Contribution of the theory of diffraction of wave beams in anisotropic media

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The diffraction of wave beams in resonant anisotropic media is investigated by analyzing the topology of the surfaces of the wave normals corresponding to possible types of normal waves in the medium. A linear differential equation of nth order is derived which describes the evolution of the slowly varying complex amplitude of an arbitrary mode to within $(\lambda/a)^n$, where λ is the wavelength of the radiation and a is a characteristic transverse dimension of the beam. Anomalous diffraction angles are predicted for the region of dipole transition frequencies in crystals and also for magnets near the magnetoacoustic resonance, with allowance for dipole interaction and spatial dispersion. For a beam traversing a plane-parallel plate the range of angles of incidence is found in which the transmitted field is related only to the response of the exciton subsystem of the crystal. An analysis of self-action effects in such media leads for the complex amplitude to a Hirota- type nonlinear equation which is a combination of a nonlinear Schrödinger equation and a complex modified Korteweg–de Vries equation.

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§1. INTRODUCTION

The problem of an approximate theoretical description of the diffraction of wave beams in anisotropic media has at present become most timely in connection with research into the linear and even nonlinear properties of crystals, in the vicinity of the absorption band, using sharply directional beams, as well as in view of the need of taking into account in this case the effects of spatial dispersion and damping. In optics, the anisotropy near the frequencies of dipole or quadrupole transitions is far from a small quantity.[1] The same holds also for the passage of acoustic beams through magnetically ordered media in the region of the magnetoacoustic-resonance frequencies; these media have, besides natural elastic anisotropy, also large magnetodipole and crystallographic anisotropy.[2] In the investigation of diffraction in anisotropic media within the framework of linear theory, it is customary to use the known Kirchoff method.[3] With an aim at generalizing the theory to include the case of nonlinear media, however, it is desirable to obtain not a complete solution in integral form, but a suitable differential equation whose solution would be a function obtainable in the corresponding approximation from the exact integral solution. Such an equation, was first obtained in the parabolic approximation in Ref. 4 for a complex envelope in the case of isotropic medium. The extensive use of this equation, especially in nonlinear optics, is due to the trailblazing papers[5,6] on the fundamentals and applications of the method of slowly varying amplitudes. For isotropic media, the use of this method in the existing form encounters considerable difficulties, for in this case the directions of the ray and wave vectors do not, generally speaking, coincide, and since the initial differential system of equations is itself quite cumbersome it is not always possible to establish in the best manner a sequence of formal small parameters for a comparison of the field components and their spatial derivatives.

A parabolic equation for a medium with the simplest uniaxial anisotropy was first obtained in Ref. 7. It did not describe, however, the obvious ellipticity that should arise upon propagation in an initially axisymmetric beam. This was achieved by another choice of the sequence of the small parameters in Ref. 8. Nonetheless, the unwieldy and, most importantly, far from obvious connection between the entire hierarchy of the introduced formal parameters and the only small parameter for the beam, $\lambda/a \ll 1$, practically excludes the possibility of investigating more complicated anisotropic media with allowance for spatial dispersion and losses.

In addition, even in the case of an isotropic medium, the usual method of obtaining the parabolic equation from the initial Helmholtz equation, if it is regarded as a first approximation, does not make it possible to construct a successive procedure for deriving the higher-approximation equations. Yet this is necessary not only for finding simple quantitative corrections, but also when resonant anisotropic media are considered. It will be shown below that in the latter there exist such special directions for which the coefficient of the second derivative, and also of the nonlinear term in the corresponding nonlinear Schrödinger equation, vanish. Clearly, the diffraction and self-action have in this case an anomalous character.

We describe in this paper a method of obtaining an approximate differential equation for the complex amplitude of the envelope (§2) when all the expansions connected with the small parameter λ/a are made after taking the Fourier transform of each normal mode in the dispersion law. The equation itself is obtained by taking the inverse transform of the dispersion relation reduced with the required accuracy. This approach makes it possible, in the presence of a small parameter, to investigate beam diffraction in any anisotropic medium with a known dispersion law. By way of example (§3) we consider the diffraction of a focused Gaussian beam in a lossy optically uniaxial crystal, obtain the beam trajec-

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tory, whose nonrectilinearity is due to the dispersion of the damping, and calculate the ellipticity parameter and the separation of the foci.

In §§4 and 5 we investigate the diffraction of a planar beam in the vicinity of an optical dipole transition in a crystal, and also at the frequencies of magnetoacoustic resonance in magnets. The fundamental role of the long-range Coulomb interaction is established for the former, and of the magnetodipole interaction for the latter. It is shown that in the region of the frequencies of the longitudinal-transverse splitting, even without allowance for the spatial dispersion, this interaction leads both to a splitting of the beam and to the existence of a certain angular region of opacity, to appearance of special directions of the anomalous diffraction, etc. In §6 are discussed effects of self-action in resonant anisotropic media. It is shown that only far from resonance these effects are well described by the usual parabolic equation with cubic nonlinearity.[5] In the vicinity of the resonance itself, on the other hand, owing to the competition between different forms of nonlinearity and the dispersion of the medium, the coefficients of this equation can vanish or reverse its sign. To describe these situations, a nonlinear equation of the Hirota type is proposed,[9] and a method of calculating its coefficients is indicated.

§2. DIFFERENTIAL EQUATION FOR THE ENVELOPE AMPLITUDE

A most important characteristic of a linear medium is its dispersion law $F(\mathbf{k},\omega)$, which determines the number and character of the normal waves. In the development and corroboration of various approximate methods of solving physical problems, it is frequently useful to investigate the topological properties of the function $F(\mathbf{k},\omega)$. From the viewpoint of the problem of the diffraction of a stationary wave beam ($\omega=$ const), the dispersion law $F(\mathbf{k},\omega)$ establishes the connection between the three components of the wave vector, two of which, for example k_x and k_y, can be regarded as real and independent. We represent the roots of the dispersion equation with respect to the third component in the explicit form

$$k_z^{(l)}=k_z^{(l)}(k_x,k_y,\omega),\qquad l=\overline{1,N}, \tag{1}$$

where the index l numbers the type of the normal waves, and the Z axis is chosen in accordance with the geometry of the problem. When considering nonequilibrium media, relations (1) become complex. In the case of an unbounded medium, if the amplitudes of all the normal waves in the $Z=0$ plane are given, the general solution of the problem can be represented in terms of the Fourier integral

$$\mathbf{E}(\mathbf{r},\omega)=\sum_{l=1}^{N}\iint_{-\infty}^{\infty}\mathbf{E}_l(k_x,k_y)\times\exp\{i[k_xx+k_yy+k_z^{(l)}(k_xk_y)z]\}\,dk_x\,dk_y, \tag{2}$$

$$\mathbf{E}_l(k_x,k_y)=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\mathbf{E}_l(\xi,\eta)\exp\{-i[k_x\xi+k_y\eta]\}\,d\xi\,d\eta. \tag{2a}$$

The topology of any of the $2N$ surfaces (1) can in the general case be very complicated, but for a beam whose transverse dimensions are much larger than the radiation wavelength, only small sections of these surfaces play a significant role in the integration in (2), and we assume these to be sufficiently smooth and nonintersecting. The essential spectral amplitudes $\mathbf{E}_l(k_x,k_y)$ are concentrated on the lth surface in the vicinity of the point $k_0^{(l)}$ with an effective transverse size $\varkappa_{nx,y}^{(l)}$ satisfying the inequality

$$\varkappa_{nx,y}^{(l)}\ll k_{z0}^{(l)}. \tag{3}$$

In the case of an unfocused beam $\varkappa_{nx,y}\sim a_{x,y}^{-1}$, where $a_{x,y}$ are its characteristic transverse dimensions.

Assume that the field structure at $z=0$ causes excitation of only one normal mode in the medium. We introduce the radius vector $\varkappa=\mathbf{k}-\mathbf{k}_0$ and write down the general solution of (2) in the form of two factors, one of which is slowly varying by virtue of the inequality (3):

$$\mathbf{E}(\mathbf{r})=\mathbf{s}A(\mathbf{r})e^{i\mathbf{k}_0\mathbf{r}},\qquad |\mathbf{s}|=1, \tag{4}$$

$$A(\mathbf{r})=\iint_{-\infty}^{\infty}A(\varkappa_x\varkappa_y)\exp\{i[\varkappa_xx+\varkappa_yy+\varkappa_z(\varkappa_x\varkappa_y)z]\}\,d\varkappa_x\,d\varkappa_y, \tag{5}$$

where $A(\varkappa_x\varkappa_y)$ is defined in analogy with (2a), the index l here and elsewhere will be left out. We expand the function $\varkappa_z(\varkappa_x\varkappa_y)$ in a Taylor series in the vicinity of the point $\varkappa_x=\varkappa_y=0$:

$$\varkappa_z(\varkappa_x\varkappa_y)=\sum_{n=1,m\leqslant n}^{M}P_{nm}\varkappa_x^{n-m}\varkappa_y^{m}, \tag{6}$$

where

$$P_{nm}=\frac{1}{(n-m)!\,m!}\frac{\partial^nk_z}{\partial k_x^{n-m}\partial k_y^m}\bigg|_{k_{x0},k_{y0}}. \tag{6a}$$

Substitution of (6) in the integrand of (5) at a fixed M and at an initial distribution $A(\xi,\eta)$ solves the problem of determining, with the required accuracy, $A(\mathbf{r})$ at all points $z\neq0$.

In a number of problems of the theory of optical resonators, and also of nonlinear optics, it is convenient to have in place of the integral solution (5) an approximate differential equation for the amplitude $A(\mathbf{r})$. For a suitably chosen number of terms of the expansion (6), the integral representation (5) makes it possible to contruct the equation

$$i\frac{\partial A}{\partial z}+\sum_{n=1,m\leqslant n}^{M}(-i)^nP_{nm}\frac{\partial^nA}{\partial x^{n-m}\partial y^m}=0, \tag{7}$$

whose Fourier transform in k-space is relation (6). Inasmuch as the medium is assumed to be unbounded in the x and y direction, and the initial distribution $\mathbf{E}(\xi,y)$ has the Fourier transform (2a), the boundary conditions as $|x|$ and $|y|\to\infty$ can be regarded as the zero conditions for all the derivatives with respect to x and y. Thus, lowering the order of the initial system of differential equations with respect to the variable z to unity within the framework of the inequality (3) brings about the appearance in (7) of derivatives of arbitrarily high order with respect to x and y, and the terms corresponding to the sth derivative are small terms of order $(\varkappa_n/k_{z0})^s$. We note that similar reasoning was used in hydrodynamics to describe with the aid of the Korteweg–de Vries equation the propagation of waves on shallow water,[10] the propagation of surface magnetostatic waves in ferromagnetic plates within the framework of the parabolic equation,[11] and others. A suitable small parameter in these cases is the ratio of the layer thickness to the wavelength. Confining ourselves in (6) to the quadratic terms of the expansion we obtain the parabolic equation

$$-i\frac{\partial A}{\partial z}+P_{20}\frac{\partial^2 A}{\partial x'^2}+P_{21}\frac{\partial^2 A}{\partial x'\,\partial y'}+P_{22}\frac{\partial^2 A}{\partial y'^2}=0, \tag{8}$$

where $x' = x + P_{10}z$ and $y' = y + P_{11}z$. The quantities P_{10} and P_{11} determine the inclination of the wave vector to the z axis, and the difference between the coefficient P_{2j} in (8) is due to the different curvatures of the surface $k_z(k_x, k_y)$ in the orthogonal directions. In the simplest case of an isotropic medium, the Z axis can be chosen as the beam propagation direction ($k_{x0} = k_{y0} = 0$). Calculation of the coefficients (6a) shows that Eq. (7) contains only terms with even n and with $m = 0$, n: $P_{20} = P_{22} = -1/2k$, $P_{40} = P_{44} = -1/8k^3$, etc.

§3. DIFFRACTION OF A FOCUSED GAUSSIAN BEAM IN A UNIAXIAL NONEQUILIBRIUM MEDIUM

To describe a nonequilibrium anisotropic medium within the framework of linear macroscopic electrodynamics, one introduces into Maxwell's equations, besides the dielectric tensor, $\hat{\varepsilon}$, also the conductivity tensor $\hat{\sigma}$. The directions of the principal axes of these tensors do not coincide in the general case. We confine ourselves here to consideration of the case in which the tensors $\hat{\varepsilon}$ and $\hat{\sigma}$ can be simultaneously diagonalized. Choosing a Cartesian system of coordinates that coincides with the principal axes of these tensors we reduce the corresponding dispersion equation to the known Fresnel equation for the wave normals,[12] in which

$$\tilde{\varepsilon}_j=\varepsilon_j+4\pi i\omega^{-1}\sigma_j,\qquad j=x,\,y,\,z. \tag{9}$$

In the particular case of an optically uniaxial crystal ($\tilde{\varepsilon}_x = \tilde{\varepsilon}_y$), the normal modes for this equation are the ordinary and extraordinary waves. Their wave surfaces in an equilibrium medium are respectively a sphere and a spheroid:

$$k_z^{(0)2}=\tilde{\varepsilon}_x k_v^2-k_x^2-k_y^2, \tag{10a}$$

$$k_z^{(e)2}=\tilde{\varepsilon}_x k_v^2-\frac{\tilde{\varepsilon}_x}{\tilde{\varepsilon}_z}(k_x^2+k_y^2),\qquad k_v=\frac{\omega}{c}. \tag{10b}$$

By virtue of the axial symmetry we can put $k_{y0} = 0$ without loss of generality. The coefficient of the expansion of (6a) for the modes (10b) take in this case the form

$$P_{10}=\frac{\tilde{\varepsilon}_x}{\tilde{\varepsilon}_z}\frac{k_{x0}}{k_{z0}},\qquad P_{11}=P_{21}=0,$$

$$P_{22}=-\frac{\tilde{\varepsilon}_x}{\tilde{\varepsilon}_z k_{z0}},\qquad P_{20}=P_{22}\left(1+\frac{\tilde{\varepsilon}_x}{\tilde{\varepsilon}_z}\frac{k_{x0}^2}{k_{z0}^2}\right), \tag{11a}$$

$$k_{z0}=\left(\tilde{\varepsilon}_x k_v^2-\frac{\tilde{\varepsilon}_x}{\tilde{\varepsilon}_z}k_{x0}^2\right)^{1/2}.$$

In an equilibrium medium ($\sigma = 0$) they are simply expressed in terms of the angles θ and θ' between the optical axis and the directions of the waves and ray normals:

$$P_{10}=-\frac{\varepsilon_x}{\varepsilon_z}\operatorname{tg}\theta=-\operatorname{tg}\theta',\qquad P_{22}=-\frac{1}{2k_v}\frac{\varepsilon_x}{\varepsilon_e^{1/2}\varepsilon_z\cos\theta},$$

$$P_{20}=P_{22}\frac{\varepsilon_x}{\varepsilon_e\cos^2\theta}, \tag{11b}$$

where $\varepsilon_e = \varepsilon_x/(1-\Delta\sin^2\theta)$ is the effective dielectric constant of the extraordinary wave, and $\Delta = 1 - \varepsilon_x/\varepsilon_z$ is the anisotropy factor.

Assume in the plane $Z = 0$ a given linearly polarized field

$$E(x,y)=E_0\exp\left[-\frac{x^2\cos^2\theta+y^2}{a^2}(1+i\mu)+ik_{x0}x\right], \tag{12}$$

that determines the contribution of a focused axisymmetric Gaussian beam at an angle θ to the optical axis Z, where a is the beam radius, R is the curvature radius of the wave front, and $\mu = ka^2/2R$ is the phase-modulation parameter. The electric-field polarization vector lies in a plane containing the optical axis and the wave vector **k**; this corresponds to the extraordinary wave (10b). On the whole, the beam (12) will excite in the medium all the normal waves, but we shall investigate the peculiarities of its propagation taking into account only the preferred excitation of one of them. The amplitude $A(\varkappa_x\varkappa_y)$, as follows from (2a), has likewise a Gaussian form:

$$A(\varkappa_x\varkappa_y)=\frac{aE_0}{4\pi\varkappa_n\cos\theta}$$

$$\times\exp\left[-\frac{\varkappa_x^2\cos^2\theta+\varkappa_y^2}{\varkappa_n^2}(1-i\mu)-i\operatorname{arctg}\mu\right], \tag{13}$$

where $\varkappa_n = 2(1+\mu^2)^{1/2}/a$ is the angular spectral halfwidth. Calculation of the integral (5) under the condition $b_{x,y} = 1 + \varkappa_n^2 P''_{20,22} > 0$ in the parabolic approximation yields

$$E(\mathbf{r})=E_1(z)\exp\left[-\frac{(x-x_1)^2\cos^2\theta}{a_x^2}(1+i\mu_x)\right.$$

$$\left.-\frac{y^2}{a_y^2}(1+i\mu_y)+i\varphi_1\right], \tag{14}$$

where

$$E_1(z)=a(a_xb_xa_yb_y)^{-1/2}E_0,\qquad \varphi_1(x,z)$$

$$=k_{x0}x+k_{z0}z-{}^1/_2(C_x+C_y-2C), \tag{15}$$

$$a_{x,y}(z)=a[(1-\mu d_{x,y})^2+d_{x,y}^2]^{1/2}, \tag{16}$$

$$\mu_{x,y}(z)=\mu(1-\mu d_{x,y})-d_{x,y}, \tag{17}$$

$$x_1(z)=(\mu_x P_{10}''-P_{10}')z, \tag{18}$$

$$r_x=2k_v\cos^2\theta\,|P_{20}'-\mu P_{20}''|,\qquad r_y=2k_v|P_{22}'-\mu P_{22}''|, \tag{19}$$

$$d_{x,y}(z)=\frac{2z}{ka^2}r_{x,y},\qquad C_j=\operatorname{arctg}\mu_j,\qquad P_{nm}=P_{nm}'+iP_{nm}''. \tag{20}$$

Thus, the curve of the maximum amplitude of the field in the XZ plane, a curve that determines the direction of propagation of the energy in the beam, is described by the equation $x = x_1(z)$. Only in an equilibrium medium ($P''_{10} = 0$) is this a straight line inclined to the optical axis at an angle that coincides with $\theta' \neq \theta$. The bending of the beam trajectory, due to the dispersion of the relaxation, also depends, as follows from (17) and (18), on the beam parameters, and in particular on the curvature of the phase front. With the aid of (16) and (17) it is easy to show that at $P''_{20}=P''_{22}=0$ the angular spectral width of the beam is a conserved quantity:

$$\varkappa_{nx,y}(z)=\frac{2}{a_{x,y}(z)}[1+\mu_{x,y}^2(z)]^{1/2}=\varkappa_n. \tag{21}$$

This property of Gaussian beams, which is valid only within the framework of the parabolic approximation, was already noted in an investigation of the propagation of Gaussian pulses in dispersive media.[13] The space-time analogy of these problems is based on the use of the dispersion law $\omega(k)$ for pulses and of its analog $k_0(k_x)$ for beams.

As a result of the difference between the rates of spreading of the beam along orthogonal directions, two foci are produced in an anisotropic medium, and their separation depends on the angle of inclination of the beam to the optical axis:

$$z_{fx}=\frac{\mu^2}{1+\mu^2}\frac{\varepsilon_z}{\varepsilon_x^2}\varepsilon_e^{1/2}R\cos\theta,\qquad z_{fx}-z_{fy}=\Delta z_{fx}\sin^2\theta. \tag{22}$$

This also leads to the appearance of ellipticity in the initially circular cross section of the beam. In a transparent medium in the absence of focusing ($R\to\infty$), the maximum ellipticity $\in(z,\theta)=a_y/a_x$ is reached according to (16) as $z\to\infty$ and is equal to $\varepsilon_e/\varepsilon_x$. In focused beams, $\in$ can reach much higher values. In particular, at $z=z_{fx}$ we have

$$\epsilon(z_{fx})=\frac{\varepsilon_e}{\varepsilon_x}(1+\Delta^2\mu^2\sin^4\theta)^{1/2}.$$

§4. ANOMALOUS DIFFRACTION OF BEAM IN THE REGION OF DIPOLE-TRANSITION FREQUENCIES

We consider now the peculiarities, in the vicinity of an electrodipole x-polarization transition, of a wave incident on and propagating through a crystal at an angle to the optical axis. The wave has the electric vector in the incidence plane. The pure states in the crystal are in this case the extraordinary wave and the Coulomb exciton.[1] Of the two independent components $\varepsilon_\perp$ and $\varepsilon_\parallel$ of the dielectric tensor, only the transverse one will then be resonant:

$$\varepsilon_\perp(\omega)=\varepsilon_x\left[1-\frac{f^2}{\omega^2-\omega_{ex}^2(\mathbf{k})}\right],\qquad \varepsilon_\parallel=\varepsilon_z, \tag{23}$$

where the pole $\omega^2=\omega_{ex}^2=\omega_0^2+\hbar\omega_0\mathbf{k}^2/m^*$ determines in the isotropic-effective-mass approximation the dispersion law of the mechanical exciton,[1] $f=4\pi e^2NF/\varepsilon_x m$; e and m are the charge and mass of the electron, N is their number per unit volume, F is the oscillator strength, m^* is the effective mass of the exciton, and $\varepsilon_x\neq\varepsilon_z$ are the background values of the dielectric constants due to the contributions of the remaining resonances. In analogy with (23), the pole $\varepsilon_e(\omega)$ determines the dispersion law of the Coulomb exciton

$$\omega_{\mathbf{k}}^2=\omega_{ex}^2(\mathbf{k})+\frac{\varepsilon_x f^2\sin^2\theta}{\varepsilon_x\sin^2\theta+\varepsilon_z\cos^2\theta}. \tag{24}$$

It follows from (24) that, without loss of generality, we can again consider propagation in the crystal only in the XZ plane ($k_{y0}=0$). For simplicity we neglect also the dissipation. Allowance for the retardation mixes the "pure" states, and we shall call the new state an extraordinary polariton. Its dispersion law is of the form

$$(\omega^2-\omega_e^2)(\omega^2-\omega_{\mathbf{k}}^2)-\frac{\varepsilon_{e0}}{\varepsilon_x}f^2\omega^2\cos^2\theta=0, \tag{25}$$

where $\omega^2=c^2k^2/\varepsilon_{e0}$ and $\varepsilon_{e0}=\varepsilon_e\ (\omega\to\infty)$. It follows from (24) and (25) that if $\theta=0$ the Coulomb exciton is transformed into a mechanical one, and if $\theta=\pi/2$ it is transformed into a purely longitudinal wave that does not interact with the electromagnetic field. To clarify the pecularities of the diffraction in the interval of angles $0<\theta<\pi/2$, we present the relation (25) in the form (1):

$$(k_z^2-k_{ze}^2)(k_z^2-k_{zk}^2)+\frac{\varepsilon_x f^2k_v^2}{\omega^2-\omega_{ex}^2(\mathbf{k})}k_z^2=0, \tag{26}$$

where k_{ze} is determined by the expression (10b),

$$k_{zk}^2=\frac{\varepsilon_x}{\varepsilon_z}\frac{\omega_{ex}^2(\mathbf{k})+f^2-\omega^2}{\omega^2-\omega_{ex}^2(\mathbf{k})}k_x^2=c_z^2k_x^2. \tag{27}$$

Neglecting the spatial dispersion in the frequency region

$$\omega_0<\omega<(\omega_0^2+f^2)^{1/2} \tag{28}$$

we have for the Coulomb excitons (27) a linear dispersion law, since $c_z^2>0$. It follows from this that the behavior of the dispersion curve, when account is taken of the retardation, is qualitatively different in the region (28) and outside it. We investigate here only the region (28). The qualitative character of the relation (26) is shown in Fig. 1. The dashed line shows the interacting branches (10b) and (27). The reverse course of curve 2 at large k_z is due to the spatial dispersion effect. The independent x component of the vector $\mathbf{k}$ is specified by the field incident on the crystal, on the basis of the equality of the tangential components of the fields on both sides of the boundary. The quantities k_z/k_x and dk_z/dk_x determined respectively the inclination angles of the wave and the ray vectors to the Z axis. Thus, the multivalued character of the function $k_z(k_x)$ corresponds to splitting of the beam in such a medium. It can be seen from Fig. 1 that the

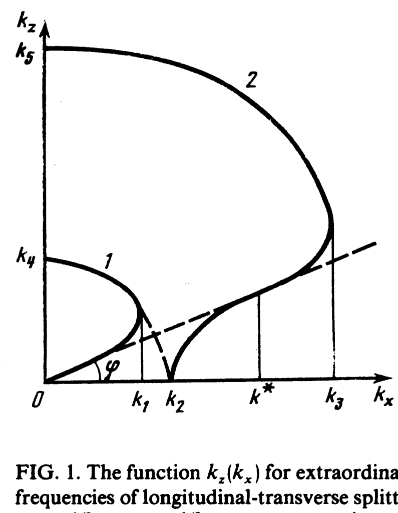

FIG. 1. The function $k_z(k_x)$ for extraordinary polaritons in the region of frequencies of longitudinal-transverse splitting ($\omega_0<\omega<(\omega_0^2+f^2)^{1/2}$); $k_2=\varepsilon_z^{1/2}k_v$, $k_4=\varepsilon_x^{1/2}k_v$, $k_5=(2m^*\hbar^{-1}(\omega-\omega_0))^{1/2}$, $\varphi=\arctan c_z$, and k_1 is defined by relation (29).

range of values of k_x at which propagating waves exist is divided by points k_1, k_2 and k_3 into three qualitatively different sections:

a) $0 < k_x < k_1$. When account is taken of the spatial dispersion, the beam splits into three; the value

$$k_1 = k_v \left(\frac{\omega_0^2 + f^2 - \omega^2}{\varepsilon_z f^2} \right)^{1/2} \tag{29}$$

determines the critical angle at which the directions of the wave vectors of the interacting waves become perpendicular to the optical axis.

b) $k_1 < k_x < k_2 = k_v \varepsilon_z^{1/2}$. The only wave that propagates in this range of angles is the exciton wave 2. If we denote by ε_s and θ_s the dielectric constant of the external medium and the incidence angle of the ray on the planar boundary of the crystal ($\varepsilon_s > \varepsilon_z$), then in the angle interval

$$(\omega_0^2 + f^2 - \omega^2)/\varepsilon_z \varepsilon_s f^2 < \sin^2 \theta_s < \varepsilon_z/\varepsilon_s$$

one can expect direct observation of the exciton response in the form of a light beam passing through the resonantly absorbing plate. A similar observation of the exciton response with the aid of ultrashort pulses of laser radiation is based on the large differences in the group velocities of the normal waves, and has been now realized in a number of experimental studies.[14,15] An alternate method of observing the response with the aid of light beams, as follows from the foregoing analysis, is possible because of the strong difference between the directions of the ray vectors of the normal waves, and seems simpler from the experimental point of view.

c) $k_2 < k_x < k_3$. We note on this interval only the anomalous character of the diffraction at the inflection point k^*, where $\partial^2 k_z / \partial k_x^2 = 0$. In the case of a plane beam ($\partial/\partial y = 0$) the corresponding differential equation for the complex amplitude (7) takes the form of a linear Korteweg–de Vries equation

$$\frac{\partial A}{\partial z} - P_{10} \frac{\partial A}{\partial x} + P_{30} \frac{\partial^3 A}{\partial x^3} = 0. \tag{30}$$

The solution of (30) for the real and imaginary parts of the amplitude is expressed in terms of an Airy function. A feature of the solution is that the intensity of the diffracted light is oscillatory even for smooth distributions of the envelope, of the Gaussian type, while the picture of the diffraction is asymmetric with respect to the propagation direction.

§5. PROPAGATION OF AN ACOUSTIC BEAM IN A FERROMAGNET UNDER CONDITIONS OF MAGNETOACOUSTIC RESONANCE

We consider a ferromagnetic ellipsoid magnetized to saturation along the Z axis by a constant magnetic field H_0. We shall assume that the energy of the magnetic crystallographic anisotropy and of the magnetoelastic interaction have cubic symmetry, and the edges of the cube coincide with the axis of the Cartesian coordinate system. The magnetodipole field, neglecting retardation, is potential and is described by the magnetostatic potential, $\mathbf{h} = \nabla \psi$. With respect to the elastic properties the medium is assumed linear and isotropic, and we neglect the inhomogeneous exchange interaction. We investigate the evolution of a planar beam of magnetoelastic waves ($\partial/\partial y = 0$) with an elastic-displacement vector lying in the XZ plane and making an angle θ with the Z axis: $\mathbf{u} = \{u \sin\theta, 0, u \cos\theta\}$. In this case the coupled system of equations for the components of the magnetization vector and of the elastic displacements can be represented, within the framework of the effective-field formalism,[2] in the form

$$\begin{aligned} &\dot{\alpha}_x + \omega_y \alpha_y - \omega_{me} \alpha_x \alpha_y \left(\sin\theta \frac{\partial}{\partial z} + \cos\theta \frac{\partial}{\partial x} \right) u = 0, \\ &\dot{\alpha}_y - \omega_x \alpha_x + \gamma \alpha_z \frac{\partial \psi}{\partial x} - \omega_{me} (\alpha_z^2 - \alpha_x^2) \left(\sin\theta \frac{\partial}{\partial z} + \cos\theta \frac{\partial}{\partial x} \right) u = 0; \end{aligned} \tag{31}$$

$$\Delta\psi = -4\pi M_0 \operatorname{div} \boldsymbol{\alpha}, \quad \alpha_z^2 = 1 - |\alpha_x|^2 - |\alpha_y|^2, \quad |\alpha_{x,y}| \ll 1, \tag{32}$$

$$\begin{aligned} &\left[c_e^2 \left(\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial z} \right)^2 \right. \\ &\left. + c_t^2 \left(\cos\theta \frac{\partial}{\partial x} - \sin\theta \frac{\partial}{\partial z} \right)^2 - \frac{\partial^2}{\partial t^2} \right] u \\ &+ \frac{B_2}{\rho} \left(\sin\theta \frac{\partial}{\partial z} + \cos\theta \frac{\partial}{\partial x} \right) \alpha_x \alpha_z = 0, \end{aligned}$$

where

$$\omega_y/\gamma = H_0 - M_0 (N_z - N_v) \alpha_z - H_{A1} (\alpha_z^2 - |\alpha_y|^2) \alpha_z + H_{A2} |\alpha_x|^2 + \partial\psi/\partial z, \tag{33}$$

$\boldsymbol{\alpha} = \mathbf{M}/M_0$, $\omega_{me} = \gamma B_2 / M_0$, $c_{e,t}$ are the respective velocities of the longitudinal and transverse sound, ρ is the density, γ is the gyromagnetic ratio, M_0 is the saturation magnetization, $N_{x,y,z}$ are the demagnetizing factors of the ellipsoid, $H_{A\,1,2} = 2K_{1,2}/M_0$ are the effective crystallographic anisotropy fields of the first and second order, and B_2 is the magnetoelastic constant. An expression for ω_x is obtained from (33) by the interchange of the indices, $x \rightleftarrows y$, the dissipation effect in the subsystems can be included by making the transition $\partial/\partial t \to \partial/\partial t + \tau_{m,e}^{-1}$. Bearing in mind the discussion of the self-action effects at resonance in §6, we have retained in (31)–(33) all the cubic nonlinear terms and discarded the quadratic ones.

For a stationary beam (u, $\alpha_{x,y} \sim \exp(-i\omega t)$, $\omega = \text{const}$), after linearizing the initial system of equation and eliminating the variables ψ and α_y, we obtain

$$\begin{aligned} &[\omega^2 - \omega_y(\omega_x + \omega_m)] \frac{\partial^2 \alpha_x}{\partial x^2} + (\omega^2 - \omega_x \omega_y) \frac{\partial^2 \alpha_x}{\partial z^2} \\ &- \omega_{me} \left(\sin\theta \frac{\partial}{\partial z} + \cos\theta \frac{\partial}{\partial x} \right) \Delta u = 0. \end{aligned} \tag{34}$$

The equation for u remains in the form (32) with the substitution $\alpha_z = 1$. The form of (32) makes it possible to analyze simultaneously magnetoelastic waves with longitudinal and transverse elastic displacements. The cases

$$\cos\theta \frac{\partial}{\partial x} = \sin\theta \frac{\partial}{\partial z}, \qquad \sin\theta \frac{\partial}{\partial x} = -\cos\theta \frac{\partial}{\partial z}$$

correspond to propagation of a longitudinal elastic wave at an angle θ and of a transverse elastic wave at an angle $\pi/$

$2-\theta$ to the direction of the field H_0.

It can be seen from (34) that without the magnetoelastic interaction ($\omega_{me}=0$) the equation for α_x in the frequency region

$$\omega_x\omega_y<\omega^2<\omega_y(\omega_x+\omega_y) \tag{35}$$

is of the hyperbolic type, i.e., it describes propagating waves with a linear dispersion law:

$$k_z=c_s k_x,\qquad c_s=\left[\frac{\omega_y(\omega_x+\omega_m)-\omega^2}{\omega^2-\omega_x\omega_y}\right]^{1/2}. \tag{36}$$

Allowance for magnetoelastic interaction with a longitudinal elastic wave leads to a biquadratic dispersion law:

$$(k_z^2-k_s^2)(k_z^2-k_e^2)+\xi_e k_x^2 k_z^2=0, \tag{37}$$

where

$$k_s=c_s k_x,\qquad k_e^2=\frac{\omega^2}{c_e^2}-k_x^2,\qquad \xi_e=\frac{4\gamma B_2}{\rho M_0}\,\frac{\omega_y}{c_e^2(\omega^2-\omega_x\omega_y)}. \tag{38}$$

It follows from (37) that far from resonance the curve $k_z(k_x)$ is an ellipse with an axis ratio that determines the birefringence. At $\omega^2>\omega_y(\omega_x+\omega_m)$ and $\omega^2<\omega_x\omega_y$ the crystal is respectively acoutically positive and acoustically negative. In the rejection band (35) the qualitative character of the dependence (37) is shown in Fig. 2. The dashes show the continuation of curve 2 without allowance for the exchange interaction. Just as in Fig. 1, there exist in this case three angle intervals within which the character of the wave-beam propagation is qualitatively different. The critical values $k_\pm$, between which there exists only one propagating exchange spin wave, are

$$k_\pm=\frac{\omega}{c_e}[1+(c_s\pm\xi_e^{1/2})^2]^{-1/2}. \tag{39}$$

In contrast to the situation considered in §4, in the interval $k_-<k_x<k_3$ there exist two directions of anomalous dispersion $k^*_{1,2}$.

It is possible to analyze in similar fashion the character of the interaction of a transverse elastic wave with a magnetic subsystem, and on the basis of Eq. (37) also extraordinary polaritons in the region of quadrupole-transition frequencies.

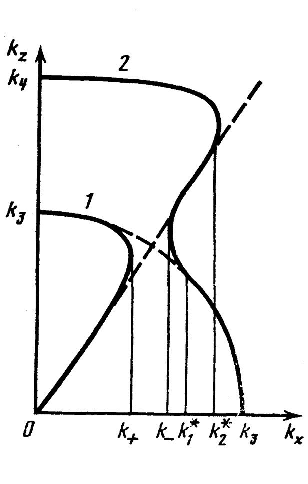

FIG. 2. The function $k_z(k_x)$ for magnetoelastic waves in magnets in the case of magnetoacoustic resonance ($\omega_x\omega_y<\omega^2<\omega_y(\omega_m+\omega_x)$), $k_3=\omega/c_e$, $k_\pm$ is determined by relation (39), k_4 is the root of the equation $(\omega_x+\eta k_4^2)(\omega_y+\eta^2_{k_4})-\omega^2=0$, and η is the exchange constant.

§6. PECULIARITIES OF EFFECTS OF SELF-ACTION IN NONLINEAR ANISOTROPIC MEDIA

The propagation of an intense beam with a narrow angle spectrum in a medium, whose parameters are functions of the field intensity will be investigated by including in the dispersion equation (1) the dependence on $|A|^2$ (Ref. 16). Assuming the beam to be planar, we confine ourselves to expansion of κ_z only in terms linear in $|A|^2$

$$\varkappa_z(\varkappa_x,|A|^2)=P_{10}\varkappa_x+P_{20}\varkappa_x^2+P_{30}\varkappa_x^3+q_1|A|^2+q_2\varkappa_x|A|^2+\dots, \tag{40}$$

where $q_1=\partial k_z/\partial|A|^2$, $q^2=\partial P_{10}/\partial|A|^2$, and the coefficients P_{j0} are defined in (6a). The corresponding differential equation takes the known Hirota form[9]

$$i\left[\frac{\partial A}{\partial z}-\frac{q_2}{3}\left(2|A|^2\frac{\partial A}{\partial x'}+A^2\frac{\partial A^*}{\partial x'}\right)+P_{30}\frac{\partial^3 A}{\partial x'^3}\right]-P_{20}\frac{\partial^2 A}{\partial x'^2}+q_1|A|^2A=0. \tag{41}$$

The reconstruction of Eq. (41) from the expansion (40) is unique, with the exception of the last term, which gives a more complicated construction because $(A^2A^*)_x=2|A|^2A_x+A^2A^*_x$.

Usually self-action effects are investigated within the framework of a nonlinear Schrödinger equation ($P_{30}=q_2=0$). Its stability to longitudinal perturbations is described by the condition $q_1P_{20}<0$ (Ref. 16). Far from the resonances $P_{20}=-1/2k$ in this case and whether the medium focuses or defocuses the beam depends exclusively on the sign of q_1. In the vicinity of the resonances the situation changes substantially, and this was shown for the first time theoretically and confirmed experimentally in an investigation of the passage of acoustic pulses through an antiferromagnet in the vicinity of the NMR frequency.[17] It turned out that on going through resonance the sign of q_1 remained constant, and owing to the reversal of the sign of the dispersion the formation of soliton took place only to the left of the resonance. The appearance of the nonlinear properties themselves in resonant media, as already noted in Refs. 18 and 19, is also a distinguishing feature. The point is that a normal wave is a superposition of modes of noninteracting subsystems. Each of them is characterized by its own nonlinear properties whose contribution is determined by the weight of the given mode in the coupled wave. We shall call these detuning nonlinearities, emphasizing by the same token the dependence of the unperturbed frequencies on the amplitude. In addition, there is also an interaction nonlinearity characterized by the dependence of the coupling parameter on the amplitude. The simplest manifestation of this nonlinearity is the self-induced transparency which occurs when a wave passes through a gas of noninteracting particles.[20] In this case the free field is described by a linear wave equation, and the particles are simulated by linear oscillators.

The number of different nonlinearities of the medium should thus be determined by the number of independent parameters of the dispersion equation (1). Their resultant action, at least qualitatively, can therefore be investigated by introducing the dependence of these parameters on the amplitude. We consider, for example, Eq. (37), which is characterized by three parameters: c_s, c_e, and ξ_e. Confining ourselves to an expansion linear in $|A|^2$ for these parameters, we obtain

$$q_1=\frac{1}{2G}\left[\alpha_s k_x^2(k_z^2-k_e^2)+\alpha_e\frac{\omega^2}{c_e^2}(k_z^2-k_s^2)-\alpha_\xi k_x^2k_z^2\right],$$

$$q_2=\frac{k_xk_z}{G^2}\left[\alpha_s\left(k_z^2-\frac{\omega^2}{c_e^2}\right)+\alpha_e c_s^2\frac{\omega^2}{c_e^2}-\alpha_\xi k_z^2\right], \qquad (42)$$

$$G=(2k_z^2-k_s^2-k_e^2-\xi_e k_x^2)\,k_z, \qquad (43)$$

where

$$\alpha_e=\frac{1}{c_e^2}\frac{\partial c_e^2}{\partial|A|^2},\qquad \alpha_s=\frac{\partial}{\partial|A|^2}\frac{\omega_y\omega_m}{\omega^2-\omega_x\omega_y},$$

$$\alpha_\xi=\frac{4\gamma B_2}{\rho M_0}\frac{\partial}{\partial|A|^2}\frac{\omega_y}{c_e^2(\omega^2-\omega_x\omega_y)}. \qquad (44)$$

A complete calculation of α_s and α_ξ can be carried out with the aid of relations (33) for $\omega_{x,y}$, by substituting there the linearized connections between u, ψ, and $\alpha_{x,y}$:

$$|\alpha_x|=\frac{\omega_y}{\omega}|\alpha_y|=\frac{\omega_m\omega_y}{\omega^2-\omega_k^2}k\sin 2\theta|u|,$$

$$\frac{\partial\psi}{\partial z}=4\pi M_0(1-\alpha_z)\cos\theta,\qquad \omega_k^2=\omega_y(\omega_x+\omega_m\sin^2\theta).$$

We note that the ellipticity $\epsilon\sim\omega_y/\omega_k$ is connected only with the dipole interaction. A giant ellipticity arises in easy-plane antiferromagnets ($H_E/H_A\sim 10^6$), and the results of this are the effects of exchange enhancement of the magnetoelastic interaction and acoustic anharmonicity.[21,22] For the same reason one should expect also singularities of the diffraction and self-action of acoustic beams in such substances, since the coefficients α_s and α_ξ are anomalously large for them.

Depending on the chosen direction of propagation and on the frequency, as follows from (43) and (44), the coefficients $q_{1,2}$ can reverse sign. In addition, $q_{1,2}$ have holes on the edges of the nonpropagation region, defined by the relation (39). By choosing the direction of propagation and the frequency within the limits of the band (35), it is possible to achieve simulataneous vanishing of the coefficients P_{20} and q_2.

In this case the self-action of the beam is described by a complex modified Korteweg–de Vries equation. In hydrodynamics this equation, for a real function, is also used in those cases when the coefficient of the nonlinear term in the Korteweg–de Vries equation vanishes for some reason.[23] The "unidirectivity" of this type of equation determines in this case, besides the asymmetry of the self-focusing phenomenon, also the effect of self-bending of the trajectory of a ray or rays, if the initial condition admits of soliton production. In the general case, however, Eq. (41) is a combination of a nonlinear Schrödinger equation and a complex modified Korteweg–de Vries equation, in which the role of each term is determined by the specified propagation direction and by the specified working frequency.

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