the wave's high degree of variability it is far from a simple matter to describe the behavior of the wave field (and, consequently, the wave reflection and transmission coefficients) within the layer of the medium. It is simpler to solve the boundary-value problem (6) directly by numerical methods. It is then convenient to use the imbedding method.[5]

Let us consider the wave field $U(x)$ as a function of the parameter L, the position of the boundary on which the wave falls, i.e., $U(x)\equiv U(x;L)$. We can then rewrite the boundary-value problem (6) as an initial-value problem

$$\frac{\partial U(x;L)}{\partial L}=i\left\{1+\frac{1}{2}\tilde{\varepsilon}(L)U_L\right\}U(x;L),\quad U(x;L)|_{L=x}=U_x, \tag{7}$$

$$\frac{dU_L}{dL}=2i(U_L-1)+\frac{i}{2}\tilde{\varepsilon}(L)U_L^2,\quad U_0=1, \tag{8}$$

$$\tilde{\varepsilon}(L)=-4\mu\cos 2(1+\Delta)L+2i\gamma,$$

where

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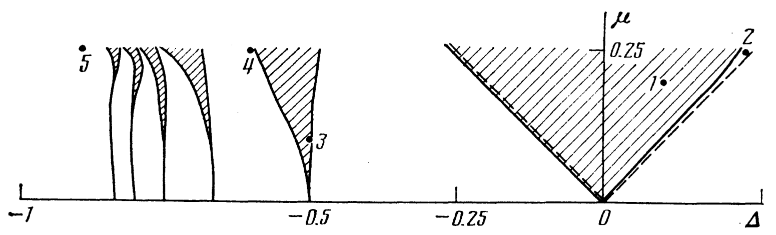

FIG. 1. Zones of parametric instability in the $\mu\Delta$ = plane for $\gamma = 0$.

and $U_L = U(L;L)$ is the field at the boundary of the layer. Equation (7) can then be integrated and consequently

$$U(x;L)=U_x\exp\left\{i(L-x)+\frac{i}{2}\int_x^L d\xi\,\varepsilon(\xi)\,U_\xi\right\}. \qquad (9)$$

Thus, the boundary-value problem (6) can be solved by solving the Riccati equation (8) and computing the quadrature (9). The intensity of the wave field within the medium $I(x;L) = |U(x;L)|^2$ is given by the expression

$$I(x;L)=|U_x|^2\exp\left\{\frac{i}{2}\int_x^L d\xi\,\varepsilon(\xi)\,(R_\xi-R_\xi^*)\right\}, \qquad (10)$$

which, in view of (8), can be rewritten in the form

$$I(x;L)=|1+R_x|^2\frac{1-|R_L|^2}{1-|R_x|^2}\exp\left\{-2\gamma\int_x^L d\xi\frac{|1+R_\xi|^2}{1-|R_\xi|^2}\right\}. \qquad (10')$$

Setting $x = 0$ in (10'), we obtain a relation between the wave reflection and transmission coefficients in the form

$$|T_L|^2=(1-|R_L|^2)\exp\left\{-2\gamma\int_0^L d\xi\frac{|1+R_\xi|^2}{1-|R_\xi|^2}\right\}. \qquad (11)$$

At $\gamma = 0$, (11) turns into (5), in which case[5]

$$I(x;L)=|1+R_x|^2\frac{1-|R_L|^2}{1-|R_x|^2}. \qquad (12)$$

Note that if we know how the square of the modulus of the reflection coefficient depends on the thickness of the layer and how the intensity is distributed within the layer of the medium at a fixed position of the boundary (for example, L_1), then, by (12), the distribution of the intensity within the layer for any other position of the boundary is described by the formula

$$I(x;\ L)=I(x;\ L_1)\,(1-|R_L|^2)/(1-|R_{L_1}|^2). \qquad (12')$$

Thus, to find the distribution of the field strength within the layer of the medium for an arbitrary position of the boundary it is necessary to know how $|R_L|^2$ depends on L and $I(x;L_1)$ for some one value L_1.

3. Let us now discuss approximation methods for solving Eq. (8). We first consider averaging over the fast oscillations. We represent the reflection coefficient R_L in the form

$$R_L=-i\rho_L\exp\{2i(1+\Delta)L\}. \qquad (13)$$

Then for the function ρ_L we have from (8) the equation

$$d\rho_L/dL=-2(\gamma+i\Delta)\rho_L+\mu(1-\rho_L^2)+\{\ldots\}, \qquad (14)$$

where $\{\ldots\}$ denotes terms containing the oscillating functions

$$\exp\{\pm 2i(1+\Delta)L\},\quad \exp\{\pm 4i(1+\Delta)L\}.$$

Assuming that ρ_L changes little during the oscillation period we can average (14) over these "fast" oscillations. As a result we obtain the approximate equation

$$d\rho_L/dL=-2(\gamma+i\Delta)\rho_L+\mu(1-\rho_L^2), \qquad (14')$$

whose solution (for the initial condition $\rho_0 = 0$) has the form

$$\rho_L=\frac{\mu}{\alpha}\frac{\operatorname{sh}\alpha L}{\operatorname{ch}\alpha L+(\gamma+i\Delta)\alpha^{-1}\operatorname{sh}\alpha L},\quad \alpha^2=\mu^2+(\gamma+i\Delta)^2. \qquad (15)$$

Let us consider in more detail the case where there is no extinction ($\gamma = 0$). Then the square of the modulus of the reflection coefficient $|R_L|^2 = |\rho_L|^2$ and, consequently,

$$|R_L|^2=\frac{\operatorname{sh}^2\alpha L}{\operatorname{ch}^2\alpha L-\Delta^2\mu^{-2}},\quad \alpha^2=\mu^2-\Delta^2. \qquad (16)$$

From (12) we have in this case an expression for the field strength inside the medium:

$$I(x;\ L)=[\operatorname{ch}(2\alpha x)-\Delta^2\mu^{-2}]/[\operatorname{ch}^2(\alpha L)-\Delta^2\mu^{-2}]. \qquad (17)$$

Formulas (16) and (17) are well known (they were obtained by a different method directly from (1), e.g., in Ref. 7). At $\mu^2 \gg \Delta^2$, as a consequence of (16) and (17), $|R_L|^2 \to 1$ and the field strength attenuates exponentially with depth. But if $\mu^2 < \Delta^2$, all the functions are periodic in the layer thickness. It is clear from our derivation of (16) and (17) that this conclusion will not hold as $\Delta \to -1$. In addition, we can expect that in the region $\mu \sim |\Delta|$, where one type of solution is replaced by another, (16) and (17) also do not hold, since they were obtained not by an asymptotic analysis, but in fact from physical considerations. The region $\mu \geqslant |\Delta|$, where our theory yields higher reflectivity for the layer of the medium, is hatched in Fig. 1.

There is another approximation method based on perturbation theory and valid for sufficiently small values of μ. Setting $U_L \sim 1$, i.e., making R_L small enough, we obtain from (8) in first-order approximation with respect to μ the equation ($\gamma = 0$)

$$dR_L/dL=2iR_L-2i\mu\cos 2(1+\Delta)L,\qquad R_0=0, \qquad (18)$$

hence it follows that

$$|R_L|^2=\mu^2\Delta^{-2}(2+\Delta)^{-2}\{[\cos 2(1+\Delta)L-\cos 2L]^2 + [(1+\Delta)\sin 2(1+\Delta)L-\sin 2L]^2\}. \qquad (19)$$

Expression (19) is applicable under the condition

$$\mu^2\Delta^{-2}(2+\Delta)^{-2}\ll 1,\ \ \Delta^2\gg 1;\qquad \mu^2\ll 1,\ \ \Delta^2\ll 1, \qquad (20)$$

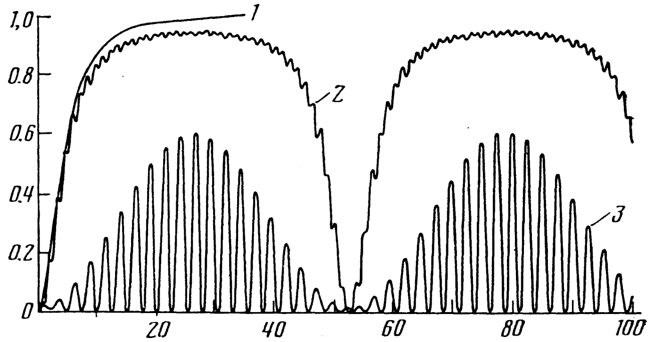

FIG. 2. $|R_L|^2$ expressed as a function of the thickness of the layer. (1) corresponds to equation (16), (2) to the computed curve, (3) to the distribution of $I(x)/40$ in the case $L = 100$ ($\mu = 0.25$, $\Delta = 0.24$, and $\gamma = 0$).

which is obtained by comparing (19) with unity. Comparing the solution of (19) with the succeeding terms of perturbation theory yields the additional constraint

$$\mu^2 \ll L^{-2}. \tag{20'}$$

4. Let us now pass to a numerical analysis of our problem. Equation (8) for the reflection coefficient R_L was solved by the Runge-Kutta method accurate to sixth order.[8] By means of this method it is possible to solve Eq. (8) with sufficient accuracy for $L \sim 100$; at the same time, no excessive computer time is required. The integration step is chosen constant: $h = 0.1$–0.01. The computations demonstrated that results obtained with steps $h = 0.1$ and 0.05 differ for $L \sim 100$ by a fraction of a percent. The wave field inside the layer of the medium was computed by means of (9). Because of the oscillatory integrand, the integral on the right side of (9) was computed by Philo's method.[8] Equation (5), which is valid for $\gamma = 0$, and Eq. (11), which is valid $\gamma \neq 0$, provide an additional test of the precision of the results. For the selected integration step, the law of conservation of the energy flux holds when $\gamma = 0$ (Eq. (5)) to within a single percent or better. The above equations were integrated up to a layer thickness of about 100 under the assumption that for this value of L the periodic behavior of L has been reliably determined in the transparency zones and that in the case of opacity zones it can be assumed that a transition to a half-space occurs. The parameters of the problem were chosen on the basis of Fig. 1.

The numerical analysis showed, first, that the solution of the problem is in fact periodic within the transparency zone if there is no absorption, and that within the opacity zones the reflectivity is increased but is highly variable. In the transparency zones (far from the boundary), the solution is well described by the approximate method of averaging. Within the first opacity zone far from the boundaries, the solution is equally well described by this method. Near the boundaries of the zones, however, the situation becomes complicated. Thus, for the parameters corresponding to point 2 in Fig. 1, the behavior of the reflection coefficent as a function of the thickness of the layer and of the distribution of the field strength in a layer are presented for $L = 100$ in Fig. 2. A sufficiently thin layer (up to $L \sim 10$) behaves as a reflecting layer and is well described by the formula (16) of the averaging method, but with further increase of its thickenss the reflectivity of the layer falls. When $L \approx 53$, the layer becomes absolutely transparent. The picture then repeats itself periodically as long as point 2 in Fig. 1 lies in the transparency zone. Analogous computations are presented in Fig. 3 (point 5 in Fig. 1 will also fall in the opacity zone).

The above computations correspond to the case in which there is no extinction. If extinction does occur, the behavior of the reflection coefficient will be qualitatively the same, whether inside or outside the transparency zones. In the case of a sufficiently thick layer, the modulus of the reflection coefficient becomes a periodic function even in the opacity zone. Figure 3 (curve 3) depicts the behavior of the reflection coefficient corresponding to point 5 in Fi.g 1 at $\gamma = 0.1$.

Note that for small values of μ the behavior of the reflection coefficient is well described by (19) for a thin layer. As the thickness of the layer increases, cumulative effects become significant and the curve begins to differ markedly from (19). Thus, condition (20) is not a sufficient test for the validity of the perturbation theory. The more rigid condition (20′) must be satisfied.

5. We considered above the incidence of a wave on a layer of a periodically inhomogeneous medium. If there is a source within the layer, the problem is described by the

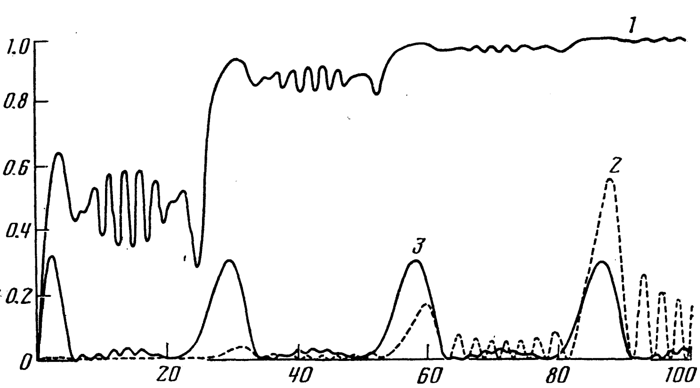

FIG. 3. $|R_L|^2$ expressed as a function of the thickness of the layer for $\mu = 0.25$, $\Delta = -0.89$. (1) corresponds to $\gamma = 0$; (3) $\gamma = 0.1$; (2) distribution of $I(x)/20$ for $L = 100$ and $\gamma = 0$.

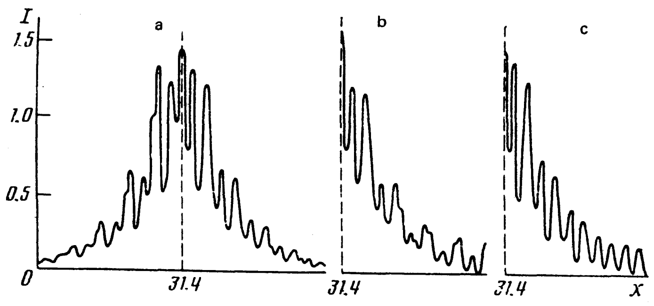

FIG. 4. Distribution of $I(x)$ within the layer as a function of the source with $x_0 = 31.4$) ($L = 62.8, \mu = 0.1, \Delta = -0.5$, and $\gamma = 0.05$) for different conditions on the boundary $x = L$ ($R_0 = 0$): (a) free transmission; (b) $dG/dx|_{x=L} = 0$; (c) $G|_{x=L} = 0$.

equation

$$d^2G(x, x_0)/dx^2+k^2[1+\varepsilon(x)]G(x, x_0)=2ik\delta(x-x_0) \quad (21)$$

with corresponding boundary conditions on the boundaries of the layer. If the solution of the boundary-value problem (21) depends on the parameter L, the problem may be restated, using the imbedding method, as the initial-value problem[9] ($k = 1$)

$$G(x, x_0; L)=G_1(x, x_0; L)+G_2(x, x_0; L), \quad (22)$$

where the function G_1 is described by the system of imbedding equations

$$\frac{\partial G_1(x,x_0;L)}{\partial L}=\frac{i}{2}\varepsilon(L)U(x;L)U(x_0;L),$$

$$G_1|_{L=\max(x,x_0)}=\begin{cases} U(x;x_0), & x_0\geqslant x,\\ U(x_0;x), & x_0\leqslant x,\end{cases} \quad (23)$$

$$\frac{\partial U(x;L)}{\partial L}=i\left\{1+\frac{1}{2}\varepsilon(L)U_L\right\}U(x;L), \quad U(x,x)=U_x,$$

$$\frac{dU_L}{dL}=2i(U_L-1)+\frac{i}{2}\varepsilon(L)U_L^2, \quad U_0=1+R_0,$$

and the quantity R_0 describes the boundary condition at the layer boundary $x = 0$. Thus, if the boundary allows the wave to pass freely through it (i.e., $\varepsilon(x) = 0$ at $x < 0$), we have $R_0 = 0$. A value $R_0 = -1$ corresponds to a reflecting boundary $U(0) = 0$, and $R_0 = 1$ corresponds to the condition $dU(0)/dx = 0$. The function $G_2(x,x_0;L)$ is described by the expression

$$G_2(x, x_0; L)=(g-U_L)^{-1}U(x_0; L)U(x; L), \quad (24)$$

where $U(x;L)$ is a solution of the system (23), and the constant g describes the boundary of the layer $x = L$. Thus, if $\varepsilon(x) = 0$ when $x > L$, i.e., if the boundary allows the wave to pass through, $g = \infty$. The values $g = 0$ and $g = 2$ correspond to reflecting boundaries with conditions $G|_{x=L} = 0$ and $dG/dx|_{x=L} = 0$.

Equations (23) and (24) were integrated numerically for a function $\varepsilon(L)$ of the same form as before, for different values of the parameters μ and Δ and different boundary conditions at the boundaries of the layer. The resulting curves attest to significant irregularity in the distribution of the intensity within the layer. An example of the behavior of the filed intensity $I(x) = |G(x,x_0)|^2$ within the layer (the field is produced by a source located in the middle of the layer) is presented in the set of Fig. 4 for different boundary conditions on the boundary $x = L$.

Note that, as follows from (22)–(24), the intensity distribution $I(x)$ in the region $x < x_0$ is independent of the boundary condition for $x = L$ in our one-dimensional problem, and only the range of variation of $I(x)$ varies. Therefore only the regions $x_0 \leqslant x \leqslant L$ are depicted in Figs. 4b and 4c.

In conclusion, note that our use of the imbedding equations for the numerical analysis of the structure of a field in a periodically inhomogeneous medium made it possible for us not only to assess the applicability or nonapplicability of the approximate analysis methods but also to readily derive this structure for arbitrary parameters of the problem. This is a particularly important advantage in problems in which the $\varepsilon(x)$ is arbitrary, and for which no approximate analysis method exists.

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