Density of induced charge in a strong Coulomb field

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(Submitted 7 September 1982)
Zh. Eksp. Teor. Fiz. **84**, 1247–1256 (April 1983)

An expression is obtained in coordinate space for the density of an induced charge in a strong Coulomb field. The treatment is based on a convenient integral representation, used earlier by the authors, for the Green's function of an electron in this field. The behavior of the potential corresponding to the obtained charge density is investigated at long and short distances.

PACS numbers: 41.10.Fs

The measurement accuracy of the energy of transitions between levels of muonic atoms is at present such (see Ref. 1 and the bibliography therein) that account must be taken of the quantum-electrodynamic corrections when this quantity is calculated theoretically. At large Z (the nuclear charge is $Z|e|$ where $e = -|e|$ is the electron charge) the Bohr radius of the muon becomes smaller than[1)] $\lambda\!\!\!^{-}_c = 1/m$ (m is the electron mass) and the polarization of the electron-positron vacuum becomes significant. At distances $\approx 1/m$ from the center the electric field is $\approx 4.4(Z\alpha)\cdot 10^{13}$ Oe ($\alpha = e^2 = 1/137$ is the fine-structure constant), i.e., the processes take place in superstrong field. The calculation must therefore be performed outside the framework of perturbation theory in the parameter[2)] $Z\alpha$.

Vacuum polarization in a strong Coulomb field was first considered in Ref. 2, where the Laplace transforms, in all orders in $Z\alpha$, were obtained for the induced-charge density $\rho(r)$ multiplied by r^2. The first term of the expansion in Z that corresponds to this density of the potential $\varphi(r)$ was first obtained earlier in Ref. 3. Starting from the results of Ref. 4, an explicit expression was obtained in the coordinate representation for the next term of the expansion of the potential in $Z\alpha$ (proportional to $(Z\alpha)^3$). The behavior of the potential at short distances was investigated in Refs. 5 and 6 by operator methods. Computer calculations of individual contributions to the polarization of vacuum were also undertaken, and reference to them can be found in the paper of Borie and Rinker.[1]

We obtain here for $\rho(r)$ an expression accurate to $Z\alpha$ directly in coordinate space. It is convenient to take the external field into account in the Furry representation. In accord with the rules of the diagram technique we obtain for $\rho(r)$

$$\tilde{\rho}(r) = -ie\,\mathrm{Tr}[G(x,x')\gamma^0],\quad x\to x'. \qquad (1)$$

Here γ^0 is a Dirac matrix and the tilde labels renormalized quantities. The expression for the Green's function $G(x,x')$ of an electron in a Coulomb field is of the form

$$G(x,x') = \int \frac{d\varepsilon}{2\pi} G(\mathbf{r},\mathbf{r}'|\varepsilon)\exp\{-i\varepsilon(t-t')\}, \qquad (2)$$

where the contour of integration with respect to ε passes in accord with Feynman's rule below the real axis in the left complex ε half-plane and above it in the right. An integral representation was recently obtained[7] for $G(\mathbf{r},\mathbf{r}'|\varepsilon|)$, valid in the entire complex plane of the energy ε and containing no contour integrals, in the contrast to papers by others (see, e.g., Ref. 8). This last circumstance makes this representation convenient for applications. The transition to the limit in (1) calls for a certain accuracy (see, e.g., Ref. 9), but the ensuing problems are eliminated upon renormalization. Putting formally $\mathbf{r} = \mathbf{r}'$ in Eqs. (19) and (20) of Ref. 7, we have

$$G^{(\pm)}(\mathbf{r},\mathbf{r}|\varepsilon) = \frac{\mp i}{4\pi r^2}\sum_{l=1}^{\infty} l \int_0^{\infty} ds\, \exp\{\pm i[2Z\alpha\varepsilon s + 2kr\,\mathrm{ctg}\,ks - \pi\nu]\}T;$$

here

$$T = xJ_{2\nu}'(x)(\gamma^0\varepsilon + m) + 2iZ\alpha\gamma^0(m(\boldsymbol{\gamma}\mathbf{n}) \mp k\,\mathrm{ctg}\,ks)J_{2\nu}(x),$$
$$\nu = [l^2 - (Z\alpha)^2]^{1/2},\quad x = 2kr/\sin ks,\quad k = (m^2-\varepsilon^2)^{1/2},\quad \mathbf{n} = \mathbf{r}/r, \qquad (4)$$

where $J_{2\nu}(x)$ is a Bessel function. For the attracting-field case considered here, $G^{(+)}$ and $G^{(-)}$ define the function G in the upper and lower complex ε half-planes. $G^{(+)}$ and $G^{(-)}$ coincide on the inteveral $(-m,m)$ of the real axis, as they should. It is shown in Ref. 7 that the expression obtained for $G(\mathbf{r},\mathbf{r}'|\varepsilon)$ has analytic properties that follow from the general theory (see, e.g., Ref. 9), i.e., it has cuts along the real ε axis from $-\infty$ to $-m$ and from m to ∞, corresponding to a continuous spectrum, and simple poles at points corresponding to the energy levels of the discrete spectrum. When account is taken of the indicated analytic properties and of the structure of Eq. (3), the contour of integration with respect to ε in (2) (where we assume $t = t'$ and $\mathbf{r} = \mathbf{r}'$) can be deformed to coincide with the imaginary axis. The integration contour with respect to s in (3) can then also be rotated to the imaginary axis so that it extends from zero to $-i\infty$ in $G^{(+)}$ and from zero to $i\infty$ in $G^{(-)}$. Making these transformations and obvious changes of variables, and calculating the trace, we obtain

$$\tilde{\rho}(r) = \frac{em^3}{\pi^2R^2}\sum_{l=1}^{\infty} l \int_0^{\infty} dx \int_0^{\infty} dt\, e^{-y\,\mathrm{ch}\,t} f(y,t), \qquad (5)$$

where

$$f(y,t) = y\frac{x}{b}\sin\mu t I_{2\nu}'(y) - 2Z\alpha\,\mathrm{cth}\,t\cos\mu t I_{2\nu}(y),$$
$$y = 2bR/\mathrm{sh}\,t,\quad b = (x^2+1)^{1/2},\quad R = mr,\quad \mu = 2Z\alpha x/b; \qquad (6)$$

where $I_{2\nu}(y)$ is a modified Bessel function of the first kind. We note that the expansion (5) of $\tilde{\rho}(r)$ in terms of $Z\alpha$ contains only odd powers of this parameter, i.e., the Furry theorem

0038-5646/83/040722-06$04.00

does not hold. Expression (5) for $\tilde{\rho}(r)$ calls for renormalization, which we carry out on the basis of the physically obvious requirement: the total induced charge must equal zero. It is technically convenient carry out the renormalization in the momentum representation. By definition:

$$\tilde{\rho}(\beta)=\int d^3r e^{i\mathbf{q}\mathbf{r}}\tilde{\rho}(r)=\frac{4\pi}{\beta m^3}\int_0^\infty dR\,R\tilde{\rho}(r)\sin\beta R, \qquad (7)$$

where $\beta = |\mathbf{q}|/m$. Substituting the expression for $\tilde{\rho}(r)$ from (5) in (7) and changing from integration with respect to R in (7) to integration with respect to y, we obtain

$$\tilde{\rho}(\beta)=\frac{4e}{\pi\beta}\sum_{l=1}^{\infty} l\int_0^\infty dx\int_0^\infty dt\int_0^\infty \frac{dy}{y}\sin\left(\frac{\beta y\,\mathrm{sh}\,t}{2b}\right)e^{-y\,\mathrm{ch}\,t}f(y,t). \qquad (8)$$

The renormalized quantity $\rho(\beta)$ must vanish at the point $\beta = 0$. To obtain $\rho(\beta)$ we must accordingly calculate the asymptotic value of $\tilde{\rho}(\beta)$ as $\beta\to 0$, retain in it only terms that do not vanish as $\beta\to 0$, and subtract the obtained asymptotic value from expression (8). As $\beta\to 0$, the terms of different order in $Z\alpha$ behave differently. Thus, the term $\tilde{\rho}_1(\beta)$ which is linear in $Z\alpha$ and is obtained from (8) by the substitution $f(y,t)\to f_1(y_1,t)$, where

$$f_1(y,\ t)=2Z\alpha[yt(x/b)^2 I_{2l}'(y)-\mathrm{cth}\,tI_{2l}(y)], \qquad (9)$$

contains $c_1/\beta^2 + c_2$ as $\beta\to 0$. In terms of perturbation theory, $\tilde{\rho}_1(\beta)$ corresponds to the diagram of the polarization operator in the lowest order in interaction:

$$\tilde{\rho}_1(\beta)=\frac{Ze}{|\mathbf{q}|^2}\tilde{\mathscr{P}}(-|\mathbf{q}|^2) \qquad (10)$$

and the subtraction of the terms $c_1/\beta^2 + c_2$ coincides with the usual procedure of renormalizing the polarization operator $\tilde{\mathscr{P}}$. Details of the calculation of $\rho_1(\beta)$ can be found in Appendix I. For the renormalized quantity $\rho_1(\beta)$ we obtain

$$\rho_1(\beta)=\frac{e(Z\alpha)}{3\pi}\left[\frac{5}{3}-\frac{4}{\beta^2}+\left(\frac{2}{\beta^2}-1\right)\left(1+\frac{4}{\beta^2}\right)^{1/2}\times\ln\left(\frac{(1+4/\beta^2)^{1/2}+1}{(1+4/\beta^2)^{1/2}-1}\right)\right]. \qquad (11)$$

As it should, $\rho_1(\beta)$ coincides (in the sense of Eq. (10), where the tildes should be omitted) with the direct calculation of the polarization operator (see Ref. 9). The potential corresponding to the distribution of the charge ρ_1, known as the Uehling potential,[3] is of the form

$$\varphi_1(r)=-\frac{2e}{3\pi}\frac{Z\alpha}{r}\left[K_0(2R)-\frac{1}{2}\int_1^\infty\frac{dx}{x^4}\frac{x^2+1}{(x^2-1)^{1/2}}e^{-2Rx}\right], \qquad (12)$$

here $K_0(2R)$ is a modified Bessel function of the third kind. This form describes lucidly the behavior of $\varphi_1(r)$ as $R\to 0$.

The terms proportional to $(Z\alpha)^3$ in (8) correspond to diagrams of the type of light scattering by light and contain prior to renormalization a logarithmic divergence. Their asymptotic form as $\beta\to 0$ depends therefore on the regularization method. If the regularization is carried out with an upper limit imposed on the integration with respect to energy (x), the asymptotic form is $c_3 + c_2\ln\beta$, as noted in Ref. 2. If an upper limit of the sum over the angular momenta (l) is imposed in the regularization, the asymptotic form is independent of β. After the renormalization the result is independent, as it should be, of the regularization method. Finally, terms of order $(Z\alpha)^5$ and higher in (8) yield a constant as $\beta\to 0$. The details of the renormalization procedure are given in Appendix I. The renormalized expression for $\rho_2(\beta)=\rho(\beta)-\rho_1(\beta)$, where $\rho_1(\beta)$ is defined in (11), is

$$\rho_2(\beta)=\frac{4e}{\pi}\left\{\frac{1}{\beta}\sum_{l=1}^{\infty}l\int_0^\infty dx\int_0^\infty dt\int_0^\infty\frac{dy}{y}\left[\sin\left(\frac{\beta y\,\mathrm{sh}\,t}{2b}\right)\times e^{-y\,\mathrm{ch}\,t}(f(y,t)-f_1(y,t))+\sin\left(\frac{\beta yt}{2b}\right)g(y,t)\right]+\Omega\right\}, \qquad (13)$$

where

$$g(y,t)=(Z\alpha)^3\left(\frac{8}{\pi y}\right)^{1/2}\left(\frac{1}{ty}-2t\frac{x^2}{b^2}+\frac{t^3y}{3}\frac{x^4}{b^4}\right)\times\exp\left(-\frac{t^2y}{2}-\frac{2l^2}{y}\right),$$

$$\Omega=\sum_{l=1}^{\infty}l\,\mathrm{Im}\left\{\ln\Gamma(\nu-iZ\alpha)+\frac{1}{2}\ln(\nu-iZ\alpha)+iZ\alpha\psi(l)+\frac{iZ\alpha}{2l}-\frac{i(Z\alpha)^3}{3l^2}\right\}; \qquad (14)$$

here $\psi(\tau) = d\ln\Gamma(\tau)/d\tau$. Taking the inverse Fourier transform, we obtain the renormalized expression for ρ_2 in coordinate space:

$$\rho_2(\mathbf{r})=\frac{em^3}{\pi^2R^2}\sum_{l=1}^{\infty}l\int_0^\infty dx\int_0^\infty dt\{e^{-y\,\mathrm{ch}\,t}[f(y,t)-f_1(y,t)]+g(s,t)\}+\frac{4e}{\pi}\delta(\mathbf{r})\Omega, \qquad (15)$$

here $y = 2bR/\sinh t$, $s = 2bR/t$, and the remaining quantities are defined in (6), (9), and (14). Having an expression for $\rho(r)$ and using the vanishing of the total induced charge, we can write the potential corresponding to $\rho(r)$ in one form or another. As indicated above, in some cases an important role is played by the behavior of the potential at small R. We present the first few terms of the expansion of $\varphi(r)$ at $R = mr \ll 1$:

$$\varphi(r)\approx em\left[-\frac{2Z\alpha}{3\pi R}\left(\ln\frac{1}{R}-C-\frac{5}{6}\right)+\frac{A}{R}+B+FR+DR^{2\nu_1}+\dots\right], \qquad (16)$$

where C is the Euler constant, $C = 0.577\ldots$, $\nu_1 = [1 - (Z\alpha)^2]^{1/2}$,

$$A=\frac{4}{\pi}\sum_{l=1}^{\infty}l\,\mathrm{Im}\left[\ln\Gamma(\nu-iZ\alpha)+\frac{1}{2}\ln(\nu-iZ\alpha)-(\nu-iZ\alpha)\psi(\nu-iZ\alpha)+\frac{iZ\alpha}{2l}-iZ\alpha l\psi'(l)\right],$$

$$F=\frac{4}{\pi}Z\alpha\sum_{l=1}^{\infty}\frac{l}{\nu}\mathrm{Re}\left[\frac{l^2\psi'(\nu-iZ\alpha)}{4\nu^2-1}-\frac{l^2+\nu}{2l^2(2\nu+1)}\right],$$

$$B=-2Z\alpha\sum_{l=1}^{\infty}\frac{l}{\nu}\int_0^{\infty}dtJ_0(2Z\alpha t)e^{-2\nu t}\left(\operatorname{cth}t-\frac{t}{\operatorname{sh}^2 t}\right), \quad (17)$$

$$D=-\frac{8Z\alpha}{\pi^{1/2}\nu_1^2}\frac{\Gamma(3/2-\nu_1)}{(4\nu_1^2-1)\Gamma(2\nu_1+2)}\int_0^{\infty}dt\,t\left(\frac{Z\alpha t}{\operatorname{sh}^2 t}\right)^{\nu_1}J_{-\nu_1}(2Z\alpha t).$$

Here J_0 and $J_{-\nu_1}$ are Bessel functions,

$$\psi'(\tau)=d\psi/d\tau=d^2\ln\Gamma(\tau)/d\tau^2.$$

The first term in the brackets in (16) stems from $\varphi_1(r)$ [Eq. (12)], and the contribution of the Uehling potential is contained in B and F. The quantity Ae corresponds to the induced point charge ($\delta Q'$ in the notation of Refs. 2 and 5) at the origin, and contains the contribution $(Z\alpha)^3$ and of higher order in the expansion in the parameter $Z\alpha$. In a different form, $\delta Q'$ was first obtained in Ref. 2. The calculation of this quantity is the subject of Ref. 5, where it is represented in a form identical with (17) and it is shown that the result coincides with that of Ref. 2. In our approach it is simplest to obtain $\delta Q'$ from (13), if it is noted that $\delta Q'$ coincides with the limit of $\rho_2(\beta)$ as $\beta\to\infty$. In this case a part of $\delta Q'$ has already resulted from the renormalization (the quantity Ω in (13)), and in the remaining integral the contribution to $\delta Q'$ is made by the region $x\gg 1$, so that we can neglect the unity in $b=(x^2+1)^{1/2}$; this corresponds to the zero-mass limit used in Ref. 5. The quantities F and D are calculated in Ref. 6. F in (17) is identical with the corresponding result of Ref. 6, and the expression for D is given in Ref. 6 in a more complicated form. The first few terms of the expansion in terms in $Z\alpha$ are identical in (17) and in Ref. 6. The singularity (pole) of D at $\nu_1=\frac{1}{2}$ is cancelled by an identical singularity in F (as $\nu_1=\frac{1}{2}$ and $R^{2\nu_1}\to R$), and this question is discussed in detail in Ref. 6. The coefficients $B, F,$ and D have a singularity proportional to $1/\nu_1$ as $\nu_1\to 0$ ($Z\alpha\to 1$). This singularity is pairwise cancelled out in $\varphi(r)$ (16), viz., B and $DR^{2\nu_1}$, FR and the terms left out of (16) and proportional to $R^{2\nu_1+1}$, a fact we verified directly. The term proportional to $(Z\alpha)^3$ in B was obtained in Ref. 4 and agrees with the expansion for B in (17). Details of the calculation of the coefficients in (17) are given in Appendix II.

At large distances $R\gg 1$, the Uehling potential decreases exponentially, and the contributions to the potential from terms $(Z\alpha)^3$ and higher decrease in power-law fashion. The nonvanishing term in the expansion of the potential at $R\gg 1$ is then proportional to R^{-5} and appears only in the $(Z\alpha)^3$ term, a fact we verified by direct calculation. The result agrees with Ref. 2 and can be easily interpreted in terms of the effective Lagrangian (see Appendix III of Ref. 2).

The authors thank V. N. Baĭer for interest in the work.

APPENDIX I

We transform the quantity $\tilde{\rho}_1(\beta)$. To this end, making in (8) the substitution $f\to f_1$, where $f_1(y,t)$ is defined in (9), we sum over l with the aid of the relation

$$\sum_{l=1}^{\infty}lI_{2l}(y)={}^1/_4 yI_1(y), \quad (I.1)$$

which can be easily derived by using the integral repesentation for $I_{2l}(y)$ (see Ref. 10, Russ. p. 972). We obtain thus for $\tilde{\rho}_1(\beta)$

$$\tilde{\rho}_1(\beta)=\frac{2eZ\alpha}{\pi\beta}\int_0^{\infty}dx\int_0^{\infty}dy\int_0^{\infty}dt\sin\left(\frac{\beta y\operatorname{sh}t}{2b}\right)\times e^{-y\operatorname{ch}t}\left[\frac{x^2}{b^2}tyI_0(y)-\operatorname{cth}tI_1(y)\right]. \quad (I.2)$$

Consider the first term in the square brackets of (I.2). Its integral with respect to t is

$$\int_0^{\infty}dt\,te^{-y\operatorname{ch}t}\sin\left(\frac{\beta y\operatorname{sh}t}{2b}\right)=\operatorname{arctg}\left(\frac{\beta}{2b}\right)K_0\left(y\left(1+\frac{\beta^2}{4b^2}\right)^{1/2}\right). \quad (I.3)$$

The integral with respect to y is of the form (Ref. 10, Russ. p. 686)

$$\int_0^{\infty}dy\,yI_0(y)K_0\left(y\left(1+\frac{\beta^2}{4b^2}\right)^{1/2}\right)=\frac{4b^2}{\beta^2}. \quad (I.4)$$

The contribution of this term to $\tilde{\rho}_1(\beta)$ is thus

$$\frac{8e(Z\alpha)}{\pi\beta^3}\int_0 dx\,x^2\operatorname{arctg}\left(\frac{\beta}{2b}\right). \quad (I.5)$$

Here $x_0\gg 1$. To regularize the expression we have restricted the region of integration with respect to energy (the latter corresponds to the variable x). If we put

$$\sin\left(\frac{\beta y\operatorname{sh}t}{2b}\right)=\frac{\beta y\operatorname{sh}t}{4b}\int_{-1}^{1}ds\exp\left(\frac{i\beta ys}{2b}\operatorname{sh}t\right), \quad (I.6)$$

the integration of the second term in (I.2) is carried out similarly and for $\tilde{\rho}_1(\beta)$ we obtain[3)]

$$\tilde{\rho}_1(\beta)=\frac{8eZ\alpha}{\pi}\int_0^{\infty}dx\left\{\frac{x^2}{\beta^3}\operatorname{arctg}\left(\frac{\beta}{2b}\right)-\frac{b}{4\beta^2}\int_{-1}^{1}\frac{ds}{s^2[1+(\beta s/2b)^2]}\right\}. \quad (I.7)$$

The renormalization reduces now to subtracting from the integrand in (I.7) the first two terms of the expansion as $\beta\to 0$; these terms are proportional to $1/\beta$ and to a constant:

$$\rho_1(\beta)=\frac{8eZ\alpha}{\pi}\int_0^{\infty}dx\left[\frac{x^2}{\beta^3}\left(\operatorname{arctg}\frac{\beta}{2b}-\frac{\beta}{2b}+\frac{\beta^3}{24b^3}\right)-\frac{b}{4\beta^2}\int_{-1}\frac{ds}{s^2}\left(\frac{1}{1+(\beta s/2b)^2}-1+\left(\frac{\beta s}{2b}\right)^2\right)\right]. \quad (I.8)$$

All the integrals in (I.8) are already convergent and x_0 can be replaced by ∞. Elementary calculations lead then to expression (11) for $\rho_1(\beta)$.

We now renormalize the terms of order $(Z\alpha)^3$ and higher. We must calculate the asymptotic form of these terms in (8) as $\beta\to 0$. We note that the contribution of $\ln\beta$ appears in the asymptotic form of the terms proportional to $(Z\alpha)^3$ following the method, indicated in the text, of regularization from the region $t\sim\beta$, $l\sim 1/\beta$, $y\sim 1/\beta^2$. In this region the integrand in (8) takes, after subtraction of the terms linear in

$Z\alpha$, the form $-y^{-1}\sin(\beta yt/2b)\,g(y,t)$, where $g(y,t)$ is defined in (14). We write $\tilde{\rho}_2(\beta)$ in the form

$$\tilde{\rho}_2(\beta)=\frac{4e}{\pi\beta}\sum_{l=1}^{\infty} l\int_0^{\infty}dx\int_0^{\infty}dt\int_0^{\infty}\frac{dy}{y}\times\left\{\left[e^{-y\,\mathrm{ch}\,t}\sin\left(\frac{\beta y\,\mathrm{sh}\,t}{2b}\right)(f(y,t)-f_1(y,t))+\sin\left(\frac{\beta yt}{2b}\right)g(y,t)\right]-\sin\left(\frac{\beta yt}{2b}\right)g(y,t)\right\}. \quad (\mathrm{I.9})$$

The integral of the expression in the square brackets in (I.9) converges, and its limit as $\beta\to 0$ is obtained by replacing $\sin(\beta y\sinh t/2b)$ and $\sin(\beta yt/2b)$ by their arguments. We represent this limit (which does not depend on β) in the form

$$\omega=\frac{4e}{\pi}\sum_{l=1}^{\infty} l[\omega_1(l)+\omega_2(l)], \quad (\mathrm{I.10})$$

where $\omega_2(l)$ corresponds to the contribution of the term proportional to g, and $\omega_1(l)$ to that of the term proportional to $(f-f_1)$. The value of $\omega_2(l)$ is calculated in elementary fashion by first integrating with respect to t and then with respect to x and y. We obtain

$$\omega_2(l)=(Z\alpha)^3/3l^2. \quad (\mathrm{I.11})$$

When calculating $\omega_1(l)$ the integrals with respect to y are evaluated with the aid of (II.4):

$$\omega_1(l)=\int_0^{\infty}\frac{dx}{b}\int_0^{\infty}dt\left[-\frac{x}{2b}\sin\mu t\frac{d}{dt}(e^{-2\nu t}\,\mathrm{cth}\,t)-Z\alpha\cos\mu t\,\mathrm{cth}\,te^{-2\nu t}+Z\alpha t\frac{x^2}{b^2}\frac{d}{dt}(e^{-2lt}\,\mathrm{cth}\,t)+Z\alpha\,\mathrm{cth}\,te^{-2lt}\right]. \quad (\mathrm{I.12})$$

After integrating the terms with the derivative in (I.12) by parts with respect to t it is easy to integrate with respect to x:

$$\omega_1(l)=\int_0^{\infty}dt\left[Z\alpha e^{-2lt}-\frac{\sin(2Z\alpha t)}{2t}e^{-2\nu t}\right]\mathrm{cth}\,t. \quad (\mathrm{I.13})$$

If $\omega_1(l)$ in (I.13) is differentiated with respect to $Z\alpha$, the ensuing integrals can be evaluated with the aid of (II.13). Integrating with respect to $Z\alpha$ with allowance for the obvious boundary condition $\omega_1(l)=0$, we obtain at $Z\alpha=0$

$$\omega_1(l)=-\mathrm{Im}\left[\ln\Gamma(\nu-iZ\alpha)+\frac{1}{2}\ln(\nu-iZ\alpha)+iZ\alpha(\psi l)+\frac{iZ\alpha}{2l}\right]. \quad (\mathrm{I.14})$$

Substituting $\omega_1(l)$ (I.14) and $\omega_2(l)$ in (I.10) we obtain $\omega=-\Omega$. Renormalization calls for subtraction of this constant from $\tilde{\rho}_2(\beta)$ (I.9).

We consider now the contribution of the last term in $\tilde{\rho}_2(\beta)$ (I.9). After substituting $t\to t/y$ we integrate with respect to y and x. We obtain (we denote the contribution of this term by χ):

$$\chi=-\frac{2ex_0}{\pi\beta}\sum_{l=1}^{l_0}\frac{(Z\alpha)^3}{l^2}\int_0^{\infty}\frac{dt}{t}\frac{1}{(t^2+1)^{5/2}}\sin\left(\frac{\beta lt}{x_0}\right), \quad (\mathrm{I.15})$$

where $x_0\gg 1$ is the upper limit of integration with respect to x, and $l_0\gg 1$. If we let $x_0\to\infty$ at fixed l_0, we obtain

$$\chi=-\frac{4e}{3\pi}(Z\alpha)^3\sum_{l=1}^{l_0}\frac{1}{l}. \quad (\mathrm{I.16})$$

If, however, we let $l_0\to\infty$ at fixed x_0 and take the sum and the integral, we get for χ:

$$\chi=-\frac{4e}{3\pi}(Z\alpha)^3\left[\ln\left(\frac{2x_0}{\beta}\right)+\frac{3}{2}\right]. \quad (\mathrm{I.17})$$

It follows from (I.16) and (I.17) that at any regularization method the quantity χ is completely eliminated by renormalization, and we arrive at Eq. (13) for $\rho_2(\beta)$.

APPENDIX II

We calculate now the coefficients B, F, and D. We consider the contribution made to these coefficients by the potential $\varphi_2(r)$ that corresponds to the charge distribution $\rho_2(r)$. The asymptotic form of $\varphi_2(r)$ as $r\to 0$ is uniquely connected with the asymptotic form of $\partial r\varphi_2(r)/\partial r$. Using the Gauss theorem we have

$$\frac{1}{me}\frac{\partial}{\partial r}r\varphi_2(r)\equiv V=\frac{4\pi}{me}\int_0^{\infty}dr'\rho_2(r')r'\vartheta(r'-r). \quad (\mathrm{II.1})$$

The terms arising in $\rho_2(r)$ as a result of renormalization (g and Ω in (15)) make no contribution to the sought coefficients. For the sake of brevity we omit also $f_1(y,t)$ in (15), and defer to corresponding subtraction of the terms linear in $Z\alpha$ to the end. We can then represent V with the aid of (15) in the form

$$V=\frac{4}{\pi}\sum_{l=1}^{\infty} l\int_0^{\infty}dx\int_0^{\infty}dt\int_0^{\infty}\frac{dy}{y}e^{-y\,\mathrm{ch}\,t}(G^{(1)}+G^{(2)}+G^{(3)})\equiv V^{(1)}+V^{(2)}+V^{(3)}, \quad (\mathrm{II.2})$$

where

$$G^{(1)}=f(y,t)-f_0(y,t),$$
$$G^{(2)}=[f_0(y,t)-f(y,t)]\vartheta(R-y\,\mathrm{sh}\,t/2b), \quad (\mathrm{II.3})$$
$$G^{(3)}=f_0(y,t)\vartheta(y\,\mathrm{sh}\,t/2b-R),\quad f_0=y\sin(\mu_0 t)I_{2\nu}'(y)-\mu_0\cos(\mu_0 t)\,\mathrm{cth}\,tI_{2\nu}(y),$$

the function $f(y,t)$ is defined in (6), $\mu_0=2Z\alpha$, and f_0 is the limit of the function f as $x\to\infty$. We use below the values of the integrals (see Ref. 10 Russ, pp. 722, 962, and 967):

$$\int_0^{\infty}dy\,I_{2\nu}(y)e^{-y\,\mathrm{ch}\,t}=e^{-2\nu t}/\mathrm{sh}\,t,$$

$$\int_0^{1}\frac{dx(\cos ax-\cos a)}{(1-x^2)^{\nu+1/2}}=\frac{\sqrt{\pi}}{2}\Gamma\left(-\nu-\frac{1}{2}\right)\left[J_{-\nu-1}(a)\left(\frac{a}{2}\right)^{\nu+1}-\frac{\cos a}{\Gamma(-\nu)}\right] \quad (\mathrm{II.4})$$

and expressions obtained from (II.4) with the aid of the recurrence relations for the Bessel functions. Integrating $G^{(1)}$

in (II.2) with respect to y and then with respect to x we obtain, using (I.14),

$$V^{(1)}=B^{(1)}=-\mu_0\sum_{l=1}^{\infty}\frac{l}{\nu}\left[\int_0^{\infty}dtJ_0(\mu_0 t)e^{-2\nu t}\left(\operatorname{cth}t-\frac{t}{\operatorname{sh}^2 t}\right)+1\right]. \tag{II.5}$$

As $r\to 0$ we have $V^{(2)}\to 0$, and the contribution of this term to the coefficient D is determined by the region $y\ll 1$. We expand the Bessel functions and integrate with respect to y and x, retaining in the sum over l the term with $l=1$, which of highest order in r in this expansion. We obtain for $D^{(2)}$

$$D^{(2)}=-\frac{8}{\pi^{1/2}\nu_1^2}\frac{\Gamma(3/2-\nu_1)}{(4\nu_1^2-1)\Gamma(2\nu_1+2)}$$
$$\times\int_0^{\infty}\frac{dt}{\operatorname{sh}^{2\nu_1}t}\left[\left(\frac{\mu_0 t}{2}\right)^{\nu_1+1}J_{-\nu_1}(\mu_0 t)-\frac{\sin\mu_0 t}{\Gamma(-\nu_1)}\right]. \tag{II.6}$$

The contribution of $V^{(2)}$ to F is determined by the region $x\gg 1$. We expand $f-f_0$ at $x\gg 1$ in powers of $1/x$ and replace b in the expression for $G^{(2)}$ in the argument of the ϑ function by x. Integrating with respect to x and y we obtain

$$F^{(2)}=\frac{2}{\pi}\sum_{l=1}^{\infty}\frac{l}{4\nu^2-1}\int_0^{\infty}dte^{-2\nu t}\left[\mu_0^2 t\sin\mu_0 t\left(\operatorname{cth}t+\frac{\operatorname{cth}^2 t}{2\nu}\right)\right.$$
$$\left.+(\sin\mu_0 t+\mu_0 t\cos\mu_0 t)(1+2\nu\operatorname{cth}t)\right]. \tag{II.7}$$

Only the argument of the ϑ function depends on x in $G^{(3)}$; integrating with respect to x we have

$$V^{(3)}=\frac{4}{\pi}\sum_{l=1}^{\infty}\int_0^{\infty}dt\int_1^{\infty}dv\left(1-\frac{1}{v^2}\right)^{1/2}e^{-2Rv\operatorname{cth}t}f_0\left(\frac{2Rv}{\operatorname{sh}t},t\right), \tag{II.8}$$

where we have changed over to the variable $v=y\sinh t/2R$. Recognizing that

$$\int_0^{\infty}dt\int_0^{\infty}dve^{-2Rv\operatorname{cth}t}f_0\left(\frac{2Rv}{\operatorname{sh}t},t\right)=0, \tag{II.9}$$

we can represent $V^{(3)}$ in the form:

$$V^{(3)}=\frac{4}{\pi}\sum_{l=1}^{\infty}l\int_0^{\infty}dt\left\{\int_1^{\infty}dv\left(\left(1-\frac{1}{v^2}\right)^{1/2}-1\right)-\int_0^1 dv\right\}e^{-2Rv\operatorname{cth}t}$$
$$\times\left[\Phi-\mu_0\operatorname{cth}tI_{2\nu}\left(\frac{2Rv}{\operatorname{sh}t}\right)\right],$$
$$\Phi=f_0\left(\frac{2Rv}{\operatorname{sh}t},t\right)+\mu_0\operatorname{cth}tI_{2\nu}\left(\frac{2Rv}{\operatorname{sh}t}\right). \tag{II.10}$$

The integral of Φ in (II.10) makes a contribution to D that is determined by the region $Rv/\sinh t\ll 1$ and a contribution to F (the region $v\gg 1$). These contributions are equal to

$$D^{(3)}=-\frac{8}{\pi^{1/2}\nu_1^2}\frac{\Gamma(3/2-\nu_1)}{(4\nu_1^2-1)\Gamma(2\nu_1+2)}\int_0^{\infty}\frac{dt\sin(\mu_0 t)}{\Gamma(-\nu_1)(\operatorname{sh}t)^{2\nu_1}},$$

$$F_1^{(3)}=\frac{2}{\pi}\sum_{l=1}^{\infty}\frac{l}{4\nu^2-1}\int_0^{\infty}dte^{-2\nu t}\left[\mu_0(\cos\mu_0 t-1)\left(\operatorname{cth}t+\frac{\operatorname{cth}^2 t}{2\nu}\right)\right.$$
$$\left.-\sin\mu_0 t(1+2\nu\operatorname{cth}t)\right]. \tag{II.11}$$

To calculate the asymptotic forms of the terms in the square brackets in (II.10) as $r\to 0$ it is convenient to match the values of t in the integral with respect to v from zero to unity and the values of v in the integral with respect to v from unity to infinity. We obtain ultimately for the contribution of this term:

$$B^{(3)}=\mu_0\sum_{l=1}^{\infty}\frac{l}{\nu},\quad F_2^{(3)}=-\frac{\mu_0}{2\pi}\sum_{l=1}^{\infty}\frac{l}{\nu^2}. \tag{II.12}$$

Summing now the individual contributions, we obtain $B=B^{(1)}+B^{(3)}$ and $D=D^{(2)}+D^{(3)}$. In the expression for $F=F^{(2)}+F_1^{(3)}+F_2^{(3)}$ we can integrate with respect to t by using Ref. 10 (Russ. p 376)

$$\int_0^{\infty}dte^{-2\nu t}\left(\frac{1}{t}-\operatorname{cth}t\right)=\psi(\nu)-\ln(\nu)+\frac{1}{2\nu}. \tag{II.13}$$

To obtain (16) and (17) we must subtract from the obtained expression the term linear in $Z\alpha$ and add the asymptotic form of $\varphi_1(r)$ (12) as $r\to 0$

$$\varphi_1(r)\approx-\frac{meZ\alpha}{\pi}\left[\frac{2}{3R}\left(\ln\frac{1}{R}-C-\frac{5}{6}\right)+\frac{\pi}{2}-R\right],\ R\ll 1. \tag{II.14}$$

The constant in the terms proportional to R in (II.14) cancel out exactly the subtraction terms for $B+FR$. As for the term $DR^{2\nu_1}$, its part linear in $Z\alpha$ takes the form D_1R^2, and it was not subtracted explicitly in (17), for this would be an exaggeration of the accuracy of this formula.

We proceed now to calculate the coefficient A (16). As indicated in the text, it is convenient to calculate it from expression (13) for $\rho_2(\beta)$, by taking the limit in the latter as $\beta\to\infty$. The value of Ω was obtained in Appendix I, where it was also shown that the contribution of the integral with $g(y,t)$ to (13) is cancelled by the last term in the expression for Ω. The contribution to the remaining integral as $\beta\to\infty$ should come from the region $x\gg 1$. Taking into account the relation (II.9), satisfaction of which ensures in fact the convergence of the integral with respect to x in (13), we can then represent this contribution in the form

$$A_1=\frac{2}{\pi}\sum_{l=1}^{\infty}l\int_0^{\infty}dx\int_0^{\infty}dt\int_0^{\infty}\frac{dy}{y}e^{-y\operatorname{ch}t}f_0(y,t)$$
$$\times\left[\sin\left(\frac{y\operatorname{sh}t}{x}\right)-y\operatorname{sh}t\sin\left(\frac{1}{x}\right)\right]. \tag{II.15}$$

We have again left out here f_1 and shall subtract the terms linear in $Z\alpha$ at the end. The integral with respect to x in (II.15) is evaluated in the form

$$\int_0^{\infty}dx\left[\sin\left(\frac{a}{x}\right)-a\sin\left(\frac{1}{x}\right)\right]=a\ln\frac{1}{a},\quad a=y\operatorname{sh}t. \tag{II.16}$$

The expression $\sinh[t\exp(-y\cosh t)]f_0(y,t)$ can be rewritten in the form

$$\operatorname{sh}te^{-y\operatorname{ch}t}f_0(y,t)=\frac{d}{dy}(\operatorname{sh}t\sin(\mu_0 t)yI_{2\nu}(y)e^{-y\operatorname{ch}t})$$

$$-\frac{d}{dt}(\operatorname{ch} t \sin \mu_0 t I_{2\nu}(y) e^{-y \operatorname{ch} t}). \qquad \text{(II.17)}$$

Next, integrating by parts and using (II.4) and (II.13), we obtain

$$A_1 = -\frac{4}{\pi} \sum_{l=1} l \operatorname{Im}[(\nu - iZ\alpha)\psi(\nu - iZ\alpha) + iZ\alpha\psi(l) + iZ\alpha l\psi'(l)]; \qquad \text{(II.18)}$$

the terms linear in $Z\alpha$ have already been subtracted here. Adding A_1 to the remaining contributions we arrive at A of Eq. (17). We emphasize that the divergence of the individual contributions to the considered coefficients in the summation over l is fictitious and drops out from the final equations.

[1] We put $\hbar = c = 1$.

[2] The results that foll are valid at $Z\alpha < 1$. The case $Z\alpha > 1$ calls for a special analysis; it becomes necessary, in particular, to take into account the finite dimensions of the nucleus.

[3] It can be assumed that the contour of interation with respect to s in (I.6) and accordingly in (I.7) does not go through zero, and the bypassing method turns out hereafter to be immaterial.

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Translated by J. G. Adashko