Dynamics of bions in long Josephson junctions

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The influence of an extraneous current and of dissipation on the dynamics of a bion in a long Josephson junction is considered. The system is described by a perturbed sine-Gordon equation. Equations are obtained for the bion parameters in first-order perturbation theory. It is shown that a constant extraneous current and dissipation can alter substantially the dynamics of the bion. If the extraneous current exceeds a certain critical value, for which an equation is obtained, the bion breaks up into a soliton and an antisolition. In the opposite case, the bion amplitude attenuates exponentially and its velocity tends to a certain limiting value that is comparable with albeit smaller than the initial value.

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1. INTRODUCTION

Wave processes in long Josephson junctions (lines) are described by a perturbed sine-Gordon (SG) equation

$$v_{tt}-v_{xx}+\sin v=\varepsilon R[v], \quad (1.1)$$

where $v = 2\pi\Phi(x,t)/\Phi_0$ is the magnetic flux normalized to the value of the magnetic-flux quantum $\Phi_0 = hc/2e$, while x and t are the dimensionless coordinates and time normalized as in Ref. 1. The right-hand side of (1.1) is the perturbing term. In this case it is the sum of the extraneous current and of terms that describe the dissipative effects. We shall assume the right-hand side to be small, as indicated by the parameter $\varepsilon \ll 1$ in (1.1). The concrete expansion for εR used in this paper will be presented below.

Important solutions of the SG equation are solitons

$$v(x,t)=4\sigma \operatorname{arctg} \exp[(x-Vt)(1-V^2)^{-1/2}],$$

where σ describes the polarity of the soliton, $\sigma = \pm 1$ (soliton,antisoliton). In the theory of Josephson junctions they are frequently called fluxons, since they can be regarded as elementary excitations of the magnetic flux. The dynamics of fluxons is being diligently investigated at the present time (see, e.g., the review[1]).

As shown by Fulton and Dynes,[2] the characteristic features of the current-voltage characteristic and of the spectrum of the microwave radiation of a long Josephson junction can be explained in most natural fashion by starting from the concept of solitons oscillating between the ends of the junction. This was followed by a number of other papers offering evidence that solitons in Josephson junctions are experimentally observable objects. Major practical applications were predicted for this group of questions (see, e.g., Refs. 3 and 4 and the literature cited therein). In Ref. 4 were experimentally investigated multisoliton systems in long junctions. One of the most interesting observed effects was the "bunching" of the solitons of like polarity σ, wherein a system of soliton is produced with constant distances between them. This effect was noted earlier in analog and numerical experiments (see, e.g., Refs. 5–9). Its theory, proposed in Refs. 10 and 11, points to the important role of the extraneous current and of dissipation in the formation of bunched systems. Indeed, it is known that solitons of like polarity, which satisfy the unperturbed SG equation, always repel each other. However, as shown in Refs. 10 and 11, it suffices to take in Eq. (1.1) the right-hand side in the form

$$\varepsilon R[v]=-f-\alpha v_t, \quad (1.2)$$

where f is the dimensionless density of the extraneous current ($f=$ const), and α is a dissipative coefficient proportional to the transverse electric conductivity of the junction, to arrive at an effect wherein a quasistationary state is established even at small f and α, such that the distance between solitons of like polarity increases so slowly that its change is not noticeable in experiment, and it is this which explains the bunching. This is evidence of the importance of taking into account even small perturbations in Eq. (1.1) when the experimental facts are interpreted.

In the present paper we consider the effect of perturbations on a bion, which at $\varepsilon = 0$ is a second localized solution of the SG equation[1]

$$v(x,t)=-4 \operatorname{arctg}\left(\frac{\nu}{\eta}\frac{\cos\varphi}{\operatorname{ch} z}\right), \quad (1.3)$$

$$\varphi=\theta(t)-(\eta/\nu)Vz, \quad z=(\nu/|\zeta|)[x-x_0(t)](1-V^2)^{-1/2}, \quad (1.4)$$

$$V=(1-4|\zeta|^2)(1+4|\zeta|^2)^{-1}, \quad \zeta=\eta+i\nu \quad (\eta>0,\ \nu>0), \quad (1.5)$$

$$dx_0/dt=V, \quad d\theta/dt=(\eta/|\zeta|)(1-V^2)^{1/2}. \quad (1.6)$$

In contrast to a soliton, a bion has besides a velocity V, also a natural-pulsation frequency $d\theta/dt$. At small η/ν, at definite time intervals, the solution (1.3) can be approximately represented as a superposition of two oppositely polarized solitons which overlap little and are attracted to each other, forming a bound system in the absence of perturbation. The presence of a perturbation can, on the contrary, cause decay of the bion into a soliton and an antisoliton or else lead to other effects. An approximate criterion of bion decay under the influence of perturbations (1.2) was obtained in a note by one of us[12] on the basis of the quasiparticle model of the bion, where the motion of the soliton and antisoliton is described by the equations of mechanics. This model is applicable when the bion can be regarded for the greater part of the time as a system of two weakly coupled overlapping solitons, so that the attraction between them is small enough ($\eta/\nu \ll 1$). The present paper is devoted to an investigation of the be-

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havior of bions at arbitrary η/v, acted upon by the perturbation (1.2), at constant f and α. The latter are assumed to be small and we use perturbation theory. Without loss of generality we can assume that $f > 0$.

Various problems of perturbation theory for Eq. (1.1) were considered in a number of papers. In particular, in Refs. 13–16 they investigated the influence of perturbations on a bion. These papers, however, leave open many questions and are frequently contradictory. It is difficult to cope with these contradictions, inasmuch as different procedures are used and the bulk of the intermediate calculations have been left out as too cumbersome. In addition, the results obtained in this case are pattently not complete enough for comparison with experiment.

We consider it therefore advisable to review the question of the influence of perturbations on a bion from a unified point of view, developing further the basic equations to a degree of completion such that it becomes possible to describe sufficiently fully the evolution of the bion at least in the case (1.2), taking into account the basic factors in the Josephson junction. The paper is organized in the following manner.

The basic equations of first-order perturbation theory for Eq. (1.1) and their concrete versions for the case (1.2) are described in Sec. 2, where a comparison is also made with equations obtained in other papers.

The perturbation-theory equations are investigated and solved in Sec. 3. It follows from the obtained solutions, in particular, that if f exceeds a certain critical value $f_{\rm cr}$, the bion breaks up into a soliton and an antisoliton. Formulas are obtained for $f_{\rm cr}$ in terms of the parameters of the unperturbed bion (at $f = 0$) both for "instantaneous" and for slow switching on of the extraneous current (at $\alpha = 0$). These results given quantitative criteria for the decay and refine the qualitative results of Ref. 12.

We next investigate in detail the evolution of the bion at $f = {\rm const}(f < f_{\rm cr})$ and $\alpha \neq 0$. It turns out that in first order the quantity f does not influence the average parameters of the bion, and their damping law is determined only by the quantity α, with the bion velocity $V(t)$ tending as $t \to \infty$ not to zero [if $V(0) \neq 0$], but to a certain finite value $V(\infty)$ is comparable with $V(0)$ but smaller. All these results together with allowance for the influence of the boundaries of the junction, considered in Ref. 12, present a complete picture of the evolution of the bion both at $f < f_{\rm cr}$ and at $f > f_{\rm cr}$. They admit of experimental verification with the aid, e.g., of methods based on the analysis of the emission spectra of long Josephson junctions, used in Ref. 4.

2. INITIAL EQUATIONS

We start with a perturbation theory based on the method of the inverse scattering problem, or more accurately on that variant of the problem which uses the formalism of variational derivatives[17] and which seems to us the simplest. As applied to the perturbed SG equation it takes the following form.

We set up in correspondence with Eq. (1.1) an eigenvalue problem of the form

$$id\Psi/dx = \hat{U}(x,\ t;\ \lambda)\Psi, \tag{2.1}$$

where

$$\hat{U} = \begin{pmatrix} \lambda/2 - (8\lambda)^{-1}\cos v & i(8\lambda)^{-1}\sin v - (v_x + v_t)/4 \\ -i(8\lambda)^{-1}\sin v - (v_x + v_t)/4 & -\lambda/2 + (8\lambda)^{-1}\cos v \end{pmatrix} \tag{2.2}$$

is a matrix, $\lambda(t)$ is the eigenvalue, $\Psi(x,t;\lambda) = \{\Psi_1, \Psi_2\}$ is a two-dimensional vector function that satisfies the usual quantum-mechanics conditions.

The matrix $\hat{U}$ is the same as for the unperturbed SG equation.[18] The difference, however, lies in the fact that in our case $v(x,t)$ satisfies Eq. (1.1). Since t is regarded in (2.1) as a parameter, those properties of Eq. (2.1) which are not connected with the time dependence are the same as in the case of the unperturbed SG equation if, of course, V satisifes the same boundary conditions

$$v(x,\ t) \to 2\pi n \quad (|x| \to \infty,\ n = 0,\ \pm 1, \ldots). \tag{2.3}$$

Obviously, these conditions can be imposed only if $R\,[v]$ vanishes as $|x| \to \infty$. It must be noted, however, that (1.2) does not satisfy this condition if f is constant. This difficulty can be avoided by assuming first that $f = 0$ outside a sufficiently large interval $-L < x < L$, and then letting $L \to \infty$. Bearing in mind the foregoing stipulations, we now cite from Ref. 18 the main properties of the spectrum and of the eigenfunctions of the problem (2.1), which will be used hereafter.[2] The continuous spectrum of Eq. (2.1) fill the entire real axis, and the discrete one consists of only complex λ. The eigenvalues of the continuous spectrum are doubly degenerate, and those of the discrete one are nondegenerate. It is useful to recall in this connection that if $\Psi(x,\lambda)$ satisfies (2.1), then $\overline{\Psi} = \{\Psi_2^*, -\Psi_1^*\}$ also satisfies the same equation but corresponds to an eigenvalue λ^*. Finally, we consider the Jost functions $F(x,\lambda)$ and $G(x,\lambda)$, defining them as eigenvalues of the problem (2.1) that satisfy at real λ the boundary conditions

$$\begin{aligned} &F \to \{0,\ 1\}\exp[i(\lambda - 1/4\lambda)x/2] \quad (x \to \infty), \\ &G \to \{1,\ 0\}\exp[-i(\lambda - 1/4\lambda)x/2] \quad (x \to -\infty) \end{aligned} \tag{2.4}$$

(here and elsewhere t is not included among the arguments so long as it is unimportant). These functions at real λ are connected by the relations

$$G = A(\lambda)\overline{F} + B(\lambda)F, \quad F = -A(\lambda)\overline{G} + B^*(\lambda)G. \tag{2.5}$$

The functions F and G are analytically continued into the upper half-plane of λ and have no singularities there. The same can be said of the coefficient $A(\lambda)$, which is connected with the components F and G by the relation

$$A(\lambda) = G_1F_2 - G_2F_1. \tag{2.6}$$

For the eigenvalues of the discrete spectrum, which are located in the upper λ half-plane and are designated λ_n ($n = 1,2,\ldots$), the following relations hold:

$$A(\lambda_n) = 0, \quad G(x,\lambda_n) = B_nF(x,\lambda_n). \tag{2.7}$$

The aggregate of the Jost coefficients A and B, as well as the quantities λ_n and B_n, are called the scattering data.

We proceed now to equations that describe the dependence of the scattering data on the time, starting from the following considerations. Equation (2.1) determines all the scattering data as certain functionals of v and v_t. Let $\mathscr{L}[v,v_t]$ by any of these functionals. Its dependence on the time is then given by

$$d\mathscr{L}/dt=\int_{-\infty}^{\infty}[v_t(x)\delta\mathscr{L}/\delta v(x)+v_{tt}(x)\delta\mathscr{L}/\delta v_t(x)]dx,$$

where $\delta\mathscr{L}/\delta v(x)$, etc. are the variational derivatives at the point x. Substituting v_{tt} from (1.1) we obtain

$$d\mathscr{L}/dt=(d\mathscr{L}/dt)_0+\varepsilon\int_{-\infty}^{\infty}R[v(x)]\delta\mathscr{L}/\delta v_t(x)dx, \tag{2.8}$$

where

$$(d\mathscr{L}/dt)_0=\int_{-\infty}^{\infty}[v_t(x)\delta\mathscr{L}/\delta v(x)+(v_{xx}-\sin v)\delta\mathscr{L}/\delta v_t(x)]dx. \tag{2.9}$$

The quantity $(d\mathscr{L}/dt)_0$ is in turn a functional of v and v_t, and this functional is determined only by the concrete form of $\mathscr{L}$ and by Eq. (2.1), which is the same for the perturbed and unperturbed SG equations. Therefore $(d\mathscr{L}/dt)_0$ as a functional of v and v_t has the same form as for the SG equation. This can also be verified with the aid of direct calculations, by transforming (2.9) into

$$(d\mathscr{L}/dt)_0=\{\mathscr{L},\mathscr{H}\}, \tag{2.10}$$

where $\mathscr{H}$ is the Hamiltonian of the unperturbed SG equation:

$$\mathscr{H}[v,v_t]={}^1/_2\int_{-\infty}[(v_t)^2+(v_x)^2+2(1-\cos v)]dx, \tag{2.11}$$

and $\{\mathscr{L},\mathscr{H}\}$ are functional Poisson brackets.[18] The latter can be expressed in terms of the canonical variables of the unperturbed problem, from which it follows, in particular, that $(d\mathscr{L}/dt)_0$ is the same functional of the canonical variables as at $\varepsilon=0$.

Applying (2.8) to the scattering data and using the known expressions for their derivatives with respect to time in the case of the unperturbed SG equation, which can be found, e.g., in Ref. 18, we obtain

$$\begin{gathered}d\lambda_n/dt=\varepsilon\int_{-\infty}^{\infty}R[v(x)]\delta\lambda_n/\delta v_t(x)dx,\\ dB_n/dt=ih(\lambda_n)B_n+\varepsilon\int_{-\infty}^{\infty}R[v(x)]\delta B_n/\delta v_t(x)dx,\end{gathered} \tag{2.12}$$

$$\begin{gathered}\partial A(t;\lambda)/\partial t=\varepsilon\int_{-\infty}^{\infty}R[v(x)]\delta A/\delta v_t(x)dx,\\ \partial B(t;\lambda)/\partial t=ih(\lambda)B+\varepsilon\int_{-\infty}^{\infty}R[v(x)]\delta B/\delta v_t(x)dx,\end{gathered} \tag{2.13}$$

where $h(\lambda)=\lambda+(4\lambda)^{-1}$.

Using next the corresponding equations from (18), with the necessary change of notation, we have for the discrete spectrum

$$\delta\lambda_n/\delta v_t(x)=-iB_n(4A_n')^{-1}[F_2^2(x,\lambda_n)-F_1^2(x,\lambda_n)], \tag{2.14}$$

$$\begin{aligned}\delta B_n/\delta v_t(x)=iB_n(4A_n')^{-1}(&F_1\partial G_1/\partial\lambda-G_1\partial F_1/\partial\lambda\\ &+G_2\partial F_2/\partial\lambda-F_2\partial G_2/\partial\lambda)_{\lambda=\lambda_n},\end{aligned} \tag{2.15}$$

where $A_n'=dA(\lambda)/d\lambda|_{\lambda=\lambda_n}$. For the continuous spectrum we have

$$\begin{gathered}\delta A/\delta v_t(x)=i(F_2G_2-F_1G_1)/4,\\ \delta B/\delta v_t(x)=i(F_1^*G_2+F_2^*G_1)/4.\end{gathered} \tag{2.16}$$

Equations (2.12)–(2.16) are the basis of the perturbation theory in the general case. Apart from the notation, they coincide with the corresponding equations obtained earlier in Refs. 19 and 14 on the basis of another more complicated procedure. We consider hereafter their applications to perturbed bions.

3. EVOLUTION OF A BION IN FIRST ORDER PERTURBATION THEORY

In first order we can write

$$v(x,t)=v_a(z,\theta(t);\zeta)+\delta v(x,t), \tag{3.1}$$

where in the case (1.2)

$$\delta v=\varepsilon v_1-f,\quad v_1\to0\quad(|x|\to\infty), \tag{3.1a}$$

$v_a(z,\theta;\zeta)$ is the expression for the bion in the adiabatic approximation, and δv is the first-order correction that determines the change of its form. According to the general scheme, $v_a(z,\theta;\zeta)$ is given by Eqs. (1.3)–(1.5), where now ζ, and consequently also all the parameters ν, η, and V, are functions of the time. In addition, the coordinate $x_0(t)$ of the center of the bion and its phase $\theta(t)$ are now determined by equations that differ generally speaking from (1.6) by terms of order ε.

To find all these functions we express the bion parameters in terms of the scattering data for the potential (3.1). In this case the discrete spectrum in the upper half-plane consists of two eigenvalues

$$\lambda_1=\zeta(t)=\eta+i\nu,\quad\lambda_2=-\zeta^*(t),\quad\nu>0,\quad\eta>0, \tag{3.2}$$

and the corresponding parameters B_n are given by

$$B_1=-B_2^*=2i\zeta(\nu/\eta)A_1'\exp[i\theta-i(\zeta-1/4\zeta)x_0]. \tag{3.3}$$

We substitute (3.2) and (3.3) in Eqs. (2.12)–(2.15). Confining ourselves to terms of first order in ε, we choose the Jost functions in the adiabatic approximation. Putting $v=v_a$ in (2.1), we have the following expressions for $F=\{F_1,F_2\}$ and G:

$$\begin{aligned}&F_1=2\eta\nu\Gamma[i|\zeta|^2\operatorname{ch}z\cos\varphi+\lambda(\eta\operatorname{ch}z\sin\varphi+\nu\operatorname{sh}z\cos\varphi)],\\ &F_2=\Gamma[\eta^2(\lambda^2-|\zeta|^2)\operatorname{ch}^2z+\nu^2(\lambda^2+|\zeta|^2)\cos^2\varphi\\ &\qquad+2i\lambda\eta\nu(\eta\operatorname{sh}z\operatorname{ch}z-\nu\sin\varphi\cos\varphi)],\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\Gamma=[(\lambda+\zeta)(\lambda-\zeta^*)(\eta^2\operatorname{ch}^2z+\nu^2\cos^2\varphi)]^{-1}\\ &\quad\times\exp[i(\lambda-1/4\lambda)x/2],\\ &\qquad G=(\lambda-\zeta)(\lambda+\zeta^*)(\lambda-\zeta^*)^{-1}(\lambda+\zeta)^{-1}\bar{F}.\end{aligned} \tag{3.5}$$

As a result we arrive at the following equations of first approximation for the parameters of the perturbed bion, defining its characteristic amplitude as

$$\gamma=\operatorname{arctg}(\nu/\eta), \tag{3.6}$$

and using V from (1.5):

$$d\gamma/dt=\varepsilon(1-V^2)^{1/2}(4\cos\gamma)^{-1}I_1, \quad (3.7)$$

$$dV/dt=-\varepsilon(1-V^2)^{3/2}(4\cos\gamma)^{-1}I_2, \quad (3.8)$$

$$d\theta/dt=\cos\gamma(1-V^2)^{1/2}-\varepsilon(1-V^2)^{1/2}[V\operatorname{ctg}\gamma I_3 +\cos^2\gamma(1-V^2)I_4-I_5](4\sin\gamma\cos^2\gamma)^{-1}, \quad (3.9)$$

$$dx_0/dt=V+\varepsilon(1-V^2)(I_3-V\operatorname{tg}\gamma I_4)(2\sin\gamma)^{-2}. \quad (3.10)$$

We have put here

$$I_1=\int_{-\infty}^{\infty}\frac{\operatorname{ch}z\sin\varphi}{\operatorname{ch}^2 z+\operatorname{tg}^2\gamma\cos^2\varphi}R[v_a(z)]dz, \quad (3.11)$$

$$I_2=\int_{-\infty}^{\infty}\frac{\operatorname{sh}z\cos\varphi}{\operatorname{ch}^2 z+\operatorname{tg}^2\gamma\cos^2\varphi}R[v_a(z)]dz, \quad (3.12)$$

$$I_3=\int_{-\infty}^{\infty}\frac{z\operatorname{ch}z\sin\varphi}{\operatorname{ch}^2 z+\operatorname{tg}^2\gamma\cos^2\varphi}R[v_a(z)]dz, \quad (3.13)$$

$$I_4=\int_{-\infty}^{\infty}\frac{z\operatorname{sh}z\cos\varphi}{\operatorname{ch}^2 z+\operatorname{tg}^2\gamma\cos^2\varphi}R[v_a(z)]dz, \quad (3.14)$$

$$I_5=\int_{-\infty}^{\infty}\frac{\operatorname{ch}z\cos\varphi}{\operatorname{ch}^2 z+\operatorname{tg}^2\gamma\cos^2\varphi}R[v_a(z)]dz, \quad (3.15)$$

where φ is a linear combination of θ and z and is expressed by Eq. (1.4).

Equations (3.7)–(3.10) determine in principle the evolution of bion parameters that enter into the adiabatic part of Eq. (3.1). The second term in (3.1) can be obtained on the basis of Eq. (2.13), which describe the evolution of the Jost coefficient of the continuous spectrum of Eq. (2.1). The corresponding procedure is similar to that used in Ref. 19 for the nonlinear Schrödinger equation. For single-soliton solutions of Eq. (1.1), the value of δv was obtained in Ref. 20; it is also shown there that general relations lead in the case of (1.2) to (3.1a). We shall not consider here this question for bions because the calculations are cumbersome. We note also that to find the adiabatic part of (3.1) as a function of the time, with an error of the order of ε at $\varepsilon t\sim 1$, it is necessary to add to Eqs. (3.7) and (3.8) terms of order ε^2; otherwise the error in the termination of $\theta(t)$ and $x_0(t)$ from (3.9) and (3.10) will be of the order of unity at $\varepsilon t\sim 1$. This situation in general and was noted earlier in Refs. 17 and 19 for other cases. To obtain physical results, however, this circumstance is not very significant; to calculate however, a sufficiently accurate profile of the wave it is simpler to integrate the initial equation (1.1) numerically than to solve Eqs. (3.7)–(3.10) with the required accuracy.

We proceed now to an investigation of the system (3.7)–(3.10), assuming that the perturbation is of the form (1.2).

We note first that in this case the system (3.7)–(3.10) has a solution with

$$V(t)=0, \quad dx_0/dt=0. \quad (3.16)$$

This follows from the fact that at $V=0$ Eq. (1.4) leads to $\varphi=\theta(t)$. In this case $R[v_a(z)]$ becomes an even function of z. Because of this $I_2=I_3=0$, which leads to relations (3.16) which have a simple physical meaning, namely, the perturbation (1.2) cannot take the bion out of the quiescent state.

The remaining equations of the system take at $V=0$ the form

$$\frac{d\gamma}{dt}=-\frac{\pi f}{4}\frac{\sin\theta}{\cos\gamma(1+\operatorname{tg}^2\gamma\cos^2\theta)^{1/2}}-\alpha\frac{\operatorname{tg}\gamma\sin^2\theta}{1+\operatorname{tg}^2\gamma\cos^2\theta}\left[1+\frac{\operatorname{ctg}\gamma\operatorname{arsh}(\operatorname{tg}\gamma\cos\theta)}{\cos\theta(1+\operatorname{tg}^2\gamma\cos^2\theta)^{1/2}}\right], \quad (3.17)$$

$$\frac{d\theta}{dt}=\cos\gamma-\frac{\pi f}{4\sin\gamma}\left[\frac{\cos\theta}{\cos^2\gamma(1+\operatorname{tg}^2\gamma\cos^2\theta)^{1/2}}-\operatorname{ctg}\gamma\operatorname{arsh}(\operatorname{tg}\gamma\cos\theta)\right]$$
$$-\alpha\frac{\cos\theta\sin\theta}{\cos^2\gamma(1+\operatorname{tg}^2\gamma\cos^2\theta)}-\alpha\frac{\operatorname{tg}\gamma\operatorname{arsh}(\operatorname{tg}\gamma\cos\theta)}{(1+\operatorname{tg}^2\gamma\cos^2\theta)^{3/2}}\sin^3\theta. \quad (3.18)$$

An equation for $d\gamma/dt$ at $V=0$, different from our (3.17), was obtained in Ref. 15. The difference is contained, first, in the term proportional to f. As for the term with α, it would coincide with (3.17) if we were to put in the latter $\theta=(\cos\gamma)t$, but this is generally speaking incorrect. No equation was obtained in Ref. 15 for $d\theta/dt$.

The phase trajectories of the system (3.17) and (3.18) at fixed f are shown in the figure. The singular points (where $d\gamma/dt=d\theta/dt=0$) are defined by the equations

$$\theta_s=2k\pi \quad (k=0,\pm1,\ldots), \quad (3.19)$$

$$\sin^2(2\gamma_s)[\sin\gamma_s-\cos^2\gamma_s\operatorname{arsh}(\operatorname{tg}\gamma_s)]^{-1}=\pi f \quad (3.20)$$

and are, as can be easily shown, saddle points. It is important that they do not depend on α. At $f\ll 1$, the solution of (3.20) takes the form

$$\operatorname{tg}\gamma_s\approx(4/\pi f)^{1/3}. \quad (3.21)$$

At $\theta=(2k+1)\pi(k=0,\pm 1,\ldots)$ all the phase curves have minima. At $\theta=2k\pi$ and $\gamma>\gamma_s$ the functions $\gamma(\theta)$ have minima, and $\gamma<\gamma_s$ they have maxima. As $\gamma\to\pi/2$ and not too close to $\gamma=\gamma_s$ we obtain from (3.17) and (3.18) an asymptotic solution of the equation for $d\gamma/d\theta$ in the form

$$\operatorname{tg}\gamma\cos\theta=\text{const}. \quad (3.22)$$

It follows therefore that $\gamma\to\pi/2$ as $\theta\to(k+1/2)\pi$. The phase curve that starts out from one such point either terminates at a neighboring point where $\gamma=\pi/2$, or goes over into an oscillating trajectory with gradually decreasing γ. The curves of the first kind correspond to a system consisting of soliton and an uncoupled antisoliton, which at $t=\pm\infty$ are at infinite distance from each other. Trajectories of the second type [they lie between lines (1) and (2)] correspond to a system that consisted at $t=-\infty$ of a soliton infinitely far from an antisoliton; as the two come closer to-

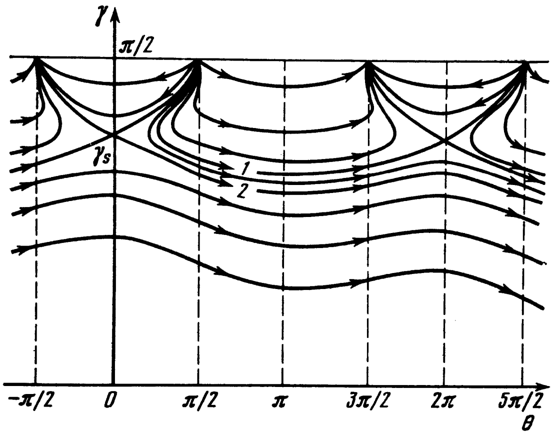

FIG. 1.

gether, they go over into a bound state, i.e., into a bion whose amplitude gradually attenuates because of dissipation.

As $\alpha \to 0$ curves (1) and (2) merge and are transformed into a single separatrix that passes through all the singular points defined in (3.19) and (3.20). In this case all the trajectories lie above this separatrix represent uncoupled soliton-antisoliton systems, and those below the separatrix represent coupled ones, i.e., bions.

Let us investigate in greater detail the phase trajectories at $\alpha = 0$. In this case the system (3.17),(3.18) is a Hamilton system with a Hamiltonian

$$H(\gamma, \theta) = \sin\gamma - \pi f \operatorname{arsh}(\operatorname{tg}\gamma\cos\theta)(4\sin\gamma)^{-1}, \quad (3.23)$$

so that the phase curves are described by the equations $H(\gamma, \theta) = \text{const}$. At $\tan\gamma \gg 1$ we get from this, in particular, (3.22) (which, however, is valid also at $\alpha \neq 0$). The equation of the separatrix can be written in the form $H(\gamma, \theta) = H_s$ where obtained from (3.23) at $\theta = 0$ and $\gamma = \gamma_s$, where H_s is obtained from (3.23) at $\theta = 0$ and $\gamma = \gamma_s$. At small f we have

$$H_s \approx 1 - (\pi f/8)[\ln(16/\pi f) + 1], \quad (3.24)$$

where we have used (3.21). In the same case, for the curves representing bions, we have $\gamma - \bar{\gamma} = O(f)$ where

$$\bar{\gamma} = (1/2\pi)\int_{-\pi}^{\pi}\gamma(\theta)\,d\theta \quad (3.25)$$

is the mean value. Thus, at $\alpha = 0$ a time-constant extraneous current f causes only periodic variations of the bion amplitude, of the order of f, which generally speaking are of little significance. However, a monotonic increase of f leads, as we shall presently show, to a decay of the bion if f exceeds a certain critical value f_{cr} that depends on the initial values of the amplitude $\gamma^{(0)} = \gamma_0$ and of the phase $\theta(0) = \theta_0$.

To verify this, we consider first the case of a "instantaneous" increase of an extraneous current from zero at $t < 0$ to f at $t > 0$. Then γ and θ have no time to change, and H changes from $H = \sin\gamma_0$ to the value given by (3.23), where $\gamma = \gamma_0$ and $\theta = \theta_0$. At $t < 0$ the separatrix was the straight line $\gamma = \pi/2$. At $t > 0$ it drops lower and its equation becomes $H = H_s$. All the phase-plane points that turn out to be above the separatrix correspond to uncoupled "decay products" of the bion. Consequently the equation that determines f_{cr} as a function of the initial state of the bion is obtained from (3.23) at $f = f_{cr}$, $\gamma = \gamma_0$, $\theta = \theta_0$ and $H = H_s$. At small f_{cr} it can be assumed that γ_0 is close to $\pi/2$, i.e., $\sin\gamma_0 \approx 1 - (1/2)\cot^2\gamma_0$. As a result we obtain from (3.23) and (3.24) the equation

$$\pi f_{cr} \operatorname{arsh}(\operatorname{tg}\gamma_0\cos\theta_0)(2\sin\gamma_0)^{-1} + \operatorname{ctg}^2\gamma_0 = (\pi f_{cr}/4)[\ln(16/\pi f_{cr}) + 1]. \quad (3.26)$$

We consider now several typical cases. At $\theta_0 = 2k\pi$ an approximate solution of (3.26) is

$$f_{cr} \approx (4/\pi)\operatorname{ctg}^2\gamma_0. \quad (3.27)$$

At $\theta_0 = (k + 1/2)\pi$ we arrive at the equation

$$(\pi f_{cr}/4)[\ln(16/\pi f_{cr}) + 1] = \operatorname{ctg}^2\gamma_0. \quad (3.28)$$

This equation is equivalent to Eq. (21) of Ref. 12. In the same reference is given a plot of $f_{cr}(\cot^2\gamma_0)$ that follows from (3.28). It can be verified that the values of f_{cr} are in this case always smaller than (3.27). Finally, the smallest value of f_{cr} at fixed γ_0 is obtained if $\theta_0 = (2k + 1)\pi$.

We consider now another limiting case, when the extraneous current is turned on adiabatically. In this case, the adiabatic invariant, which according to (3.25) coincides with $\bar{\gamma}$, shoud be conserved. Thus, in adiabatic switching $\bar{\gamma} = \gamma_0$. The condition of the bion decay is obtained approximately by putting $\bar{\gamma} \approx (\bar{\gamma})_s$, where $(\gamma)_s$ is the average value of γ on the separatrix. From the equation for the separatrix we find that $\cot^2(\bar{\gamma})_s \approx (\pi f/4)[\ln(16/\pi f) + 1]$. Putting here $(\bar{\gamma})_s = \gamma_0$ and $f = f_{cr}$, we again arrive at (3.28). Thus, when the extraneous current is turned on adiabatically, f_{cr} agrees approximately with the value of f_{cr} for instantaneous switching at the same γ_0 and $\theta_0 = (k + 1/2)\pi$.

These results agree well with the bion-decay criteria obtained in Ref. 12 on the basis of a quasiparticle model; this can be regarded as a verification of the latter from a more rigorous point of view. (In comparisons with Ref. 12 it must be borne in mind that $2\cot^2\gamma_0 = -E_0$, where E_0 is the sum of the kinetic and potential energies of the solitons that make up the bion at $f = 0$.) Notice should also be taken of the good quantitative agreement between the described experiment and the numerical experiment. Thus, a direct numerical solution of (1.1) under conditions corresponding to instantaneous switching on of f and at $\alpha = 0$, $\cot^2\gamma_0 = 0.0785$, and $\theta_0 = 3\pi/2$ yields $f_{cr} \approx 0.0136$ (Ref. 21), as against $f_{cr} \approx 0.015$ obtained from (3.28).

At $\alpha > 0$ the value of f_{cr} should be larger than at $\alpha = 0$, inasmuch as it follows from the averaged system (3.17) and (3.18) that $f_{cr}(\alpha)$ is a monotonically increasing function. In addition, at $\alpha > 0$ there arises a substantially new effect: A soliton and antisoliton moving towards each other from $x = \pm\infty$ can *form* a bound state even at constant f; for this prupose it is necessary that the phase trajectory of the system lie between curves 1 and 2 in the figure. It can be seen from the same figure that the produced bion attenuates in time because of dissipation.

We consider now in greater detail the dynamics of the bion in the general case when $f \neq 0$, $\alpha \neq 0$, and $V \neq 0$. Noting that the right-hand sides of (3.7)–(3.10) are periodic functions of θ, with $d\gamma/dt \sim dV/dt \sim \varepsilon$, we can assume for a bound state that is not too close the separatrix (i.e., where it is meaningful to separate the slow dependence on θ from the rapid variations with period 2π):

$$\theta=\bar{\theta}+\varepsilon\varphi_3(\bar{\gamma},\bar{V},\bar{\theta})+O(\varepsilon^2),\quad x_0=\bar{x}_0+\varepsilon\varphi_4(\bar{\gamma},\bar{V},\bar{\theta})+O(\varepsilon^2),\tag{3.29}$$

$$\gamma=\bar{\gamma}+\varepsilon\varphi_1(\bar{\gamma},\bar{V},\bar{\theta})+O(\varepsilon^2),\quad V=\bar{V}+\varepsilon\varphi_2(\bar{\gamma},\bar{V},\bar{\theta})+O(\varepsilon^2),$$

$$\dot{\bar{\gamma}}=\varepsilon\psi_1(\bar{\gamma},\bar{V})+O(\varepsilon^2),\quad \dot{\bar{V}}=\varepsilon\psi_2(\bar{\gamma},\bar{V})+O(\varepsilon^2),\tag{3.30}$$

$$\dot{\bar{\theta}}=\cos\bar{\gamma}(1-\bar{V}^2)^{1/2}+\varepsilon\psi_3(\bar{\gamma},\bar{V})+O(\varepsilon^2),$$
$$\dot{\bar{x}}_0=\bar{V}+\varepsilon\psi_4(\bar{\gamma},\bar{V})+O(\varepsilon^2),\tag{3.31}$$

where the functions φ_i and ψ_i are to be determined. Equations (3.29)–(3.31) correspond to the first approximation of the Krylov-Bogolyubov method. It is implied in them that $\bar{\gamma}$ and $\bar{V}$ are slow variables, and $\bar{\theta}$ is a renormalized fast variable. Substituting (3.29) in (3.7)–(3.10) we solve the obtained equations under the following condition: The functions $\varphi_i(\bar{\gamma},\bar{V},\bar{\theta})$ should be periodic in $\bar{\theta}$ with a period 2π. As a result we obtain

$$\dot{\bar{\gamma}}=-\alpha\bar{\gamma},\quad \dot{\bar{\theta}}=\cos\bar{\gamma}(1-\bar{V}^2)^{1/2},$$
$$\dot{\bar{V}}=-\alpha\bar{V}(1-\bar{V}^2)(1-\bar{\gamma}\operatorname{ctg}\bar{\gamma}),\quad \dot{\bar{x}}_0=\bar{V}.\tag{3.32}$$

It is useful to note that because of the periodicity condition $\varphi_i(\bar{\theta})$ the right-hand side of (3.32) turns out to be automatically the average, over $\bar{\theta}$, of Eqs. (3.7), (3.8), and (3.10), provided the substitutions $\gamma\to\bar{\gamma}, V\to\bar{V},\theta\to\bar{\theta}$ are made in the latter. This is a consequence of the general properties of the lowest approximation of the Krylov-Bogolyubov method (see, e.g., Ref. 22). We note also that in (3.32) there are no terms with f, since the terms in I_1 and I_2, which contain f, vanish after averaging over θ. Thus, a constant extraneous current exerts no influence on the averaged parameters of the bion (provided, of course, that $f<f_{cr}$; in the opposite case the bion decays in accordance with the theory expounded above).

For lack of space we shall not present here the rather long expressions for φ_i $(i=1,2,3,4)$. which are obtained immediately once we find $\varepsilon\psi_1$ and $\varepsilon\psi_2$ which coincide with the right-hand sides of (3.32). We indicate only that φ_i are automatically periodic in $\bar{\theta}$, as required. Therefore at $t\lesssim 1/\varepsilon$ we have $|\gamma\sim\bar{\gamma}|\sim|V-\bar{V}|\sim\varepsilon$, $|\theta-\bar{\theta}|\sim\varepsilon$, i.e., the terms $\varepsilon\varphi_i$, after substitution of (3.29) in (3.1), lead in the expression for v to terms of order ε that can be unified with the term δv; it is therefore meaningful to discuss them only together with δv as a combined correction of order ε to the form of the bion. The latter, however, is not considered in the present paper.

In connection with the foregoing we shall discuss Ref. 16, where a study was made of the action of the perturbation (1.2) on a bion, using a different approach, and where a much more complicated system than (3.7)–(3.15) was obtained and then solved numerically at $V=0$. Our analysis has shown that the equations of Ref. 16 contain many errors. After eliminating these errors and solving the obtained system we have arrived again at (3.32), but obtained for φ_i expressions different from those that follow from (3.7) and (3.8). Thus, the small oscillating additions to $\bar{\gamma}$, $\bar{V}$, $\bar{\theta}$, and $\bar{x}_0$ are not uniquely determined. This ambiguity is offset by a suitable change of δv. As for the phase trajectories at $\gamma>\gamma_s$, for $f\ll 1$ we obtain from the corrected equations of Ref. 16 the same results as from ours.

We return now to the system (3.32). Its solution is

$$\bar{\gamma}=\gamma(0)\exp(-\alpha t),\tag{3.33}$$

$$\bar{V}=\left\{1+\frac{1-V^2(0)}{V^2(0)}\frac{\gamma^2(0)}{\sin^2\gamma(0)}\frac{\sin^2\bar{\gamma}(t)}{\bar{\gamma}^2(t)}\right\}^{-1/2}.\tag{3.34}$$

Equation (3.33) was obtained earlier in Ref. 15 at $\bar{\gamma}(t)\ll 1$. It follows from our reasoning that this formula is valid not only at small $\bar{\gamma}(0)$ but at all $\bar{\gamma}(0)$ not too close to the values of γ_s defined in (3.21).

We turn next to (3.34). We can see from it that

$$\bar{V}(\infty)=V(0)\sin\gamma(0)\{V^2(0)\sin^2\gamma(0)+[1-V^2(0)]\gamma^2(0)\}^{-1/2}.\tag{3.35}$$

Thus, in contrast to the amplitude $\bar{\gamma}(t)$, the bion velocity $\bar{V}(t)$ does not vanish if $V(0)\neq 0$. In this case $V(\infty)/V(0)$, while less than unity, is generally speaking not very small. It must, however, be noted that as $t\to\infty$, when the amplitude is small, perturbation theory is no longer valid, since both terms in (3.1) become comparable. In addition, the quantitative applicability of (3.33) and (3.34) is strictly speaking, limited, since at $\varepsilon t\sim 1$ it is necessary, as already mentioned, to take into account terms of order ε^2 in $d\gamma/dt$ and dV/dt, in order to obain $\theta(t)$ accurate to terms of order ε at $\varepsilon t\sim 1$. This, however, is not very important for the qualitative aspect of the situation, since t in (3.33) and (3.34) can be regarded as a certain parameter that increases monotonically together with the true time and has the same order of magnitude. [It would be possible to get along without introducing t at all, and solve the equations for $d\gamma/d\theta$ and $dV/d\theta$ which follow from (3.7)–(3.9). Since $\theta(t)$ is a monotonic function and $\theta(\infty)=\infty$, all the quantitative results described above remain in force.]

We have thus shown that the action of the perturbation (1.2) at constant (or at sufficiently slowly varying in time) f consists in the following. If f exceeds the critical value f_{cr} [see (3.26)], the bion decays into a soliton and an uncoupled antisoliton. If $f<f_{cr}$, then f has practically no influence on the average bion parameters $\bar{\gamma}(t)$ and $\bar{V}(t)$, which decrease according to (3.33) and (3.34). In this case, if $V(0)\neq 0$, then $\bar{V}(\infty)/V(0)$, while less than unity, is nevertheless not very small. These results can be experimentally verified in principle by studying the emission spectra of long Josephson junctions in analogy with Ref. 4.

We have assumed above that the Josephson line is infinitely long. A bion reflected from the end of the junction is preserved, so that the influence of the boundaries reduces to losses to radiation, so long as $f<f_{cr}$. At $f>f_{cr}$ the bion decays, and reflection of individual solitons reverses their polarization. This circumstance is important, since it can make the produced solitons equally polarized within a certain time. In this case they can, according to Refs. 10 and 11, form a bunched system. As a result, at $f>f_{cr}$ the bion is transformed in final analysis into a bunched pair of silitons with identical polarization.[12] This effect was observed in numerical experiments[12] that simulated the conditions in real Josephson junctions. From the results of Refs. 10, 11, and 21 it can be seen in this case that the bunched pair is completely restored after reflection. The foregoing explains also the formation of like-polarized bunched pairs out of the initially differently polarized solitons in laboratory experiments.[4]

[1)]In the English literature such a formation is most frequently called "breather."

[2)]It must be kept in mind here that Ref. 18 has as the independent variables $\xi = -x$ and $\tau = t$, and contains a misprint: On p. 104 in Eq. (14) the last matrix should be preceded by a plus sign.

[1]R. D. Parmentier, in Solitons in Action, K.Lonngren and A. Scott, eds. Academic, 1978. Russ. transl. Mir, 1981, p. 185.
[2]T. A. Fulton and R. C. Dynes, Sol. State Commun. **12**, 57 (1973).
[3]T. A. Fulten and L. N. Dunkleberger, Rev. Phys. Appl. **9**, 299 (1974).
[4]B. Dueholm, E. A. Levring, J. Mygind, N. F. Pederson, O. H. Soerensen, and M. Cirillo, Phys. Rev. Lett. **46**, 1299 (1981).
[5]K. Nakajima, T. Yamashita, and Y. Onodera, J. Appl. Phys. **45**, 3141 (1974).
[6]K. Nakajima, Y. Onodera, T. Nakamura, and R. Sato, J. App. Phys. **45**, 4095 (1974).
[7]S. N. Erné and R. D. Parmentier, J. Appl. Phys. **52**, 1091 (1981).
[8]P. L. Christiansen, P. S. Lomdahl. A. C. Scott, O. H. Soerensen, and J. C. Eilbeck, Appl. Phys. Lett. **39**, 108 (1981).
[9]M. Cirillo, R. D. Parmentier, and B. Savo, Physica **3D**, 565 (1981).
[10]V. I. Karpman, N. A. Ryabova and V. V. Solov'ev, Phys. Lett. **85A**, 251 (1981).
[11]V. I. Karpman, V. A. Ryabova, and V. V. Solov'ev, Zh. Eksp. Teor. Fiz. **81**, 1327 (1981) [Sov. Phys. JETP **54**, 705 (1981)].
[12]V. I. Karpman, Phys. Lett. **88A**, 207 (1982).
[13]J. P. Keener and D. W. McLaughlin, Phys. Ref. A **16**, 777 (1977).
[14]D. J. Kaup and A. C. Newell, Proc. R. Soc. Lond. A **361**, 413 (1978).
[15]D. W. McLaughlin and A. C. Scott, Phys. Rev. A **18**, 1652 (1978).
[16]M. Inoue, J. Phys. Soc. Japan **47**, 1723 (1979).
[17]V. I. Karpman, Phys. Scripta **20**, 462 (1979).
[18]V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskii, Teoriya solitonov (Soliton Theory), Nauka, 1980.
[19]V. I. Karpman and E. M. Maslov, Zh. Eksp. Teor. Fiz. **73**, 537(1977); **75**, 504 (1978) [Sov. Phys. JETP **46**, 281 (1977); **48**, 252 (1978)].
[20]V. I. Karpman and V. V. Solov'ev, Phys. Lett. **84A**, 39 (1981).
[21]V. I. Karpman, N. A. Ryabova, and V. V. Solov'ev, Breather Decay into Fluxon-Antifluxon Pair and Its Transformation in a Long Josephson Junction, IZMIRAN preprint No. 52a, 1981.
[22]N. N. Moiseev, Asimptoticheskie metody nelineinoĭ mekhaniki (Asymptotic Methods of Nonlinear Mechanics), Nauka, 1981.

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