New cases of integrability of the Landau-Lifshitz equations

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A method is proposed for constructing multiparameter potentials that lead to complete integrability of the stationary Landau-Lifshitz equations. The singled-out class of potentials corresponds to a magnetic-medium anisotropy represented not only by a quadratic form, but also by forms of higher order. The first two integrals of the Landau-Lifshitz equations are obtained in an explicit form that makes possible a complete description of the change of the equilibrium states and of the structure of domain walls, with account taken of two angular degrees of freedom of the magnetic moment as a function of the structural parameters of the model.

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1. Progress in the theory of domain walls in magnetic media, as well as in the theory of magnetic phase transition, makes it necessary to find solutions of the Landau-Lifshitz equations for the case of an anisotropy energy represented not only by a quadratic form of the moment **m**, but also by forms of higher order. For example, for an anisotropy energy in the form

$$U=\sum \alpha_{ij}(C)\, m_i m_j+\sum \beta_{ijkl}(C)\, m_i m_j m_k m_l+\ldots \quad (1.1)$$

great interest attaches to the solution of the problem of how the structure and classification of the domain walls varies with the parameter C (which is generally speaking vectorlike) that determines the anisotropy tensors α, β,... . A change in the parameter C can cause a change of both the number and the type of equilibrium states (of the singular points of the Landau-Lifshitz equations) and an associated change in the domain walls. A complete classification of domain walls should be accompanied by an analysis of the solutions of the Landau-Lifshitz equations with account taken of the two degrees of freedom of the magnetic-moment unit vector. As a rule, such a problem cannot be solved exactly for an anisotropy energy of the type (1.1). The known cases of complete integrability of the Landau-Lifshitz equations with account taken of two degrees of freedom correspond to a quadratic form of the anisotropy energy. In the more general case, even if exact solutions are known, they make allowance for only one degree of freedom (e.g., domain walls with rotation of the magnetic moment in a definite plane).

It is therefore of interest to search for cases of complete integrability of the Landau-Lifshitz equations for an anisotropy energy in the form (1.1). Such examples can serve as the basis for the construction of models of magnetic phase transitions that admit of the solution of the problem of the complete classification of the domain walls.

We propose below a method of finding the solutions of the Landau-Lifshitz equations

$$\mathbf{m}\times[d^2\mathbf{m}/dx^2+\partial U/\partial \mathbf{m}]=0, \quad (\mathbf{m}, \mathbf{m})=1, \quad (1.2)$$

which entails the construction of a definite class of potentials of the form (1.1) and leads to a complete integrability of the Landau-Lifshitz equations (1.2). The latter means that Eqs. (1.2) lead, for a singled-out class of potentials $U(\mathbf{m})$, to two independent first integrals, which will in fact be derived below. It will be shown furthermore that the singled-out class of potentials, which depends on arbitrary parameters, will make it possible to determine of how both the type and the number of the equilibrium states (singular points) vary.

It was noted earlier[1] that the problem of integrability of the Landau-Lifshitz equation (12) is connected with the integrability of the problem of the motion of a material point on the a unit-sphere surface in a potential field of forces. A solution for a potential that is a homogeneous quadratic form was obtained by Neumann in 1859.[2,3] The main significance of Neumann's generalized problem is that the problem remains completely integrable in the case of multiparameter potentials. This uncovers a possibility of analyzing weak bifurcations of the solutions in complete integrable models of various physical problems.

For example, in the theory of magnetic phase transitions one determines critical values of the parameter C such that new types of domain (interphase) walls are produced. The mathematical representation of a domain (interphase) wall is a separatrix common to a pair of equilibrium states, and a mathematical representation of the nucleation center is a separatrix loop that is closed on one of the equilibrium states. All types of domain (interphase) walls that can exist at a given value of the parameter C correspond to a so-called separatrix contour made up of separatrices that are common to different pairs of equilibrium states. Corresponding to a magnetic phase transition (relative to the parameter C) is a restructuring (bifurcation) of the separatrix contour, due to the decay of the old separatrices and creation of new ones that go from equilibrium state to another.

In the general case, the Landau-Lifshitz equations (1.2) admit of the existence of only one first integral connected with a translation group, while the separatrix contour that determines the admissible types of domain walls is made up of separatrix curves that join equilibrium states having one and the same constant of the only first integral.

For the class of multiparameter potentials obtained below, the Landau-Lifshitz equations (1.2) admit of the existence of two independent first integrals. In such cases the separatrix contour is made up of separatrices that connect different pairs of equilibrium states, for which the constants of the two first integrals should coincide. Thus, in the general case there appears for the class of integrable potentials an additional constraint on the separatrices that connect dif-

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ferent equilibrium-state pairs, and consequently on the admissible domain-wall types.

The examples of the integrable potentials considered below points to the existence of completely integrable problems in which the presence of two first integrals does not lead to additional constraints when the separatrix contour is constructed. In such cases, the elements of the separatrix contour are separatrices that correspond to degenerate solutions with one angular degree of freedom and are due to the fact that the Landau-Lifshitz equations have a discrete symmetry group. In these particular cases the classification of the admissible domain walls does not depend essentially on the integrability of the problem. In addition, we present here an example of a separatrix contour for which, both in the construction of the constituent separatrices and in the investigation of the investigation of the bifurcations of this contour, essential use is made of the presence of the first two integrals.

We note that a complete classification of domain (interphase) walls and nucleation centers is possible only in the case of integrable potentials, for otherwise the construction of the separatrix contours inevitably calls for a numerical analysis of the problem. It must also be emphasized that in the case of equilibrium states of the saddle-saddle type the integrable potentials are characterized by the presence of a continuous set of separatrices, of which only a finite number remains on going over to a nonintegrable potential. An enumerable set of separatrices is then additionally obtained, and these have already a more complicated internal structure. An example of such a restructuring is given in Ref. 4, where the transition from an integrable to a nonintegrable potential of the Neumann problem was investigated.

2. In a spherical coordinate frame, the Landau-Lifshitz equations (1.2) correspond to a system of canonical variables

$$\frac{dp_\theta}{dx}=-\frac{\partial H}{\partial\theta},\quad \frac{d\theta}{dx}=\frac{\partial H}{\partial p_\theta},\qquad \frac{dp_\varphi}{dx}=-\frac{\partial H}{\partial\varphi},\quad \frac{d\varphi}{dx}=\frac{\partial H}{\partial p_\varphi} \tag{2.1}$$

with a Hamiltonian in the form

$$H=\frac{1}{2}p_\theta^2+\frac{1}{2}\frac{p_\varphi^2}{\sin^2\theta}+U(\theta,\varphi). \tag{2.2}$$

Here θ and φ are the azimuthal and polar angles, while p_θ and p_φ are the corresponding canonical momenta.

We transform to sphero-conical coordinates u_1 and u_2, defined on the unit-sphere surface by the relations[5]

$$\sin^2\theta\cos^2\varphi=(1+\varepsilon+u_1)(1+\varepsilon+u_2)/\varepsilon(1+\varepsilon),$$
$$\sin^2\theta\sin^2\varphi=-(1+u_1)(1+u_2)/\varepsilon,\quad \cos^2\theta=u_1u_2/(1+\varepsilon) \tag{2.3}$$

or by relations of the type

$$u_1u_2=(1+\varepsilon)\cos^2\theta,\quad u_1+u_2=(1+\varepsilon\cos^2\varphi)\sin^2\theta-(2+\varepsilon), \tag{2.4}$$

where $\varepsilon>0$ is a parameter of the sphero-conical coordinate frame. The range of variation of the u_1 and u_2 is defined by the inequalities

$$-1-\varepsilon<u_1<-1<u_2<0 \tag{2.5}$$

and is represented in Fig. 1a by the rectangular region *abcd*. The metric on the unit-sphere surface is defined in terms of the orthogonal variables u_1 and u_2 by the expression[3,5]

$$ds^2=g_1(u_1,u_2)(du_1)^2+g_2(u_1,u_2)(du_2)^2. \tag{2.6}$$

Here

$$g_1(u_1,u_2)=-(u_1-u_2)/4f(u_1),\quad g_2(u_1,u_2)=-(u_2-u_1)/4f(u_2),$$
$$f(z)=(z+1+\varepsilon)(z+1)z,\quad f(u_1)>0,\ f(u_2)<0. \tag{2.7}$$

In the new variables, the Hamiltonian takes the form

$$H=\frac{1}{2}\left\{\frac{4f(u_1)}{u_2-u_1}p_1^2-\frac{4f(u_2)}{u_2-u_1}p_2^2\right\}+U(u_1,u_2). \tag{2.8}$$

Here

$$p_1=g_1(u_1,u_2)du_1/dx,\quad p_2=g_2(u_1,u_2)du_2/dx \tag{2.9}$$

are the momenta conjugate to the coordinates u_1 and u_2.

To transform the Hamiltonian (2.8) to the Liouville form[6] we represent the potential in the form

$$U(u_1,u_2)=\frac{1}{2}\sum_{n\geq1}C_nU^{(n)}(u_1,u_2). \tag{2.10}$$

Here

$$-U^{(n)}(u_1,u_2)=u_1^n+u_1^{n-1}u_2+\ldots+u_1u_2^{n-1}+u_2^n \tag{2.11}$$

is a homogeneous polynomial of degree n and C_n are arbitrary constants. The singled-out class of potentials (2.10) satisfies the relation

$$(u_2-u_1)U(u_1,u_2)=\frac{1}{2}\sum_{n\geq1}C_n(u_2^{n+1}-u_1^{n+1}). \tag{2.12}$$

When (2.12) is taken into account, the Hamiltonian (2.8) can be written in the form

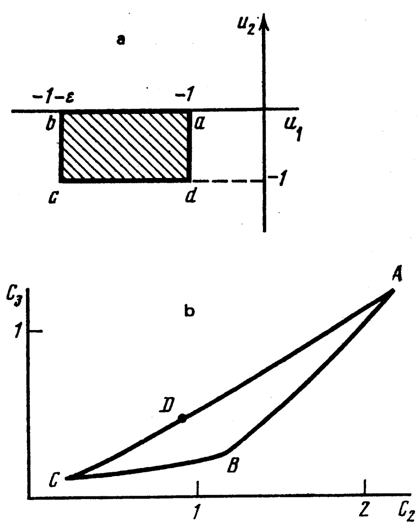


FIG. 1.

$$H=[H_1(p_1, u_1)+H_2(p_2, u_2)]/(u_2-u_1),$$
$$H_1(p_1,u_1)=\frac{1}{2}\left\{4f(u_1)p_1^2-\sum_{n\geq 1}C_n u_1^{n+1}\right\}, \tag{2.13}$$
$$H_2(p_2,u_2)=\frac{1}{2}\left\{-4f(u_2)p_2^2+\sum_{n\geq 1}C_n u_2^{n+1}\right\},$$

which indicates explicitly the separation of the variables in the corresponding Hamilton-Jacobi equation.[3,6]

Thus, the class of potentials (2.10), (2.11) corresponds to a fully integrable problem. One first integral is the Hamiltonian (2.8). It can be shown (see the Appendix) that the second independent first integral for the class of potentials (2.10), (2.11) is given by the expression

$$Q=\frac{1}{2}\left\{\frac{4u_2 f(u_1)}{u_2-u_1}p_1^2-\frac{4u_1 f(u_2)}{u_2-u_1}p_2^2\right\}$$
$$-\frac{u_1u_2}{2}\sum_{n\geq 1}C_n U^{(n-1)}(u_1,u_2). \tag{2.14}$$

In the simplest case ($C_1 = 1$, $C_n = 0$ for $n > 1$) the first integrals (2.8) and (2.14) coincide with the first integrals of the aforementioned Neumann problem. An essential aspect of the generalization of the Neumann problem is that the singled-out class of "integrable" potentials (2.10) and (2.11) depends on arbitrary constants..., C_n, ... Another representation of the class of integrable potentials of the problem considered, defined accurate to an arbitrary function, is given in the Appendix.

The expressions for the first three potentials (2.11), which lead to complete integrability of the Landau-Lifshitz equations, are of the form (apart from inessential constants)

$$U^{(1)}=(\mathbf{m}, A_\varepsilon\mathbf{m}),$$
$$U^{(2)}=-(\mathbf{m}, A_\varepsilon\mathbf{m})^2-2(2+\varepsilon)(\mathbf{m}, A_\varepsilon\mathbf{m})+(1+\varepsilon)(\mathbf{m}, A_0\mathbf{m}),$$
$$\begin{aligned}U^{(3)}=&(\mathbf{m}, A_\varepsilon\mathbf{m})^3+3(2+\varepsilon)(\mathbf{m}, A_\varepsilon\mathbf{m})^2\\&-2(1+\varepsilon)(\mathbf{m}, A_\varepsilon\mathbf{m})(\mathbf{m}, A_0\mathbf{m})+(10+10\varepsilon+3\varepsilon^2)\\&\times(\mathbf{m}, A_\varepsilon\mathbf{m})-2(2+\varepsilon)(1+\varepsilon)(\mathbf{m}, A_0\mathbf{m}).\end{aligned} \tag{2.15}$$

We have introduced here for brevity the diagonal matrix

$$A_\varepsilon=-\mathrm{diag}(1+\varepsilon, 1, 0). \tag{2.16}$$

Recall that a sum of potentials (2.15) of the form

$$2U=U^{(1)}+C_2U^{(2)}+C_3U^{(3)} \tag{2.17}$$

with arbitrary parameters C_2 and C_3 also leads to complete integrability of the Landau-Lifshitz equations. Corresponding to the potential $U^{(1)}$ is a quadratic form of the anisotropy energy with orthorhombic symmetry. It was shown earlier[1] that for the case of the potential $1/2U^{(1)}$ the supplementary first integral can be written in the form

$$Q=\mu_z^2-\varepsilon\mu_x^2+\varepsilon m_y^2. \tag{2.18}$$

Here $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$ is the "kinetic" moment of the magnetic-moment vector

$$\boldsymbol{\mu}=[\mathbf{m}\times\mathbf{m}'].$$

Following Ref. 1, we can show that a generalization of (2.18) to the case, say, of the potential $1/2U^{(1)} + C_2U^{(2)}$ is the expression

$$Q=\mu_z^2-\varepsilon\mu_x^2+\varepsilon m_y^2-\varepsilon C_2[(3+\varepsilon)m_y^2-m_y^4-(1+\varepsilon)m_x^2m_y^2]. \tag{2.19}$$

We note that the derivation of the expresions for the supplementary first integral (2.18) or (2.19) in the ($\boldsymbol{\mu}$,$\mathbf{m}$) representation entails only algebraic operations and does not involve the choice of any particular coordinate frame on the sphere. In particular, expresions (2.18) and (2.19) are valid at any choice of the sign of the parameter ε.

3. We shall show that the obtained class of "integrable" potentials (2.10) and (2.11) makes it possible to investigate the bifurcation of the solutions of the Landau-Lifshitz equations, which is connected with the creation (or annihilation) of certain equilibrium states when the parameters (...C_n...) change.

We consider a two-parameter potential (2.17) in the variables u_1 and u_2:

$$\begin{aligned}U=-{}^1/_2\{&(u_1+u_2)+C_2(u_1^2+u_1u_2+u_2^2)\\&+C_3(u_1^3+u_1^2u_2+u_1u_2^2+u_2^3)\}.\end{aligned} \tag{3.1}$$

At all values of the parameters C_2 and C_3 (including $C_2 = C_3 = 0$), the Landau-Lifshitz equations have equilibrium states (singular points) corresponding to the values of the angle

$$\theta=0,\ \pi/2,\ \pi, \tag{3.2}$$

but the type of the singular point can vary, depending on the values of the parameters C_2 and C_3. Moreover, new equilibrium states are produced in a certain region of values of C_2 and C_3. Indeed, the equilibrium conditions

$$\partial U/\partial u_1=\partial U/\partial u_2=0 \tag{3.3}$$

for the potential (3.1) lead to the relations

$$\tilde{u}_1+\tilde{u}_2=-C_2/2C_3,\quad \tilde{u}_1\tilde{u}_2=(4C_3-C_2^2)/8C_3^2, \tag{3.4}$$

which determine, according to (2.4), the equilibrium values of the angles (θ,φ) in a spherical coordinate system. By virtue of relations (3.4) and (2.3), each new equilibrium state on the (u_1, u_2) plane

$$\tilde{u}_1(C_2,\ C_3),\ \tilde{u}_2(C_2,\ C_3), \tag{3.5}$$

corresponds on the unit sphere to eight equilibrium points (four points located on the upper and lower hemisphere symmetrically about the equatorial plane), defined by the relations

$$\begin{aligned}&\cos^2\theta=(4C_3-C_2^2)/8(1+\varepsilon)C_3^2,\\&\varepsilon\cos^2\varphi=[8(1+\varepsilon)^2C_3^2-4(1+\varepsilon)C_2C_3-C_2^2+4C_3]\\&\qquad\times[8(1+\varepsilon)C_3^2+C_2^2-4C_3]^{-1}.\end{aligned} \tag{3.6}$$

On the plane of the plane of the parameters C_2 and C_3, the new equilibrium states are produced in the region bounded by the contour *ABCD* (see Fig. 1b). This contour is the mapping of the contour *abcd* (Fig. 1a) on the plane of the parameters C_2 and C_3 specified by relations (3.4). We present expressions for the coordinates of the points of the contour *ABCD*:

$$A(C_2=2,\ C_3=1),\ B(C_2={}^2/_3,\ C_3={}^1/_6),$$

$$C\left(C_2=\frac{2(2+\varepsilon)}{6+6\varepsilon+\varepsilon^2},\ C_3=\frac{1}{6+6\varepsilon+\varepsilon^2}\right),$$

$$D\left(C_2=\frac{2}{1+\varepsilon},\ C_3=\frac{1}{(1+\varepsilon)^2}\right). \qquad (3.7)$$

We note that to the only point C of the contour $ABCD$ there corresponds an angle $\theta = 0$ or π if $C_2 \neq 0$ simultaneously. We recall that at each point located inside the contour $ABCD$ there exist, besides the new equilibrium states (3.5), the old equilibrium states (3.2).

For the new equilibrium states (3.5), the constant of the new integral (2.8) is

$$H(\tilde{u}_1,\ \tilde{u}_2)=U(u_1,\ u_2)=-{}^1/_2C_2(4C_3-C_2^2)/8C_3^2, \qquad (3.8)$$

whereas for the old equilibrium states (3.2) we have

$$H(u_1=-1-\varepsilon,\ u_2=-1)=U(-1-\varepsilon,\ -1)$$
$$=-{}^1/_2\{(2+\varepsilon)+(2+\varepsilon)(2+2\varepsilon+\varepsilon^2)C_3-(3+3\varepsilon+\varepsilon^2)C_2\}. \qquad (3.9)$$

It is easy to verify that for the point C of the contour $ABCD$ the constants (3.8) and (3.9) coincide.

Besides the equilibrium state (3.5), at definite values of the parameters C_2 and C_3 there arise additional equilibrium states for which $\theta = \pi/2$ and the quanitity $X = 1 + \varepsilon \cos^2 \varphi$, which defines the polar angle φ, satisfies the quadratic equation

$$-3C_3X^2-2[C_2-3(2+\varepsilon)C_3]X+[-1+2(2+\varepsilon)C_2-3(2+\varepsilon)^2C_3]$$
$$=0. \qquad (3.10)$$

Finally, there are regions of the parameters C_2 and C_3 in which the additional equilibrium states are defined by the conditions

$$\varphi_k=k\pi/2,\ k=0,\ 1,\ 2,\ 3,$$

and the quantity $Y = \sin^2 \theta$ satisfies the quadratic equation

$$-3C_3X^3Y^2-2[C_2X^2-3(2+\varepsilon)C_3X^2+2(1+\varepsilon)C_3X]\,Y$$
$$+[-X+2(2+\varepsilon)C_2X-(1+\varepsilon)C_2-(10+10\varepsilon+3\varepsilon^2)C_3X$$
$$+2(1+\varepsilon)(2+\varepsilon)C_3]=0. \qquad (3.11)$$

4. We proceed now to investigate the separatrix contours that arise for a potential in the form

$$U={}^1/_2U^{(1)}+{}^1/_2C_2U^{(2)}. \qquad (4.1)$$

In this case, besides the equilibrium states

$$I_x=(1,0,0),\ I_y=(0,1,0),\ I_z=(0,0,1), \qquad (4.2)$$

that exist at all values of the parameters ε and C_2, there can arise additional equilibrium states. Thus, on the equator $(\theta = \pi/2)$ of the unit sphere there arise four equilibrium states $E_i (i = 1 \text{ to } 4)$, for which

$$\cos^2 \varphi=[2C_2(1+\varepsilon)-1]/2\varepsilon C_2. \qquad (4.3)$$

In addition, two equilibrium states $M_i (i = 1 \text{ to } 4)$ appear on each of the meridians $\varphi = 0$ and $\varphi = \pi$; for these states

$$\sin^2 \theta=[C_2(3+2\varepsilon)-1]/2C_2(1+\varepsilon), \qquad (4.4)$$

Finally, on the meridians $\varphi = \pi/2$ and $\varphi = -\pi/2$ there appear equilibrium states $N_i (i = 1 \text{ to } 4)$ for which

$$\sin^2 \theta=[C_2(3+\varepsilon)-1]/2C_2. \qquad (4.5)$$

The cases considered above cover all the equilibrium states that can arise for a potential defined by Eq. (4.1). Bifurcations of these equilibrium states are listed in the table. It is assumed in this table that $0 < \varepsilon < 1$. For the case $\varepsilon > 1$ it is necessary to interchange rows 3 and 4 as well as 5 and 6. We note that all four equilibrium states E_i (in analogy with M_i and N_i) correspond to one and the same set of characteristic exponents, and consequently pertain to one and the same type.

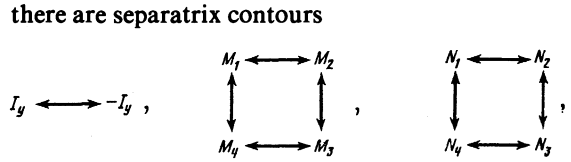
It can be seen from the table that the change of the singular points $\pm I_x$, $\pm I_y$, and $\pm I_z$ is due to the splitting from them, or to merging with them, of equatorial (E_i) or meridional (M_i, N_i) groups of equilibrium states. Thus, for ecample when the line $C_2 = 1/(3 + 2\varepsilon)$ passes from left to right the singular points $\pm I_z$ becomes centers in the m_x direction (plane m_x, m'_x of the phase space) and a saddle point in the direction of m_y (plane m_y, m'_y). In this case four new equilibrium states M_i split away from them and are saddle points in both the θ and φ direction. The remaining equilibrium states remain of the same type.

We consider now some of the resultant separatrix contours. Analysis shows that in the case of a potential of the type (4.1) the separatrix can connect only related equilibrium states, i.e., equilibrium states belonging to one of the six groups in the table. We recall that among the solutions of the Landau-Lifshitz equations (1.2) there are degenerate solutions corresponding to motion with one angular degree of freedom, namely solutions corresponding to motion with either $\theta = \pi/2$ or $\varphi = k\pi/2 (k = 0,1,2,3)$. In the case of a potential of type (4.2), these trajectories play an important role in the formation of the separatrix contours.

Thus, for example, in the parameter ranges

$$0<\varepsilon<1,\ 1/(3+\varepsilon)<C_2<1/2(1+\varepsilon) \qquad (4.6)$$

there are separatrix contours



(4.7)

where the separatrices with $\theta \equiv \pi/2$ correspond to the first contour, and those with $\varphi \equiv 0$, π and $\varphi \equiv \pm\pi/2$ respectively to the two others. In addition, there is also a continuous set of the separatrices near the singular points M_i. Separatrix contours are similarly constructed also in other region of values of the parameters ε and C_1. We emphasize that in the foregoing analysis of the separatrix contours we used nowhere the fact that there exists an additional first integral Q (2.19), and the existence of three separatrices was ensured only by the presence of a discrete symmetry group.

We consider now a case when the potential U is given by

$$U=-{}^1/_2U^{(1)}-{}^1/_2C_2U^{(2)}. \qquad (4.8)$$

In this case, to obtain a table of the bifurcations of the singu-

TABLE I. Bifurcation of the types of singular points for the potential (4.1)

		$\pm I_x$	$\pm I_y$	$\pm I_z$	E_i	M_i	N_i
1	$C_2=-\infty$	center φ – center θ	saddle φ – center θ	saddle m_x – saddle m_y	no	no	no
2	$C_2=\frac{1}{3+2\varepsilon}$	–	–	center m_x – saddle m_y	–	split-off from $\pm I_z$ of type saddle θ–saddle φ	–
3	$C_2=\frac{1}{3+\varepsilon}$	–	–	center m_x – center m_y	–	–	split off from $\pm I_z$ of type saddle θ–center φ
4	$C_2=\frac{1}{2(2+\varepsilon)}$	–	center φ – center θ	–	split off from $\pm I_y$ of type saddle φ–center θ	–	–
5	$C_2=\frac{1}{2}$	saddle φ – center θ	–	–	merge with $\pm I_x$	–	–
6	$C_2=\frac{1}{1+\varepsilon}$	–	center φ – saddle θ	–	–	–	merge with $\pm I_y$
7	$C_2=1$	saddle φ – saddle θ	–	–	–	merge with $\pm I_x$	–

lar point it suffices to interchange in our table the words "saddle point" and "center."

Now transitions become possible not only between related states. But this requires that the values of both first integrals H (2.2) and Q (2.19) for the corresponding equilibrium states coincide (this requirement is always satisfied for related equilibrium states). Thus, for example, on the line

$$C_2=1/(2+\varepsilon) \tag{4.9}$$

the values of the integral H and the values of the integral Q are pairwise equal for all three groups of equilibrium states $\pm I_x$, $\pm I_y$, and $\pm I_z$, and the separatrix contour shown in Fig. 2 appears on the line (4.9).

We consider now the equilibrium states I_x and N_i. The first integrals H for these states coincide on the line

$$(1-6\varepsilon-3\varepsilon^2)C_2^2-2(1-\varepsilon)C_2+1=0, \tag{4.10}$$

whereas the supplementary first integral Q coincides only at the instant when the equilibrium constant N_i break away from the states $\pm I_z$ (i.e., when the N_i coincide with $\pm I_z$). Thus the supplementary first integral Q excludes the existence of separatrices that connect groups of equilibrium states $\pm I_x$ and N_i.

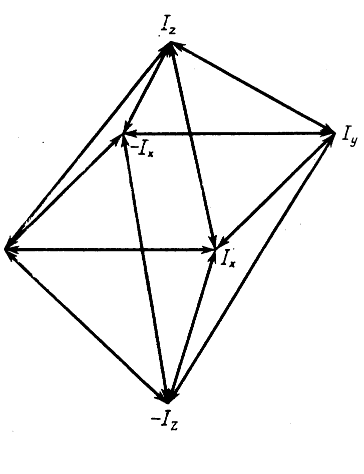


FIG. 2.

We present now an example in which the construction of the separatrices is substantially connected with the complete integrability of the Landau-Lifshitz equations. To this end we consider again an integrable potential such as (4.1). In this case, in the region

$$1/(3+2\varepsilon)<C_2<1 \tag{4.11}$$

the equilibrijm states M_i are equilibrium states of the saddle-saddle type. In the analysis of the Landau-Lifshitz equations[1] for the case $C_n=0$, $n\geqslant 2$, we have noted that the solutions corresponding to Bloch and Néel walls (to separatrices that connect the equilibrium states I_z and $-I_z$) acquire a functional dependence of the first integrals H and Q. Next, in the papers of Lerman and Umanskiĭ, devoted to topological structures of integrable Hamiltonian systems with two degrees of freedom, it was proved in particular that in the general case, in the vicinity of the singular point of the saddle-saddle type, there are exactly two one-dimensional integral manifolds on which the first integrals are dependent, i.e.,

$$\text{grad } H=\rho \text{ grad } Q, \tag{4.12}$$

where ρ is a function defined in phase space. These manifolds correspond to trajectories that emerge from the equilibrium state M_i. There are analogously two manifolds corresponding trajectories that enter the equilibrium state M_i.

In our case it is convenient to employ the Lerman-Umanskiĭ theorem using the variables u_1 and u_2 in terms of which the first integrals H and Q are specified respectively by Eqs. (2.2) and (2.14). It is easy to show that at $C_2\neq 1/3$ the integral manifolds indicated above are singled out by the respective conditions $p_1=0$ and $p_2=0$. Consequently at $C_2\neq 1/3$ there exist separatrices that connect equilibrium states for which either u_1 or u_2 is constant. In the variables θ and φ these separatrices are described by the equation

$$\frac{(1-C_2)^2}{4C_2^2}+\frac{1-C_2}{2C_2}[(1+\varepsilon\cos^2\varphi)\sin^2\theta-2-\varepsilon]+(1+\varepsilon)\cos^2\theta=0. \tag{4.13}$$

We note that at $C_2 < 1/3$ the equilibrium states M_i are located on the side cd (see Fig. 1a), while at $C_2 > 1/3$ they are on the side da. Finally at $C = 1/3$ they are at the point d.

The separatrix contour becomes restructured on going through the critical parameter value $C_1 = 1/3$. This bifurcation is shown in Fig. 3. Figure 4 shows for the Landau-Lifshitz equations the solutions corresponding to these two cases. The presence of first two integrals enables us to determine the values of φ^* and θ^* (see Fig. 4):

$$\sin^2\varphi^*=(1-3C_2)/2\varepsilon C_2 \quad \left(C_2<\frac{1}{3}\right);$$

$$\sin^2\theta^*=(3C_2-1)/2C_2 \quad \left(C_2>\frac{1}{3}\right).$$

Corresponding to the separatrix solutions shown in Fig. 4 are domain walls of a new type, which are closely tied to the rotation of the magnetic-moment vector relative to the two angle variables.

The examples considered above demonstrate the peculiarities that arise in the analysis of separatrix contours in integrable systems. We note that a situation is also possible wherein, for two equilibrium states, the line of equal values of the integral H cross, in the space of the parameters ε and C_2, the line of equal values of the integral Q in one or several points. Then the presence of the supplementary first integral Q imposes only a partial hindrance on the existence of separatrix contours, i.e., it makes possible the existence of separatrix contours only at individual points of the space of the parameters ε and C_2 (or on a line, if we consider the three-dimensional space of the parameters ε, C_2, and C_3).

In conclusion, the authors are deeply grateful to L. M. Lerman and Ya. L. Umanskiĭ for acquainting them with the results of their research and for helpful discussions.

APPENDIX

We consider a dynamic system with two degrees of freedom and with a Hamiltonian in the form

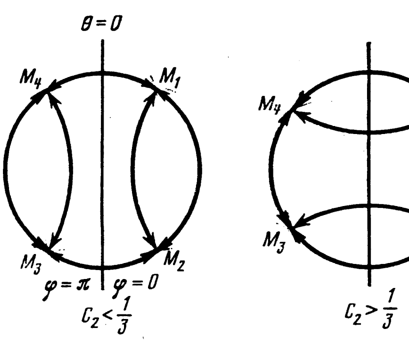


FIG. 3.

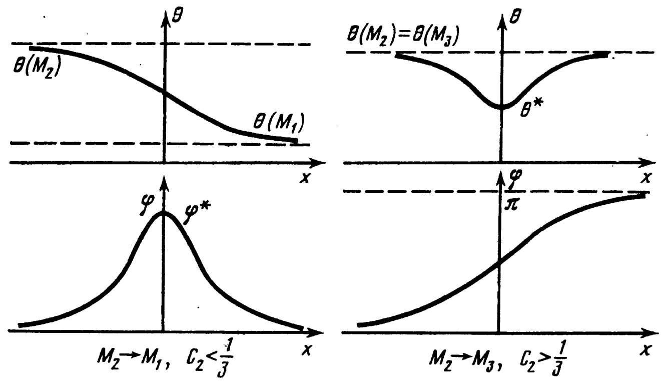


FIG. 4.

$$H={}^1/_2G_1(q_1,\ q_2)p_1^2+{}^1/_2G_2(q_1,\ q_2)p_2^2+U(q_1,\ q_2). \tag{A1}$$

The system (A1) has an obvious first integral

$$H(p_1,\ q_1;\ p_2,\ q_2)=\text{const}. \tag{A2}$$

Assume that the system is completely integrable, and let the supplementary first integral be of the form

$$Q=a_1(q_1,\ q_2)p_1^2+a_2(q_1,\ q_2)p_2^2+V(q_1,\ q_2). \tag{A3}$$

The condition that the Poisson brackets of the first integrals (A1) and (A3) vanish leads to the relations

$$G_1\frac{\partial a_1}{\partial q_1}-\frac{\partial G_1}{\partial q_1}a_1=0,\quad G_2\frac{\partial a_2}{\partial q_2}-\frac{\partial G_2}{\partial q_2}a_2=0,$$
$$G_2\frac{\partial a_1}{\partial q_1}-\frac{\partial G_1}{\partial q_2}a_2=0,\quad G_1\frac{\partial a_2}{\partial q_1}-\frac{\partial G_2}{\partial q_1}a_1=0, \tag{A4}$$
$$G_1\frac{\partial V}{\partial q_1}=2a_1\frac{\partial U}{\partial q_1},\quad G_2\frac{\partial V}{\partial q_2}=2a_2\frac{\partial U}{\partial q_2}.$$

The first four states in (A4) establish the correspondence between the metric coefficients $G1$, $G2$ and $a1$, $a2$ of the first integrals (A1) and (A3), while the last two establish the correspondence between the conjugate potentials U and V. Assuming

$$a_1=A_1G_1,\ a_2=A_2G_2, \tag{A5}$$

we find that by virtue of the first relations of (A4)

$$A_1=A_1(q_2),\ A_2=A_2(q_1), \tag{A6}$$

$$G_1(q_1,q_2)=\frac{f_1(q_1)}{A_1(q_2)-A_2(q_1)},\quad G_2(q_1,q_2)=\frac{f_2(q_2)}{A_1(q_2)-A_2(q_1)}. \tag{A7}$$

Here A and f are arbitrary functions of one variable. Next, according to (A6), we find that

$$a_1=\frac{A_1(q_2)f_1(q_1)}{A_1(q_2)-A_2(q_1)},\quad a_2=\frac{A_2(q_1)f_2(q_2)}{A_1(q_2)-A_2(q_1)}. \tag{A8}$$

Thus the first four of the relations (A4) determine (accurate to two functions A, f of a single variable) the structure of the metric coefficients $G1$, $G2$ and a_1, a_2. Next, the condition that the last two relations of (A4) be compatible, namely, the

Frobenius condition

$$\partial^2 V/\partial q_1\partial q_2=\partial^2 V/\partial q_2\partial q_1,$$

leads to the condition

$$\frac{\partial}{\partial q_2}\left(A_1(q_2)\frac{\partial U}{\partial q_1}\right)=\frac{\partial}{\partial q_1}\left(A_2(q_1)\frac{\partial U}{\partial q_2}\right), \qquad (A9)$$

which determines the class of potentials $U(q_1,q_2)$ in the Hamiltonian (A1) under the condition that the supplementary first integral (A3) exists.

Putting

$$U(q_1,\ q_2)=W(q_1,\ q_2)/[A_1(q_2)-A_2(q_1)], \qquad (A10)$$

we find that the function $W(q_1,q_2)$ should satisfy the equation

$$\partial^2 W/\partial q_1\partial q_2=0. \qquad (A11)$$

Consequently the potential $U(q_1,q_2)$ should take the form

$$U(q_1,\ q_2)=[W_1(q_1)+W_2(q_2)]/[A_1(q_2)-A_2(q_1)]. \qquad (A12)$$

When account is taken of (A7) and (A12), the Hamiltonian (A1) takes a form typical of systems of the Liouville type,[6] and the variables in the corresponding Hamilton-Jacobi equation are separable. The potential $V(q_1,q_2)$ and hence the supplementary first integral (A2) are determined then by solving the first-order equations

$$\frac{\partial V}{\partial q_1}=2A_1(q_2)\frac{\partial U}{\partial q_1},\quad \frac{\partial V}{\partial q_2}=2A_2(q_1)\frac{\partial U}{\partial q_2}. \qquad (A13)$$

In particular, for the Hamiltonian (2.8) and the potential (2.10) Eqs. (A13) lead to the supplementary first integral (2.14).

Putting in Eq. (A9)

$$U(q_1,\ q_2)=W(q_1,\ q_2;\ \lambda)/[A_1(q_2)-\lambda]\,[A_2(q_1)-\lambda], \qquad (A14)$$

where λ is an arbitrary parameter, we find that Eq. (A9) admits of solutions of the type

$$U(q_1,\ q_2)=\int\frac{W(\lambda)\,d\lambda}{[A_1(q_2)-\lambda][A_2(q_1)-\lambda]}. \qquad (A15)$$

Expression (A15) determines the class of potentials [defined accurate to the function $W(\lambda)$] that leads to a completely integrable dynamic system with a Hamiltonian (A1). The supplementary first integral is then

$$Q=\frac{A_1(q_2)f_1(q_1)}{A_1(q_2)-A_2(q_1)}p_1^2+\frac{A_2(q_1)f_2(q_2)}{A_1(q_2)-A_2(q_1)}p_2^2 +2\int d\lambda W(\lambda)\frac{A_1(q_2)+A_2(q_1)-\lambda}{[A_1(q_2)-\lambda][A_2(q_1)-\lambda]}. \qquad (A16)$$

We note that the sum of the potentials of the type (A12) and (A15) is also an "integrable" potential.

[1]V. M. Eleonskii, N. N. Kirova, and N. E. Kulagin, Zh. Eksp. Teor. Fiz. **77**, 409 (1979) [Sov. Phys. JETP **52**, 209 (1979)].

[2]C. Neumann, J. reine angew. Math. **55**, 46 (1859).

[3]Yu. Mozer, Usp. Mat. Nauk **36**, 109 (1981).

[4]V. M. Eleonskii, N. N. Kirova, and N. E. Kulagin, Zh. Eksp. Teor. Fiz. **75**, 2210 (1978) [Sov. Phys. JETP **48**, 1113 (1978)].

[5]Matematicheskii analiz (differentsirovanie i integrirovanie), Spravochnaya matematicheskiya biblioteka (Mathematical Analysis (Differentiation and Integration), Mathematical Handbood Library), Fizmatgiz, 1961, p. 145.

[6]J. L. Synge, Classical Dynamics [Russ. transl.] Fizmatgiz, 1963, p. 256.

[7]L. M. Lerman and Ya. L. Umanskiĭ, Abstracts, 9-th Internat. Conf. on Nonlinear Oscillations, Kiev, 1981, p. 190. L. M. Lerman and Ya. L. Umanskiĭ, Deposited Manuscript No. 342-81, VINITI, 1981.

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