The classical inverse scattering problem for a spherically symmetric gravitational field

I. V. Bogdanov

Institute of Precision Mechanics and Optics, Leningrad
(Submitted 27 October 1982)
Zh. Eksp. Teor. Fiz. **84**, 1921–1929 (June 1983)

The inverse problem for the classical scattering of a particle by a spherically symmetric gravitational field is formulated and solved within the framework of the general theory of relativity. The method used in the solution generalizes the inversion algorithms employed in classical mechanics in the case of a central potential in a flat space to the case of a curved space-time. The hydrodynamic model of a static fluid sphere is considered. In the problem the well-known dependence of the classical cross section for scattering of ultrarelativistic particles on the angle of deflection is used to reconstruct the radial density and pressure distributions of gravitating matter. The solution of the problem reduces to the integration of a nonlinear second-order differential equation whose explicit form is determined from the scattering data. The established method allows the determination of the matter distributions that correspond to ideally focusing gravitational systems. For weak gravitational fields the problem reduces to the Firsov inversion algorithm in geometrical optics. The latter case is of interest in connection with the recently detected phenomenon of gravitational distortion of the images of distant galaxies. The constructed method affords us the only (albeit remote) possibility of directly determining the internal structure of stars in astrophysics. Some results obtained for the case of scattering of relativistic particles of finite mass are discussed.

PACS numbers: 95.30.Sf

1. INTRODUCTION

At present there are in the approximation of classical mechanics three methods of solving the inverse scattering problem (i.e., of reducing it to quadratures), that allow, upon the fulfillment of certain uniqueness conditions, the reconstruction of a static, spherically symmetric potential $V(r)$ from the given angle χ of deflection of the particle. These are the algorithms obtained by Firsov in Ref. 1 (the angle χ is given in the form of a function of the impact parameter b, with the energy ε fixed), by Hoyt in Ref. 2 (here the energy dependence of the angle for a fixed angular momentum is known), and by the present author and Demkov in Ref. 3. In the last method the energy dependence of the angle of deflection is prescribed for a constant impact parameter. In Ref. 4 the Firsov inversion formula is extended to relativistic particles.

There naturally arises the question how we can generalize the indicated inversion algorithms, which are applicable in a flat space, to the case of a strong gravitational field. Here we shall formulate and solve the inverse problem for the classical scattering of a particle by a static, spherically symmetric gravitational field that is also considered in a purely classical (non-quantum) manner in the approximation of the Einstein equations.

As probing particles we shall, for simplicity, use zero-mass particles (we shall arbitrarily call them neutrinos) scattered by a spherically symmetric distribution of gravitating matter having a finite radius a. The problem will be investigated within the framework of the hydrodynamic model of a static fluid sphere.[5]

The assumption that the particles have zero mass is not essential to the solution of the indicated inverse problem. The case of the scattering of particles of finite mass is more tedious, since here we must construct three methods of solving the inverse problem in a curved space: for the cases of fixed energy E, angular momentum l, and impact parameter b (in confirmity with the inversion algorithms found inRefs. 1–3 for a flat space), whereas in the ultrarelativistic case the differences between the first two problems disappear and the last problem has no meaning. These three variants of the inverse problem require special treatment.

As the initial datum in the problem under discussion, we choose the dependence of the neutrino-scattering cross section σ on the scattering angle χ. Then with the aid of the formula[6]

$$\int_{\chi}^{\pi} \sigma'(\chi)\,d\chi=\pi b^2(\chi) \quad (1)$$

we determine the functions $b(\chi)$ and $\chi(b)$. The gravitational inverse problem in the indicated approximations consists in the reconstruction of the radial density $\rho(r)$ and pressure $p(r)$ distributions of matter (i.e., the parameters of the state of a fluid sphere) from a given angle $\chi(b)$ of deflection of neutrinos probing the matter. Eliminating next the radial variable, we find the equation of state $p=p(\rho)$ of the matter.

The method established here of solving the inverse problem in the theory of gravitation offers us in essence the only possibility of directly determining the internal structure of stars (e.g., neutron stars) within the framework of the

 0038-5646/83/061119-05$04.00

fluid-sphere model from the cross section for classical neutrino scattering.

As is well known,[3,4,7,8] we can, by using the inversion algorithms, obtain force fields possessing the various focusing properties connected with the intrinsic symmetry of the corresponding problem of mechanics (classical or quantum-mechanical).[9] This circumstance pertains in equal measure to the inverse problem solved here for the gravitational field, a fact which allows us to construct gravitational lenses (e.g., "cat's eyes") that focus relativistic particles in a specified fashion. We shall say a few words in the Conclusion about focusing systems for ultrarelativistic particles. The question of the symmetry groups of such focusing metrics of space remains for the present open.

2. METHOD OF SOLVING THE PROBLEM

To describe the gravitational field inside matter that can be probed, we shall use the inner Schwarzschild metric[10]

$$ds^2=e^\nu dt^2-e^\lambda dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2), \quad (2)$$

where the velocity of light in free space $c = 1$. As usual, we denote differentiation with respect to time by a dot and with respect to the radial variable by a prime. The condition for the distribution to be static (i.e., to be an equilibrium one) is given by the equality[5]

$$\dot\lambda=\ddot\lambda=\dot\nu=T_t{}^r=0.$$

Here T_t^r is an element of the energy-momentum tensor of the matter. Its diagonal components are given by the Pascal law for an isotropic (fluid) medium:

$$T_r{}^r=T_\theta{}^\theta=T_\varphi{}^\varphi=-p, \quad T_t{}^t=\rho.$$

The Einstein equations for the metric (2) in the indicated approximations have the form[5]

$$e^{-\lambda}(\nu'/r+1/r^2)-1/r^2=\varkappa p, \quad (3)$$

$$e^{-\lambda}(\nu''+\nu'^2/2-(\nu'-\lambda')/r-\nu'\lambda'/2)=2\varkappa p, \quad (4)$$

$$e^{-\lambda}(\lambda'/r-1/r^2)+1/r^2=\varkappa\rho, \quad (5)$$

where the Einstein gravitational constant $\varkappa = 8\pi k$ (k is the "usual" gravitational constant).

The system of field equations (3)–(5), together with the boundary conditions, which require the regularity of the metric functions at the origin and their vanishing at infinity, can be reduced to quadratures. Furthermore, we can derive from this system an integro-differential relation between the pressure and the density, which expresses the condition for hydrostatic equilibrium.[5] Let us set $f\equiv e^{-\lambda}$, and represent this solution (the inner Schwarzschild solution) in the form

$$f=1-\frac{\varkappa}{r}\int_0^r \rho r^2\,dr, \quad (6)$$

$$\nu=\int_r^\infty [\lambda'-\varkappa(\rho+p)re^\lambda]dr. \quad (7)$$

Let us introduce the function

$$\mu(r)=4\pi\int_0^r \rho r^2\,dr. \quad (8)$$

The value $\mu(a) = M$ has the meaning of the mass of the distribution. The hydrostatic-equilibrium equation then has the form

$$p'=-\frac{k(\rho+p)(\mu+4\pi r^3p)}{r^2(1-2k\mu/r)}; \quad (9)$$

it is a consequence of the field equations. The expression (9) is called the Oppenheimer-Volkoff equation.

From (5) and (7) we find

$$\nu=\int_r^\infty \frac{dr}{r}\frac{f-1-\varkappa r^2p}{f}. \quad (10)$$

Let us now consider the scattering problem. Let us find the equation of the plane trajectory (we fix the "plane" $\theta = \pi/2$) of particles of zero mass in a gravitational field with the metric (2) in the integral form. To do this we employ the Hamilton-Jacobi formalism.[10] The ultrarelativistic Hamilton-Jacobi equation for the classical action S is

$$g^{ik}\partial_iS\partial_kS=0,$$

or, with allowance for (2),

$$e^{-\nu}(\partial_tS)^2-f(\partial_rS)^2-r^{-2}(\partial_\varphi S)^2=0. \quad (11)$$

Let us seek the solution to Eq. (11) in the form $S= -Et + l\varphi + S_r(r)$; then for the radial part S_r of the action we obtain an ordinary equation, which can be integrated immediately. After this we find S_r:

$$S_r=\int drf^{-1/2}(E^2e^{-\nu}-l^2/r^2)^{1/2}.$$

The trajectory of a particle in a spherically symmetric field is given by the condition $\partial S/\partial l=$ const. From this we determine the trajectory equation in polar coordinates:

$$\varphi(r)=\int\frac{l\,dr/r^2}{f^{1/2}(E^2e^{-\nu}-l^2/r^2)^{1/2}}. \quad (12)$$

Let us introduce the frequency $\omega = E$ and the impact parameter $b = l/\omega$ of the incoming particles. Then the trajectory has the form

$$\varphi(r)=\int\frac{b\,dr/r^2}{f^{1/2}(e^{-\nu}-b^2/r^2)^{1/2}}. \quad (13)$$

In the absence of a gravitational field (i.e., as $f\to1$, $\nu\to0$) the expression (13) goes over into the well-known[6] formula for a free particle in a flat space.

The metric function $\nu(r)$ is a monotonically increasing function (as can be seen from (10)); it is known from the general theory[10] that it is nonpositive in all of space. This guarantees the existence of a single zero of the denominator in (13).

If r_0 is the distance of closest approach of the particle to the center of symmetry of the problem (the zero of the denominator in (13)), then the polar angle corresponding to this distance is

$$\varphi_0(b)=\frac{1}{2}[\pi+\chi(b)]=\int_{r_0}^\infty\frac{dr/r^2}{f^{1/2}(b^{-2}e^{-\nu}-r^{-2})^{1/2}}. \quad (14)$$

In the inverse problem the equality (14) is a nonlinear integral equation for the metric functions $f(r)$ and $\nu(r)$ for a given scattering angle $\chi(b)$.

Let us split the last integral into a sum of two integrals: from r_0 to a and from a to ∞, and let us denote them by φ_1 and φ_2 respectively. In this case the angle φ_2 is determined by the outer Schwarzschild solution, and can easily be found; it can be expressed in terms of the Legendre elliptic integrals. Below we shall assume that the function e^{ν}/r^2 is monotonic in the interval from r_0 to a. We can, using (10), show that this requirement will be fulfilled if the function

$$H(r) \equiv 1-3f+\varkappa r^2 p$$

does not have zeros in the indicated interval.

Let us set $x \equiv b^{-2}$ in (14) and introduce the functions $u(r)$ and $F(r)$ such that

$$u \equiv r^{-2}e^{\nu}, \quad F' \equiv r^{-2}f^{-1/2}e^{\nu/2}. \tag{15}$$

Going over from r to u, we have

$$\varphi_1(x) = -\int_{u(a)}^{x} (x-u)^{-1/2} \frac{d}{du} F(r(u))\, du. \tag{16}$$

This is the Abel integral equation for the function $F(r(u))$. It can be solved analytically.[11] The result of the inversion has the form

$$F(r) = -\frac{2}{\pi}\left\{[(u-u(a)]^{1/2}\varphi_1(u(a)) + \int_{u(a)}^{u} (u-x)^{1/2}\varphi_1'(x)\, dx\right\}.$$

After integration by parts, we find that

$$F(r) = h(u), \tag{17}$$

where

$$h(u) \equiv \frac{1}{\pi}\int_{u(a)}^{u} (u-x)^{-1/2}\varphi_2\, dx - \frac{1}{\pi}\int_{u(a)}^{u} (u-x)^{-1/2}\varphi_0\, dx. \tag{18}$$

For a given gravitational radius r_g of the distribution, the polar angle in this expression is given by

$$\varphi_2(x) = \int_0^{1/a} \frac{d\zeta}{(r_g\zeta^3-\zeta^2+x)^{1/2}}, \tag{19}$$

and the value $u(a) = a^{-3}(a - r_g)$ (it is determined from the condition for matching with the outer Schwarzschild metric). Thus, the function $h(u)$ is known in the inverse problem.

Let us now differentiate both sides of (17) with respect to r. Taking (15) into account, and setting $dh/du \equiv \psi$, we find

$$f = e^{-\nu}(\nu'-2/r)^{-2}\psi^{-2}(e^{\nu}/r^2). \tag{20}$$

This expression can be called the inversion formula for the gravitational field. It gives a relation, determinable from the scattering data, between the metric functions f and ν.

If the radial function f has been found, the fluid density is

$$\rho = (1-f-rf')/\varkappa r^2, \tag{21}$$

as follows from (6).

We determine the pressure $p(r)$ of the fluid by differentiating (10):

$$p = (f-1+rf\nu')/\varkappa r^2. \tag{22}$$

In the static-distribution approximation, the metric functions f and ν are connected by some differential equation: it can easily be obtained from the Oppenheimer-Volkoff equation (9). Indeed, with allowance for the formulas (6), (8), (21), and (22), this equation has the form

$$p' = \nu'(f'-f\nu')/2\varkappa r. \tag{23}$$

Now calculating p' from (22), and comparing with (23), we find that

$$(\nu'+2/r)f' + (2\nu''+\nu'^2-2\nu'/r-4/r^2)f + 4/r^2 = 0. \tag{24}$$

The last equation can also be derived directly from the field equation; it must be considered together with the inversion formula (20).

Substituting the values of f and f' determined from the formula (20) into (24), we obtain a closed—with respect to $\nu(r)$—nonlinear second-order equation:

$$2r^2\psi\nu'' + e^{\nu}(\nu'-2/r)^2[(\psi_u-2\psi^3)r\nu'/2+\psi_u+2\psi^3] + r^2\psi\nu'^2 = 0. \tag{25}$$

The explicit form of this equation is determined from the scattering data after the construction of the function ψ and its derivative ψ_u.

Let us introduce in (25) the new function

$$\xi \equiv e^{\nu/2}/r \equiv u^{1/2} \tag{26}$$

and the logarithmic derivative $\Phi \equiv \psi_u/\psi$. Then the indicated equation assumes the form

$$\xi'' + r\left[\Phi(\xi^2) - 2\exp 2\int \Phi(\xi^2)\, d(\xi^2)\right]\xi'^3 + 2\Phi(\xi^2)\xi\xi'^2 + 2\xi'/r = 0. \tag{27}$$

It must be integrated with the boundary conditions

$$\xi(a) = a^{-3/2}(a-r_g)^{1/2}, \qquad \xi'(a) = a^{-5/2}(a-r_g)^{-1/2}(3r_g-2a)/2,$$
$$r_g = 2kM, \tag{28}$$

which follow from the condition for matching with the outer Schwarzschild metric.

The boundary-value problem for the nonlinear second-order differential equation (27), (28) is the final result of the above-established algorithm for reconstructing the hydrodynamic parameters (the density and pressure) of gravitating matter from the classical cross section for scattering of ultrarelativistic particles. It can be solved, for example, by numerical methods.

Let us consider the weak gravitational field approximation in the method obtained; this can be done by going over to the equivalent geometrical optics problem of the propagation of ultrarelativistic particles in a spherically inhomogeneous optical medium with refractive index $n(r)$ in a flat space.

In this space the local velocity of light will depend on the local gravitational potential. Thus, the gravitational field can be replaced by an equivalent refracting medium in the flat space, and the effective refractive index can be computed. For a weak spherically symmetric gravitational field with a small gravitational potential U the equivalent refractive index[12] $n = 1 - U$.

Further, let us proceed from the inversion formula (20) for the gravitational field. It is clear that the method constructed in the present paper should, in the weak field limit, give the Firsov inversion algorithm, by which a spherically symmetric refractive index can be reconstructed in geometrical optics[7] (or a spherically symmetric potential, in classical mechanics[1]) from the scattering data. Entering into the inversion formula (20) are the metric tensor elements $g^{00} \equiv e^{-\nu}$ and $g^{11} \equiv f$. It is well known that, in the limiting case of a weak field, $g_{00} = 1 + 2U$, and the correction in g_{11} is of the same order of magnitude as the correction in g_{00}. But the correction in g_{11}, though of the same order of smallness as the correction in g_{00}, gives rise in the Lagrangian (and, consequently, in the equations of motion) of the particle to terms of higher order in smallness (see Ref. 10, p. 246 of the English translation).

The Firsov formula is a consequence of the equations of motion[6]; therefore, we set $g^{00} = 1 - 2U$ and $g^{11} = 1$ in the expression (20) in the case under discussion. Then it assumes the form

$$2r\xi'\psi(\xi^2) = 1, \tag{29}$$

where $\xi \equiv g_{00}^{1/2}/r \approx (1 + U)/r$, and, after introducing the equivalent refractive index, we have $\xi \approx r^{-1}n^{-1} \equiv q$.

Let us extend the inner Schwarzschild metric to all space. Under the assumptions made, $u \approx q^2$, $\varphi_2 \to 0$, $u(a) \to 0$, and $a \to \infty$ in the relation (18), and, after integrating by parts, we have

$$h(q^2) = -q - \frac{1}{\pi} \int_0^{q^2} (q^2 - x)^{1/2} \chi'(x)\, dx. \tag{30}$$

Now, remembering that $\psi = dh/du$, we obtain after separating the variables in Eq. (29) and integrating the relation

$$\ln rq = -\frac{1}{\pi} \int dq \int_0^{q^2} (q^2 - x)^{-1/2} \chi'(x)\, dx. \tag{31}$$

Then, changing the order of the integrations in (31), going over to the variables b and n, and integrating by parts, we finally find that

$$n = \exp \frac{1}{\pi} \int_{rn}^{\infty} (b^2 - r^2 n^2)^{-1/2} \chi(b)\, db. \tag{32}$$

This is the well-known Firsov inversion formula,[1,7] which solves the inverse problem in geometrical optics.

Let us now summarize the above-presented method of solving the inverse problem for a spherically symmetric gravitational field. First, we must calculate $b(\chi)$ (and then $\varphi_0(x)$) from the given scattering cross section $\sigma(\chi)$, using the formula (1), and construct the function $h(u)$ with the aid of the definition (18). The explicit form of Eq. (27) is then known, and, solving the boundary-value problem (27), (28), we obtain the function $\xi(r)$ and, on the basis of the formula (26), one of the metric functions, namely, $\nu(r)$. Further, using the inversion formula (20), we calculate the other metric function $f(r)$. Then the expressions (21) and (22) allow us to determine the radial density and pressure distributions of the matter. Finally, eliminating the variable r, we find the equation of state of the matter.

Thus, the solution scheme for the considered inverse problem has the following form:

$$\sigma(\chi) \to b(\chi) \to \varphi_0(x) \to h(u)$$

$$\to \Phi(\xi^2) \to \xi(r) \to \nu(r) \to f(r) \to \rho(r)$$

$$\nu(r) \to p(r) \to p(\rho); \quad f(r) \to p(r); \quad \rho(r) \to p(\rho).$$

3. CONCLUSION

The results obtained here for ultrarelativistic particles show that the classical spherically-symmetric inverse problem for the case of a strong gravitational field can also be solved. The basic nonlinear integral equation (14) of the problem reduces to the Abel equation.

In the case of scattering of particles of finite mass the inversion algorithms for the gravitational field can be constructed, using a method similar to the one found here. As for massless particles, the inversion process leads there to boundary-value problems for nonlinear second-order differential equations whose explicit forms are determined from the scattering data. This circumstance is valid for all the three inverse problems indicated in the Introduction: for fixed energy, angular momentum, and impact parameter. In the ultrarelativistic limit, the above-named equations for the problems with fixed E and l go over into an equation (27) for neutrino scattering. In the Newtonian (i.e., weak field) approximation, we can derive from the relativistic equations obtained for the cases of constant energy, constant angular momentum, and constant impact parameter the formulas established respectively by Firsov,[1] Hoyt,[2] and the present author and Demkov[3] for inversion in a flat space.

We shall assume that the inner Schwarzschild metric is specified in all space (i.e., that $a \to \infty$). Then the analogue of the Firsov formula for the gravitational field has the form

$$rg \exp\left\{-\int f^{-1/2} \frac{dr}{r}\right\} = \exp \frac{1}{\pi} \int_{rg}^{\infty} (b^2 - r^2 g^2)^{-1/2} \chi\, db,$$

$$g \equiv (E^2 - m^2)^{-1/2} (E^2 e^{-\nu} - m^2)^{1/2}. \tag{33}$$

The system of equations composed of the inversion formula (33) and the equilibrium equation (24) allows us to find the metric functions f and ν. In the weak field limit we take $j(r) \approx 1$, $g(r) \approx (1 - V/\varepsilon)^{1/2}$, and then the expression (33) yields the usual Firsov formula.[1] For the case of a finite radius of the matter distribution, the relation of the type (33) is slightly more complicated.

Thus, the spherical symmetry of the field allows us to construct an explicit algorithm for solving the classical inverse problem, even for the general case of strong gravitation, i.e., of curved space-time, as well as for the simplest potential scattering. In this connection, the range of centrosymmetric inverse problems considered in various papers turns out to be, to a certain extent, settled and completed (although additional generalizations are, of course, possible, e.g., for charged-particle scattering in the presence of a gravitational field).

As has already been noted in the Introduction, the described method of solving the inverse problem can be used to construct gravitational fields that focus ultrarelativistic particles in a specified manner. Thus, for example, we can calculate the metric of the space inside gravitating matter that reflects a particle incident on it strictly backwards (a gravitational "cat's eye"). For this focusing system, the function $\Phi(\xi^2)$ in Eq. (27) can be expressed in terms of the Legendre elliptic integrals, and can be integrated numerically.

The passage to the weak-field limit in algorithm obtained for ultrarelativistic-particle scattering yields the Firsov inversion formula[1] in the form (32), i.e., in geometrical optics (in terms of the refractive index). In this form the Firsov formula is valid for both nonrelativistic particles[7] (in view of the existence of an optical-mechanical analogy) and relativistic particles in classical mechanics (generalized optical-mechanical analogy[4]) and, of course, for ultrarelativistic particles in geometrical optics. This case of a weak gravitational field is of interest in connection with the recently discovered phenomenon of gravitational distortion of the images of distant galaxies.

Let us extend the inner Schwarzschild metric to all space. Then, just as in Maxwell's "fisheye" problem,[9] we can seek a gravitational field in which every particle with energy $E = m$ moves along a circle. The establishment of the intrinsic symmetry in the corresponding classical-mechanics or quantum-mechanical problem is apparently tied with the possibility of determining such a metric (the metric of a gravitational "fisheye") in its analytic form.

The author is grateful to Yu. N. Demkov for the formulation of the problem and for useful discussions and A. A. Grib for interest in and attention to the work.

[1]O. B. Firsov, Zh. Eksp. Teor. Fiz. **24**, 279 (1953).
[2]F. Hoyt, Phys. Rev. **55**, 664 (1939).
[3]I. V. Bogdanov and Yu. N. Demkov, Zh. Eksp. Teor. Fiz. **82**, 1798 (1982) [Sov. Phys. JETP **55**, 1037 (1982)].
[4]I. V. Bogdanov and Yu. N. Demkov, Vestn. Leningr. Univ. **22**, 5 (1982).
[5]Ya. B. Zel'dovich and I. D. Novikov, Teoriya tyagoteniya i évolyutsiya zvezd (Theory of Gravitation and Stellar Evolution), Nauka, Moscow, 1971, Chap. 10, §3.
[6]L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics), Nauka, Moscow, 1973 (Eng. Transl., Pergamon, Oxford, 1976).
[7]Yu. N. Demkov, V. N. Ostrovskiĭ, and N. B. Berezina, Zh. Eksp. Teor, Fiz. **60**, 1604 (1971) [Sov. Phys. JETP **33**, 867 (1971)].
[8]Yu. N. Demkov and V. N. Ostrovskiĭ, Zh. Eksp. Teor. Fiz. **62**, 125 (1972) [Sov. Phys. JETP **35**, 66 (1972)].
[9]Yu. N. Demkov and V. N. Ostrovskiĭ, Zh. Eksp. Teor. Fiz. **60**, 2011 (1971) [Sov. Phys. JETP **33**, 1083 (1971)].
[10]L. D. Landau and E. M. Lifshitz, Teoriya polya (Classical Theory of Fields), Nauka, Moscow, 1973 (Eng. Transl., 3rd. ed., Pergamon, Oxford, 1971).
[11]G. A. Korn and T. M. Korn, Mathematical Handbook for Scientists and Engineers, McGraw-Hill, New York, 1961 (Russ. Transl., Nauka, Moscow. 1974, p. 469).
[12]M. G. Bowler, Gravitation and Relativity, Pergamon, New York, 1976 (Russ. Transl., Mir, Moscow, 1979, p. 96).

Translated by A. K. Agyei